# Collective State Spaces

Noriaki Kiguchi

(joint with Takashi Hayashi and Norio Takeoka)

April 23, 2025

# What We Did

- Dekel, Lipman, and Rustichini's (2001) (DLR) preferences:
  - for a set $X$, which is referred to as a *menu*,

$$U(X) = \sum_{s \in S} \pi(s) \max_{x \in X} u(x, s),$$

  where

    - $S$: a subjective state space,
    - $\pi$: a probability measure over $S$,
    - $u : A \times S \to \mathbb{R}$: a state dependent utility function.

- We **aggregate individual DLR preferences into social DLR preferences**.
  - No paper has tackled this aggregation problem yet.

# Brief Explanation of DLR Preferences

- A DM buys food for tomorrow's lunch.
  - Relevant states in their mind: $\{\text{sunny}, \text{rainy}\}$.
  - Tastes over food:
    - $u\,(\text{ice cream}, \text{sunny}) > u\,(\text{apple pie}, \text{sunny})$,
    - $u\,(\text{apple pie}, \text{rainy}) > u\,(\text{ice cream}, \text{rainy})$.
- The DM wants to buy both today:

$$\begin{aligned} & U\,(\{\text{ice cream}, \text{apple pie}\}) \\ = & \pi\,(\text{sunny})\,u\,(\text{ice cream}, \text{sunny}) + \pi\,(\text{rainy})\,u\,(\text{apple pie}, \text{rainy}) \\ > & \pi\,(\text{sunny})\,u\,(\text{ice cream}, \text{sunny}) + \pi\,(\text{rainy})\,u\,(\text{ice cream}, \text{rainy}) \\ = & U\,(\{\text{ice cream}\})\,. \end{aligned}$$

# What We Did (Reprinted)

- Dekel, Lipman, and Rustichini's (2001) (DLR) preferences:
  - for a set $X$, which is referred to as a *menu*,

$$U(X) = \sum_{s \in S} \pi(s) \max_{x \in X} u(x, s),$$

  where

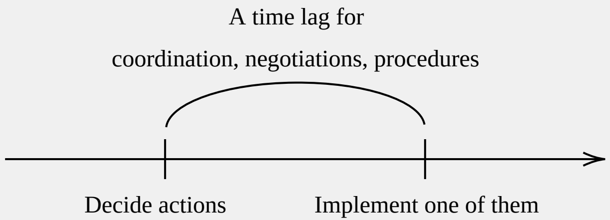
    - $S$: a subjective state space,
    - $\pi$: a probability measure over $S$,
    - $u : A \times S \to \mathbb{R}$: a state dependent utility function.

- We **aggregate individual DLR preferences into social DLR preferences**.
  - No paper has tackled this aggregation problem yet.

# Motivating Story

- Consider a meeting in a large company, which is held by
  - CEO (= society),
  - division heads (= individuals).
    - E.g., automobiles, social networking services (SNS), artificial intelligence (AI).
- They decide on the next action,
  - e.g., determining which another company to acquire.

# Why Menu Preferences?




- A large group needs to make decisions well before implementation.
  - But, effectiveness of actions is uncertain at the decision stage:
    - It depends on the circumstances during their implementation.

$\Rightarrow$ At the decision stage, multiple actions are required as candidates for the best option.

- Multiple actions = a menu.

# Why Subjective States? and Why Aggregation?

- The relevant states differ entirely across the divisions.
  - Automobile industry: $\{\text{gasoline engines}, \text{hydrogen engines}\}$.
  - AI industry: $\{\text{Google}, \ \text{Apple}\}$.

$\Rightarrow$ The devision heads hold different preferences over menus of actions.

- How should the CEO aggregate these preferences?
  - Especially, how should the CEO construct a comprehensive state space?

$$\begin{matrix} \{\text{gasoline engines}, \text{hydrogen engines}\} \\ \{\text{Google}, \ \text{Apple}\} \end{matrix} \Longrightarrow ?$$

# Model

- $A$: a finite set.
  - We refer to $a \in A$ as an *outcome*.
- $\Delta(A)$: the set of probability distributions over $A$.
  - We refer to $l = (l(a))_{a \in A} \in \Delta(A)$ as a *lottery* (= action).
- $\mathcal{K}(\Delta(A))$: the set of nonempty and compact subsets in $\Delta(A)$, which is endowed with the Hausdorff topology.
  - We refer to $X \in \mathcal{K}(\Delta(A))$ as a *menu*.

# Individual and Social Preferences

- $N = \{1, \ldots, n\}$: a set of individuals.
- Index 0 represents society.
- $\succsim_i$: a complete and transitive binary relation on the set of menus, $\mathcal{K}(\Delta(A))$.
  - $X \succsim_i Y$: Individual $i$ evaluates that $X$ is at least as good as $Y$.

# Aggregation Problem

- $(\succsim_i)_{i\in N}$ and $\succsim_0$ admit the DLR representation:

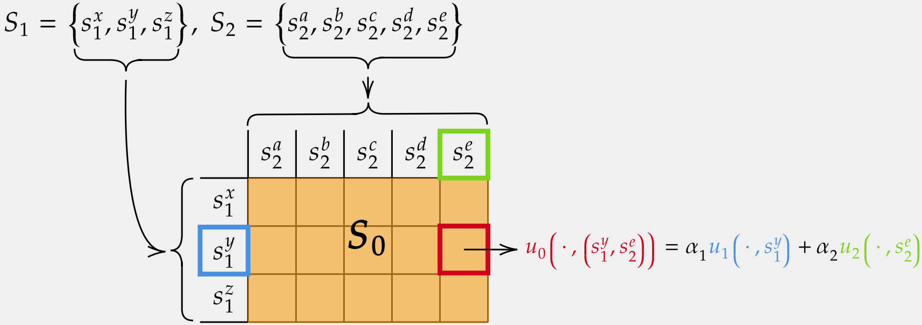
$$U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i).$$

  - $S_i$: a finite set.
  - $\pi_i$: a full support probability measure over $S_i$.
  - $u_i : \Delta(A) \times S_i \to \mathbb{R}$: a state dependent utility function.
    - Each $u_i(\cdot\,, s_i)$ is mixture-linear.

**Question:**

- How should society aggregate $(S_i, \pi_i, u_i)_{i\in N}$ into $(S_0, \pi_0, u_0)$?

# Representation: Rough Preview




- The following **4 axioms** characterize this representation:
    1. two restricted Pareto conditions,
    2. a violation of Pareto indifference,
    3. a rationality axiom.

# Outline of the Remaining Part

1. Preliminary clarifications on DLR preferences
2. A benchmark Pareto indifference
   2.1 An impossibility theorem
   2.2 Discussion
3. Our axioms
   3.1 Two axioms from the above discussion
   3.2 Two further axioms
4. Representation theorem
5. Proof

# Features of DLR Preferences

**DLR Representation:** $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$.

- For all $X \supset Y$, $X \succsim_i Y$ must hold.
- $X \cup \{l\} \succ_i X$: "Individual $i$ has a possibility to need option $l$."

$\iff$ There exists $s_i \in S_i$ such that $u_i(l, s_i) > u_i(l', s_i)$ for all $l' \in X$.

  - We do not know whether $u_i(l, s_i') \gtreqless u_i(l', s_i')$ under other $s_i' \in S_i$.

- $X \cup \{l\} \sim_i X$: "Individual $i$ will never need option $l$."

$\iff$ For each $s_i \in S_i$, there exists $l_{s_i} \in X$ such that $u_i(l_{s_i}, s_i) \geq u_i(l, s_i)$.

# Benchmark Pareto Indifference

## Expanding Pareto Indifference

For all menus $X \in \mathcal{K}(\Delta(A))$ and all lotteries $l \in \Delta(A)$,

$$X \cup \{l\} \sim_i X \text{ for all } i \in N \Longrightarrow X \cup \{l\} \sim_0 X.$$

**Interpretation:**

- If no one needs option $l$, then neither does society.

# Benchmark Theorem: Ex-post Dictatorship

**DLR Representation:** $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$.

Theorem

*The DLR preference profile,* $(\succsim_i)_{i \in N}$ *and* $\succsim_0$*, satisfies Expanding Pareto Indifference if and only if for each* $s_0 \in S_0$*, there exist* $i \in N$ *and* $s_i \in S_i$ *such that* $u_0(\cdot\,, s_0) = u_i(\cdot\,, s_i)$*.*

**Interpretation:**

- It says $S_0 \subset S_1 \cup \cdots \cup S_n$.
  - $\rightarrow$ Society plans to focus exclusively on one aspect.

# Discussions about Expanding Pareto Indifference

**Example:**

- $N = \{1, 2\}$, $S_1 = \{s_1\}$, and $S_2 = \{s_2\}$.

  $\Rightarrow\ U_i(X) = \max_{l \in X} u_i(l, s_i)$.

- $u_1(l, s_1) > u_1(l'', s_1) >> u_1(l', s_1)$ and $u_2(l', s_2) > u_2(l'', s_2) >> u_2(l, s_2)$.

$$\Downarrow$$

- $\{l, l', l''\} \sim_i \{l, l'\}$ for $i = 1, 2$.
- However, $\{l, l', l''\} \succ_0 \{l, l'\}$ seems desirable.

  $\because$ Option $l''$ is highly regarded by everyone.

**Lesson:**

- If an ex-post disagreement will occur, society may need a compromise option.

# Axiom 1: Weaker Pareto Indifference

**Idea:** If an option is surely Pareto dominated ex-post, society does not need it.

## Pareto Indifference for Dominated Options

For all menus $X \in \mathcal{K}(\Delta(A))$ and all lotteries $\hat{l} \in \Delta(A)$, if

(1) $X \cup \{\hat{l}\} \sim_i X$ for some $i \in N$ and

(2) $\{\hat{l}, l\} \sim_j \{l\}$ for all $l \in X$ and all other individuals $j \in N \setminus \{i\}$,

then $X \cup \{\hat{l}\} \sim_0 X$.

- Under DLR preferences: $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$,
  - (1) $\iff$ In every $s_i \in S_i$, $\hat{l}$ is not the best among $X \cup \{\hat{l}\}$.
  - (2) $\iff$ In every $s_j \in S_j$, $\hat{l}$ is the worst among $X \cup \{\hat{l}\}$.

# Axiom 2: Violation of Pareto Indifference

**Idea:** $\{l, l', l''\} \succ_0 \{l, l'\}$ if an ex-post disagreement between $l$ and $l'$ is sufficiently large.

## Expansion toward Moderate Options

For all lotteries $\hat{l}, l_1, \ldots, l_n \in \Delta(A)$, if for each individual $i \in N$,

- $\{\hat{l}, l_j\} \sim_i \{\hat{l}\} \sim_i \{l_j\}$ for all $j \neq i$ and
- $\{\hat{l}, l_i\} \succ_i \{l_i\}$,

there exists $l^* := \sum_{i=1}^n \lambda_i l_i + (1 - \sum_{i=1}^n \lambda_i) \hat{l}$ ($(\lambda_i)_i \in (0,1)^n$ with $\sum_{i=1}^n \lambda_i < 1$) such that

$$\{l^*, l_1, \ldots, l_n\} \succ_0 \{l_1, \ldots, l_n\}.$$

# $n = 2$ Case

## Expansion toward Moderate Options (when $n = 2$)

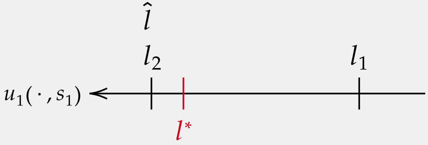
For all lotteries $\hat{l}, l_1, l_2 \in \Delta(A)$, if

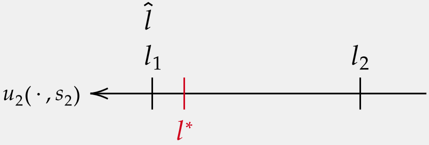
- $\{\hat{l}, l_2\} \sim_1 \{\hat{l}\} \sim_1 \{l_2\}$ and $\{\hat{l}, l_1\} \succ_1 \{l_1\}$,
- $\{\hat{l}, l_1\} \sim_2 \{\hat{l}\} \sim_2 \{l_1\}$ and $\{\hat{l}, l_2\} \succ_2 \{l_2\}$,

there exists $l^* := \lambda_1 l_1 + \lambda_2 l_2 + (1 - \lambda_1 - \lambda_2)\,\hat{l}$ ($\lambda_1, \lambda_2 \in (0,1)$ with $\lambda_1 + \lambda_2 < 1$) such that

$$\{l^*, l_1, l_2\} \succ_0 \{l_1, l_2\}.$$

**Interpretation**: when $S_1 = \{s_1\}$, $S_2 = \{s_2\}$, and $\lambda_1$ and $\lambda_2$ are sufficiently small,



# Axiom 3: Commitment Pareto

**Commitment Pareto**

For all lotteries $l, l' \in \Delta(A)$, if $\{l\} \succsim_i \{l'\}$ for all $i \in N$, then $\{l\} \succsim_0 \{l'\}$.

# Preliminary for Axiom 4

**DLR Representation:** $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$.

## Normalization Assumption

For $(\succsim_i)_{i \in N}$, take $(S_i, \pi_i, u_i)_{i \in N}$ so that there exists $b, w \in \Delta(A)$ such that $u_i(b, s_i) = 1$ and $u_i(w, s_i) = 0$ for all $i \in N$ and all $s_i \in S_i$.

- In the paper, we ensure the existence of $b$ and $w$ that satisfy $u_i(b, s_i) > u_i(w, s_i)$ for all $i \in N$ and all $s_i \in S_i$.
- Given this, the assumption imposes that the evaluation of $b$ and $w$ are the same across all individuals' possible tastes, respectively.

# Axiom 4: Rationality Requirement

**DLR Representation:** $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$.

## Exclusion of Redundant Flexibility

For all lotteries $l, l' \in \Delta(A)$, if for each $i \in N$

- either $\{b, l\} \sim_i \{b\} \sim_i \{l\}$ or $\{w, l\} \sim_i \{w\} \sim_i \{l\}$, and
- either $\{b, l'\} \sim_i \{b\} \sim_i \{l'\}$ or $\{w, l'\} \sim_i \{w\} \sim_i \{l'\}$,

then either $\{l, l'\} \sim_0 \{l\}$ or $\{l, l'\} \sim_0 \{l'\}$ holds.

- $\{b, l\} \sim_i \{b\} \sim_i \{l\} \Longrightarrow u_i(l, s_i) = u_i(b, s_i) = 1$ for all $s_i \in S_i$.
  - i.e., everyone foresees with certainty the evaluations of $l$ and $l'$.

$\Rightarrow$ No multiple possibilities exist for future tastes.

$\Rightarrow$ One lottery is sufficient.

# Representation Theorem

**DLR Representation:** $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$.

## Theorem

*Fix the representation $(S_i, \pi_i, u_i)_{i \in N}$ that satisfies the normalization assumption, arbitrarily. Then, the DLR preference profile, $(\succsim_i)_{i \in N}$ and $\succsim_0$, satisfies the four axioms if and only if*

1. $S_0 = S_1 \times \cdots \times S_n$;
2. *there exists* $(\alpha_i)_{i \in N} \in (0,1)^n$ *such that for each* $s_0 = (s_i)_i \in S_0$,
$$u_0(\cdot\,, (s_i)_i) = \sum_{i \in N} \alpha_i u_i(\cdot\,, s_i);$$
3. *for each* $i \in N$ *and each* $s_i^* \in S_i$,
$$\sum_{s_0 = (s_j)_j \in S_0 : s_i = s_i^*} \pi_0(s_0) = \pi_i(s_i^*).$$

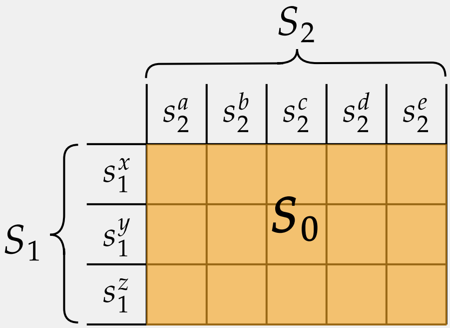
# Interpretation

**DLR Representation:** $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$.

### Theorem

*The DLR preference profile, $(\succsim_i)_{i \in N}$ and $\succsim_0$, satisfies the four axioms if and only if*

1. $S_0 = S_1 \times \cdots \times S_n$;

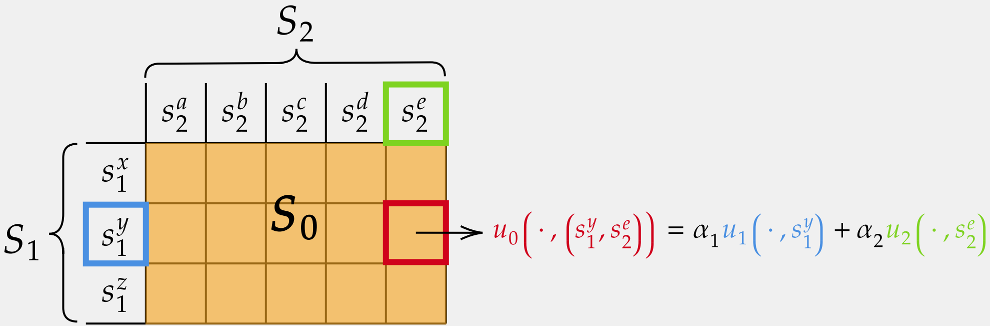


**DLR Representation:** $U_i\left(X\right)=\sum_{s_i\in S_i}\pi_i\left(s_i\right)\max_{l\in X}u_i\left(l,s_i\right)$.

## Theorem

*The DLR preference profile,* $\left(\succsim_i\right)_{i\in N}$ *and* $\succsim_0$*, satisfies the four axioms if and only if*

2. $u_0\left(\cdot\ ,\left(s_i\right)_i\right)=\sum_{i\in N}\alpha_i u_i\left(\cdot\ ,s_i\right)$ *for each* $s_0=\left(s_i\right)_i\in S_0$*;*



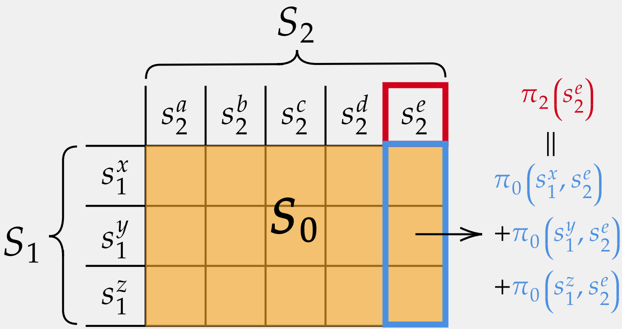
# Interpretation

**DLR Representation:** $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$.

## Theorem

*The DLR preference profile,* $(\succsim_i)_{i \in N}$ *and* $\succsim_0$*, satisfies the four axioms if and only if*

3. *for each* $i \in N$ *and each* $s_i^* \in S_i$, $\sum_{s_0=(s_j)_j \in S_0 : s_i = s_i^*} \pi_0(s_0) = \pi_i(s_i^*)$.



# Proof Intuition (1/4)

## Pareto Indifference for Dominated Options

(1) $X \cup \{\hat{l}\} \sim_i X$ for some $i \in N$ and

(2) $\{\hat{l}, l\} \sim_j \{l\}$ for all $l \in X$ and all other individuals $j \in N \setminus \{i\}$,

$\Rightarrow X \cup \{\hat{l}\} \sim_0 X$.

$\fallingdotseq$ a Pareto principle for tastes over lotteries

$\Rightarrow$ For each $s_0 \in S_0$, there exists some $(s_i)_{i \in N}$ and $(\alpha_{i,s_0})_{i \in N} \in [0,1]^n$ such that

$$u_0(\cdot\ , s_0) = \sum_{i \in N} \alpha_{i,s_0} u_i(\cdot\ , s_i)\,.$$

# Proof Intuition (2/4)

## Expansion toward Moderate Options (when $n = 2$)

1. $\{\hat{l}, l_2\} \sim_1 \{\hat{l}\} \sim_1 \{l_2\}$ and $\{\hat{l}, l_1\} \succ_1 \{l_1\}$,
2. $\{\hat{l}, l_1\} \sim_2 \{\hat{l}\} \sim_2 \{l_1\}$ and $\{\hat{l}, l_2\} \succ_2 \{l_2\}$,

$\Rightarrow$ ${}^{\exists} l^* := \lambda_1 l_1 + \lambda_2 l_2 + (1 - \lambda_1 - \lambda_2)\, \hat{l}$ such that $\{l^*, l_1, l_2\} \succ_0 \{l_1, l_2\}$.

- "Any $(u_i(\cdot\ , s_i))_{i \in N}$ has a disagreement $\implies$ society needs a compromise lottery."

$\Rightarrow$ Society considers all of the combinations $S_1 \times \cdots \times S_n$.

$\Rightarrow$ $S_0 \supset S_1 \times \cdots \times S_n$.

# Proof Intuition (3/4)

## Exclusion of Redundant Flexibility

- either $\{b, l\} \sim_i \{b\} \sim_i \{l\}$ or $\{w, l\} \sim_i \{w\} \sim_i \{l\}$, and
- either $\{b, l'\} \sim_i \{b\} \sim_i \{l'\}$ or $\{w, l'\} \sim_i \{w\} \sim_i \{l'\}$

for each $i \in N$, then either $\{l, l'\} \sim_0 \{l\}$ or $\{l, l'\} \sim_0 \{l'\}$ holds.

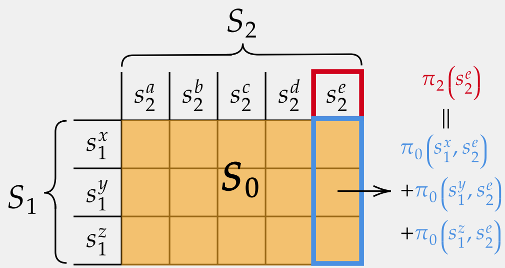
- This axiom is violated if
  - $1 = u_1(l, s_1) > u_1(l', s_1) = 0$ for all $s_1 \in S_1$;
  - $0 = u_2(l, s_2) < u_2(l', s_2) = 1$ for all $s_2 \in S_2$;
  - society has $s_0, s_0' \in S_0$ such that $u_0(\cdot, s_0) = u_1(\cdot, s_1)$ and $u_0(\cdot, s_0') = u_2(\cdot, s_2)$.
- As a result, the axiom implies $S_0 \subset S_1 \times \cdots \times S_n$.

# Proof Intuition (4/4)

Commitment Pareto: $\{l\} \succsim_i \{l'\}$ for all $i \in N \Longrightarrow \{l\} \succsim_0 \{l'\}$.

$\Rightarrow$ In the evaluation, society has to maintain the ratio $\pi_i(s_i)/\pi_i(s_i')$.

$\Rightarrow$ $\sum_{s_0=(s_j)_j \in S_0 : s_i = s_i^*} \pi_0(s_0) = \pi_i(s_i^*)$ for each $s_i^* \in S_i$.



# Proof of the Core Part

# Two Core Axioms

- We only see the implications of the first two axioms:

## Pareto Indifference for Dominated Options

(1) $X \cup \{\hat{l}\} \sim_i X$ for some $i \in N$ and

(2) $\{\hat{l}, l\} \sim_j \{l\}$ for all $l \in X$ and all other individuals $j \in N \setminus \{i\}$,

$\Rightarrow X \cup \{\hat{l}\} \sim_0 X$.

## Expansion toward Moderate Options (when $n = 2$)

1. $\{\hat{l}, l_2\} \sim_1 \{\hat{l}\} \sim_1 \{l_2\}$ and $\{\hat{l}, l_1\} \succ_1 \{l_1\}$,
2. $\{\hat{l}, l_1\} \sim_2 \{\hat{l}\} \sim_2 \{l_1\}$ and $\{\hat{l}, l_2\} \succ_2 \{l_2\}$,

$\Rightarrow {}^{\exists} l^* := \lambda_1 l_1 + \lambda_2 l_2 + (1 - \lambda_1 - \lambda_2)\,\hat{l}$ such that $\{l^*, l_1, l_2\} \succ_0 \{l_1, l_2\}$.

# Technical Assumption for the Proof

## Richness Condition

For each $i \in N$ and each $s_i \in S_i$, there exist lotteries $l_{s_i}, l'_{s_i} \in \Delta(A)$ such that

- $u_i(l_{s_i}, s_i) > u_i(l'_{s_i}, s_i)$,
- $u_i(l_{s_i}, t_i) = u_i(l'_{s_i}, t_i)$ for all $t_i \neq s_i$, and
- $u_j(l_{s_i}, s_j) = u_j(l'_{s_i}, s_j)$ for all $j \neq i$ and all $s_j \in S_j$.

- In the paper, we adopt a weaker richness condition.
  - But here, we impose the above condition to simplify the proof.

# Lemma (1/2): Ex-Post Utilitarianism

**DLR Representation:** $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$.

## Lemma

*If the DLR preference profile, $(\succsim_i)_{i \in I}$ and $\succsim_0$, satisfies Pareto Indifference for Dominated Options, then for each $s_0 \in S_0$, there exist $(s_i)_i \in S_1 \times \cdots \times S_n$ and $(\alpha_i)_i \in [0,1]^n$ with $\sum_{i \in N} \alpha_i = 1$ such that*

$$u_0(\cdot\ , s_0) = \sum_{i \in N} \alpha_i u_i(\cdot\ , s_i)\,.$$

**Remarks:**

- Under some $s_0 \in S_0$, society may assign zero weight to some individuals.
- For some profile $(s_i)_i \in S_1 \times \cdots \times S_n$, there may be no corresponding $s_0$.

# Proof

**DLR Representation:** $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$.

- When $X = \{l\}$ in Pareto Indifference for Dominated Options,
  - $\{l, \hat{l}\} \sim_i \{l\}$ for all $i \in N \implies \{l, \hat{l}\} \sim_0 \{l\}$.

  $\Leftrightarrow$ $u_i(l, s_i) \geq u_i\left(\hat{l}, s_i\right)$ for all $s_i \in S_i$ and all $i \in N \implies u_0(l, s_0) \geq u_0(l', s_0)$ for all $s_0 \in S_0$.

$\Rightarrow$ For each $s_0 \in S_0$, by applying Harsanyi's Theorem,

$$u_0(\cdot\ , s_0) = \sum_{i \in N} \sum_{s_i \in S_i} \alpha_{s_i} u_i(\cdot\ , s_i)\,.$$

# Proof (Continued)

- Suppose that for some $s_0 \in S_0$,

$$u_0\left(\cdot\,,s_0\right)=\underbrace{\alpha_{s_i}}_{>0}u_i\left(\cdot\,,s_i\right)+\underbrace{\alpha_{s_i'}}_{>0}u_i\left(\cdot\,,s_i'\right)+\sum_{j\neq i}\sum_{s_j\in S_j}\alpha_{s_j}u_j\left(\cdot\,,s_j\right).$$

- Take $l, l', l'' \in \Delta\left(A\right)$ so that
  - $u_i\left(l, s_i\right) = u_i\left(l'', s_i\right) > u_i\left(l', s_i\right)$,
  - $u_i\left(l', s_i'\right) = u_i\left(l'', s_i'\right) > u_i\left(l, s_i'\right)$,
  - $u_j\left(l'', s_j\right) = u_j\left(l, s_j\right) = u_j\left(l', s_j\right)$ for all $s_j \in \bigcup_{j\in N} S_j \backslash \left\{s_i, s_i'\right\}$.

1. Pareto Indifference for Dominated Options $\Longrightarrow \left\{l, l', l''\right\} \sim_0 \left\{l, l'\right\}$.
2. But, $u_0\left(l'', s_0\right) > u_0\left(l, s_0\right)$ and $u_0\left(l'', s_0\right)$. $\Longrightarrow \left\{l, l', l''\right\} \succ_0 \left\{l, l'\right\}$: a contradiction.

# Lemma (2/2): Responsiveness to Every Profile of Individual States

**DLR Representation:** $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$.

Lemma

*Suppose that for each $s_0 \in S_0$, there exist $(s_i)_i \in S_1 \times \cdots \times S_n$ and $(\alpha_i)_i \in [0,1]^n$ with $\sum_{i \in N} \alpha_i = 1$ such that*

$$u_0(\cdot\,, s_0) = \sum_{i \in N} \alpha_i u_i(\cdot\,, s_i)\,. \tag{1}$$

*Then, if the DLR preference profile, $(\succsim_i)_{i \in I}$ and $\succsim_0$, satisfies Expansion toward Moderate Options, for each profile $(s_i)_i \in S_1 \times \cdots \times S_n$, there exists $s_0 \in S_0$ such that equation (1) holds where $\alpha_i > 0$ for all $i \in N$.*

**Remarks:**

- Still, for some $(s_i)_i \in S_1 \times \cdots \times S_n$, there may exist multiple corresponding social states.

# Proof

- Take any $s_1 \in S_1$, $s_2 \in S_2$ and $\hat{l}, l_1, l_2 \in \Delta(A)$ so that
  - $u_1(\hat{l}, s_1) = u_1(l_2, s_1) > u_1(l_1, s_1)$,
  - $u_2(\hat{l}, s_2) = u_2(l_1, s_2) > u_2(l_2, s_2)$,
  - $u_i(\hat{l}, s_i) = u_i(l_1, s_i) = u_i(l_i, s_i)$ for all $s_i \in (S_1 \cup S_2) \setminus \{s_1, s_2\}$.

1. Expansion toward Moderate Options $\Longrightarrow \{l^*, l_1, l_2\} \succ_0 \{l_1, l_2\}$.
2. $\nexists s_0 \in S_0$ such that $u_0(\cdot\,, s_0) = \alpha_1 u_1(\cdot\,, s_1) + \alpha_2 u_2(\cdot\,, s_2)$.

   $\Rightarrow$ $\nexists s_0 \in S_0$ such that $u_0(l^*, s_0) > u_0(l_1, s_0)$ and $u_0(l_2, s_0)$.

   $\Rightarrow$ $\{l^*, l_1, l_2\} \sim_0 \{l_1, l_2\}$: a contradiction.

# Connection to the Literature

# Previous Study: Preferences over Lotteries

- **Domain**: lotteries $l \in \Delta(A)$.
- **Preferences**: $\succsim_i$ for each individual $i \in N$ and social $\succsim_0$ is represented by

$$U_i(l) = \sum_{a \in A} l(a)\, u_i(a).$$

Theorem (Harsanyi (1955))

*$(\succsim_i)_{i\in}$ and $\succsim_0$ satisfy the Pareto condition if and only if $u_0 = \sum_{i \in N} \alpha_i u_i$.*

# Previous Study: Preferences over Acts

- **Domain**: acts $f : S \to A$.
- **Preferences**: $\succsim_i$ for each individual $i \in N$ and social $\succsim_0$ is represented by

$$U_i\left(f\right) = \sum_{s \in S} \pi_i\left(s\right) u_i\left(f\left(s\right)\right).$$

Theorem (Mongin (1995))

*$(\succsim_i)_{i\in}$ and $\succsim_0$ satisfy the Pareto condition if and only if $u_0 = u_i$ and $\pi_0 = \pi_i$ for some $i$.*

Theorem (Gilboa et al. (2004))

*$(\succsim_i)_{i\in}$ and $\succsim_0$ satisfy a certain restricted Pareto condition if and only if $u_0 = \sum_{i\in N} \alpha_i u_i$ and $\pi_0 = \sum_{i \in N} \beta_i \pi_i$.*

# Features of This Paper

- Previous study:
  - The probability measure $\pi_i$ over $S$ is different across individuals.
  - But, the state space $S$ is common.
- This paper:
  - Relevant states are different among individuals.
  - We consider menu preferences.
    - Only a few studies exist: Ahn and Chambers (2010), Qu (2016), Hayashi (2021), Hayashi et al. (2024).

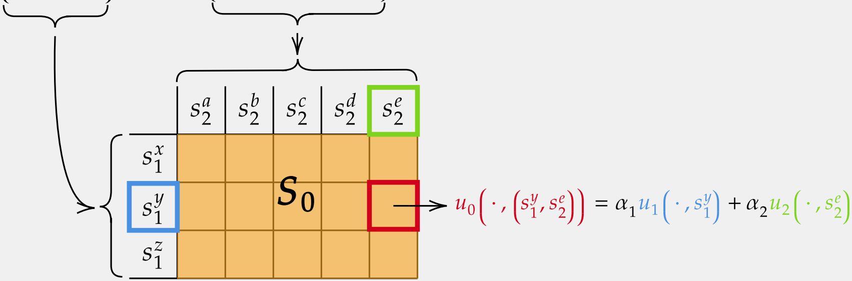
# Summary

**Question:**

- How should society aggregate preferences over menus of options?
    - Especially, how should society construct a comprehensive state space?

**Answer:**

$$S_1 = \{s_1^x, s_1^y, s_1^z\},\ S_2 = \{s_2^a, s_2^b, s_2^c, s_2^d, s_2^e\}$$



# Reference I

**Ahn, David and Christopher Chambers**, "What's on the Menu? Deciding What is Available to the Group?," *Working Paper*, 2010.

**Dekel, Eddie, Barton L. Lipman, and Aldo Rustichini**, "Representing Preferences with a Unique Subjective State Space," *Econometrica*, 2001, *69* (4), 891–934.

**Gilboa, Itzhak, Dov Samet, and David Schmeidler**, "Utilitarian Aggregation of Beliefs and Tastes," *Journal of Political Economy*, 2004, *112* (4), 932–938.

**Harsanyi, J. C.**, "Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility," *Journal of Political Economy*, 1955, *63* (4), 309–321.

**Hayashi, Takashi**, "Collective Decision under Ignorance," *Social Choice and Welfare*, 2021, *57* (2), 347–359.

# Reference II


___ **, Noriaki Kiguchi, and Norio Takeoka**, "Temptation and Self-Control for the Impure Benevolent Planner: The Case of Heterogeneous Discounting," *Journal of Public Economic Theory*, 2024, *26* (1), e12674.

**Mongin, Philippe**, "Consistent Bayesian Aggregation," *Journal of Economic Theory*, 1995, *66* (2), 313–351.

**Qu, Xiangyu**, "Commitment and Anticipated Utilitarianism," *Social Choice and Welfare*, 2016, *47* (2), 349–358.