Strategy-proof rules in object allocation problems with hard budget constraints and income effects *

Yuya Wakabayashi † Ryosuke Sakai ‡ Hiroki Shinozaki §

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Abstract

We consider the single-object allocation problem with monetary transfers. Agents may have hard budgets and their utility functions may exhibit income effects. When hard budget constraints are present, it is known that *efficiency* and *strategy-proofness* are incompatible along with *individual rationality* and *no subsidy*. Our objective is to clarify what forms of partial *efficiency* are compatible with *strategy-proofness* alongside *individual rationality* and *no subsidy for losers*. We focus on *constrained efficiency* as a weak *efficiency* condition, and introduce *truncated Vickrey rules with endogenous reserve prices*. We show that some of them are the only rules that satisfy *constrained efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. Whether these rules satisfy *constrained efficiency* or *strategy-proofness* critically depends on the structure of the tie-breaking rule. We identify what structures are necessary and sufficient for truncated Vickrey rules with endogenous reserve prices to satisfy these properties. Moreover, we also show the parallel characterization results for several *fairness* properties instead of *constrained efficiency*.

JEL classification: D47, D63, D82.

Keywords: Single-object allocation problem, Non-quasi-linear preference, Hard budget constraint, Efficiency, Fairness, Strategy-proofness.

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†Graduate School of Economics, Osaka University. Email: u690485a@ecs.osaka-u.ac.jp

‡School of Engineering, Institute of Science Tokyo. Email: sakai.r.al@m.titech.ac.jp

§Hitotsubashi Institute for Advanced Study, Hitotsubashi University. Email: shinozaki.hiroki@r.hit-u.ac.jp

1 Introduction

In real-life auctions, participants often face *budget constraints*, which is the maximum amount of money they can spend on auctioned objects. An important example of budget-constrained auctions is spectrum license auctions (Bulow et al. 2009), where firms must put aside money in advance to acquire spectrum licenses.

In addition to budget constraints, we cannot ignore *income effects* experienced by agents. In large-scale auctions such as spectrum license auctions, substantial payments can diminish agents' capacity to afford certain complementary goods related to the auctioned objects, thereby leading to significant income effects (Saitoh and Serizawa 2008).

It is one of crucial goals for auctioneers to allocate objects *efficiently* to agents. Moreover, to accurately evaluate *efficiency* based on true utility functions, auctioneers require *strategy-proofness*, where reporting true utility functions becomes a dominant strategy. If utility functions are quasi-linear, and so there are no budget constraint and income effect, then Groves rules are the only rules satisfying *efficiency* and *strategy-proofness* (Holmström, 1979). However, Dobzinski et al. (2012) demonstrate that if private budgets exist, which undermines quasi-linearity of utility functions, then no rule satisfies these conditions along with *individual rationality* and *no subsidy*.

In this paper, we consider an environment where both budget constraints and income effects are present, aiming to derive positive results by relinquishing *efficiency*. A natural approach to achieving positive results is to moderate *efficiency*. While full *efficiency* is incompatible with *strategy-proofness*, certain degrees of partial *efficiency* may be reconcilable. Our objective in this paper is to clarify what forms of partial *efficiency* are compatible with *strategy-proofness*, while ensuring *individual rationality* and *no subsidy*. Furthermore, we investigate the relation between *constrained efficiency* and several *fairness* properties.

1.1 Results

We consider the single-object allocation problem with monetary transfers. Each agent has a utility function on the pairs of the object assignment and the payment. Note that agents' utility functions can exhibit *income effects*. Additionally, each agent has a *hard budget constraint*, wherein payments exceeding the budget render their utility as negative infinity. The willingness to pay for the object is called the *valuation*. We define the minimum between the willingness to pay and the budget as the *truncated valuation*. A *utility profile* is a vector consisting of agents' utility functions. An *allocation* specifies who gets the object and how much agents pay.

A *rule* is a mapping from a set of utility profiles to the set of allocations. A rule satisfies *individual rationality* if each agent's outcome is at least as good as receiving no object and paying nothing. A rule satisfies *no subsidy for losers* if each agent who receives no object makes the nonnegative payment.

We decompose *efficiency* into two properties for *individually rational* allocations (Proposition 1).[1] The first property is *no wastage*, which ensures that the object is always assigned to an agent. The second property is *constrained efficiency*, which demands that no reallocation can improve both agents' welfare and revenue simultaneously. In this study, we forgo *no wastage* and focus on developing a rule that satisfies *constrained efficiency*, in addition to *individual rationality*, *no subsidy for losers*, and *strategy-proofness*.

We also consider *weak envy-freeness for equals* as a *fairness* property, introduced by Sakai (2013a). It requires that if two agents with identical utility functions differ in outcome (one being a winner, the other a loser), then the loser does not prefer the winner's outcome to her own. This property is so weak that it is implied by many other *fairness* properties, such as *equal treatment of equals*, *envy-freeness*, and *anonymity in welfare*.

We introduce two rules. The first is a *threshold-price rule*. In this rule, each agent's threshold-price is determined by the utility functions of the other agents. An agent receives the object if her truncated valuation is higher than the threshold-price and does not if her truncated valuation is smaller than the threshold-price. If the agent's truncated valuation equals the threshold-price, she may either receive the object or not, depending on a tie-breaking rule. When the agent receives the object, her payment is equal to the threshold-price. Importantly, the choice of a tie-breaking rule in cases where the truncated valuation equals the threshold-price can significantly impact the properties satisfied by threshold-price rules.

The second rule is a *truncated Vickrey rule with endogenous reserve prices*. This rule is a special case of threshold-price rules, where each agent's threshold-price is always at least as large as the second-highest truncated valuation. Note that when an agent's threshold-price exceeds the second-highest truncated valuation, the threshold-price equals the reserve price attached to the rule.

All threshold-price rules, including all truncated Vickrey rule with endogenous reserve prices, satisfy *individual rationality* and *no subsidy for losers*. However, some of them fail to satisfy one or more of the following properties: *constrained efficiency*, *weak envy-freeness for equals*, and *strategy-proofness*. The satisfaction of these properties depends on the tie-breaking rules and endogenous reserve prices that are applied. In this paper, we identify the necessary and sufficient conditions under which these properties hold. Our main results consist of the following three parts.

The first is about *strategy-proofness*. We show that some of the threshold-price rules are the only rules satisfying *individual rationality*, *no subsidy for losers*, and *strategy-proofness* (Theorem 1). Each rule in this class must adopt a *prioritized tie-breaking rule*,[2] which is the necessary and sufficient condition for the rule to satisfy *strategy-proofness* (Proposition 2).

[1] For *any* allocation, *within budget*, in addition to *no wastage* and *constrained efficiency*, characterizes *efficiency*. This property requires that the payment does not exceed the budget, and it is weaker than *individual rationality*.

[2] See Section 5.1 for the formal definition.

The second is about *efficiency*. We show that some of the truncated Vickrey rules with endogenous reserve prices are the only rules satisfying *constrained efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness* (Theorem 2). Each rule in this class must adopt an *exclusive tie-breaking rule*,[3] which is the necessary and sufficient condition for the rule to satisfy *constrained efficiency* (Proposition 3).

The third is about *fairness*. We show that if *individual rationality*, *no subsidy for losers*, and *strategy-proofness* are satisfied, then *weak envy-freeness for equals* implies *constrained efficiency* (Theorem 3). This result indicates that some of the truncated Vickrey rules with endogenous reserve prices are the only rules that satisfy *weak envy freeness for equals*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness.* Moreover, we show that *non-negligibility*[4] of reserve prices is the necessary and sufficient condition for the rule to satisfy *weak envy-freeness for equals* (Proposition 5).

In real-life auctions, auctioneers often disclose tie-breaking rules or reserve price before conducting the auction. As a result, these tie-breaking rules or reserve prices become fixed and may fail to satisfy the conditions we have identified; the *prioritized condition*, the *exclusive condition*, and the *non-negligibility condition.* Consequently, our results imply that such fixed tie-breaking rules or reserve prices may violate desirable properties; *constrained efficiency*, *weak envy-freeness for equals*, and *strategy-proofness.* This paper reveals what structure of tie-breaking rules and reserve prices is necessary and sufficient to satisfy the desirable properties, emphasizing the importance of proper operation of tie-breaking rules and reserve prices.

1.2 Organization

The structure of this paper is as follows: Section 2 reviews the related literature. Section 3 presents the fundamental components of the model. Following this, in Section 4, we introduce truncated Vickrey rules with endogenous reserve prices. Section 5 explains the results, comprising three parts concerning *strategy-proofness*, *weak efficiency*, and *fairness.* Lastly, Section 6 provides a conclusion. All proofs are relegated to the Appendix.

2 Related literature

Our contributions include: (i) enabling agents to have hard budget constraints, (ii) allowing for income effects in agents' utility functions, and (iii) exploring alternative properties instead of *efficiency*. Previous studies have often focused on one or two of these factors while neglecting the other(s), resulting in few studies that consider them comprehensively. In the following, to facilitate understanding of our contribution, we categorize previous studies that consider at least one factor above into four strands.

[3] See Section 5.2 for the formal definition.

[4] See Section 5.3 for the formal definition.

2.1 Budget-constrained auctions

On the domain where agents' utility functions do not exhibit income effects, hereafter we call it the *no income effect domain*, Dobzinski et al. (2012) consider the allocation problem of homogenous objects. They show that if utility functions exhibit constant marginal valuation and budgets are private information, then there is no rule satisfying *efficiency*, *individual rationality*, *no subsidy*, and *strategy-proofness*. Building upon their work, the subsequent literature has investigated budget-constrained auctions, seeking to achieve positive results. There are various approaches to reconcile the conflict between *efficiency* and *strategy-proofness*.

One approach is to restrict the domain of utility functions. Le (2018) examines the domain where there is the unique agent with the highest truncated valuation, and show that truncated Vickrey rules are the only rules satisfying *efficiency*, *individual rationality*, *no subsidy*, and *strategy-proofness* on the domain. Similarly, other studies focus on settings with public budgets (Fiat et al. 2011; Dobzinski et al. 2012)[5] or consider unit-demand settings (Aggarwal et al. 2009; Dütting, Henzinger, and Weber 2015; Mackenzie and Zhou 2022).[6]

Another line of research relaxes the properties to get the positive results. Our work aligns with this strand. Some studies give up *efficiency* (Dobzinski and Leme 2014; Le 2017), while others relax *strategy-proofness* (Baisa 2017; Shinozaki 2023).

With the exception of Baisa (2017) and Shinozaki (2023), the studies mentioned above assume no income effect, which distinguishes them from our research. As for Baisa (2017) and Shinozaki (2023), they weaken *strategy-proofness* into the mild conditions, which contrasts our study in that we consider alternative properties instead of *efficiency* while maintaining *strategy-proofness*.

Finally, we elucidate the contrast between our findings and those of Le (2018), as his work closely relates to ours. He focuses on the domain where there is the unique agent with the highest truncated valuation, thus excluding the situation where ties exist. Furthermore, he does not allow the presence of income effects. In contrast, we adopt a broader domain without such restrictions, but attain *constrained efficiency* and *strategy-proofness* by imposing additional conditions on tie-breaking rules. This is the main difference from Le (2018) to overcome the problem stemming from the existence of ties. Moreover, our approach always ensures *constrained efficiency* and *strategy-proofness*, unlike Le's (2018) result. This distinction highlights that, while a social planner cannot freely restrict the domain, she can design tie-breaking rules as necessary to achieve desirable outcomes.

[5] On the domain violating constant marginal valuations, the impossibility result returns even if budgets are public information (Lavi and May 2012; Ting and Xing 2012; Yi 2024).

[6] Dütting, Henzinger, and Starnberger (2015) establish the negative result when the multi-demand preferences are allowed.

2.2 Auctions with income effects

Some studies examine the situation where there is no hard budget constraint but income effects exist. In such cases, whether we can attain the possibility result highly depends on whether agents exhibit unit-demand preferences or not. When agents have unit-demand preferences, there is a unique rule satisfying *efficiency*, *individual rationality*, *no subsidy*, and *strategy-proofness* (Saitoh and Serizawa 2008; Sakai 2008; Morimoto and Serizawa 2015; Zhou and Serizawa 2018; Wakabayashi et al. 2025). On the other hand, when agents have multi-demand preferences, there is no rule satisfying the same four properties (Kazumura and Serizawa 2016; Baisa 2020; Malik and Mishra 2021; Shinozaki et al. 2022). The difference between our results and theirs lies in that we consider a hard budget constraint while they do not.

2.3 Vickrey auctions with reserve prices

Several studies explore Vickrey auctions with reserve prices. Some of these works characterize Vickrey rules with reserve prices by *weak efficiency* or *anonymity in welfare*, alongside *strategy-proofness* and other mild properties (Sakai 2013b; Basu and Mukherjee 2023; Basu and Mukherjee 2024). The difference between our results and theirs is that in their findings, the reserve price is determined by the rule itself rather than by other agents' utility functions, meaning the reserve price is fixed. Additionally, the tie-breaking rule is also fixed in their results. Furthermore, their results assume neither hard budget constraints nor income effects.

Kazumura et al. (2017) consider variable reserve prices. On the general domain with a single object, they show that Vickrey rules with endogenous reserve prices are the only rules satisfying *anonymity in welfare*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*.[7] Their result does not include the case with hard budget constraints, which contrasts our result. Moreover, we employ a weaker property, *weak envy-freeness for equals*, rather than *anonymity in welfare* as they do. Thus, their result does not imply ours.

2.4 Fair and strategy-proof auctions

Several studies examine *fair* and *strategy-proof* rules. Some of these works characterize Vickrey rules by *fairness* properties such as *weak envy-freeness for equals* and *anonymity in welfare*, alongside *no wastage*, *strategy-proofness*, and other mild properties (Ohseto 2006; Ashlagi and Serizawa 2012; Sakai 2013a; Adachi 2014). These studies mentioned above do not account for hard budget constraints, unlike our research. Additionally, our

[7]Precisely, they use *loser payment independence* which is weaker than *individual rationality* and *no subsidy for losers*.

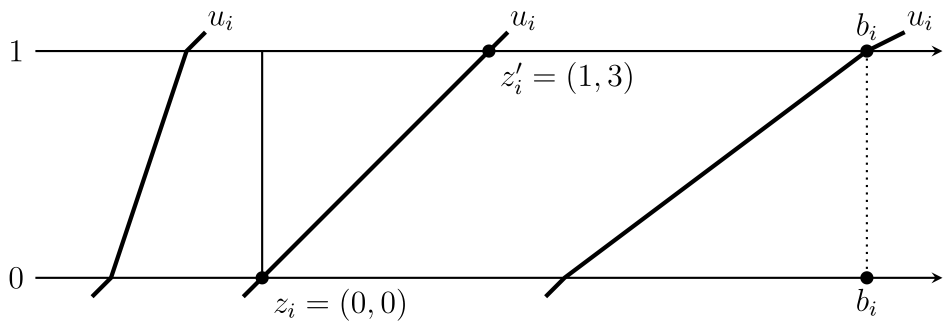


Figure 1: Illustration of a utility function

results regarding the *fairness* properties do not impose *no wastage*. These distinctions constitute the main differences between our findings and theirs.

3 The model

There are n agents and a single object. Let $N = \{1, 2, \cdots, n\}$ be the set of agents. We denote consuming the object and not consuming the object by 1 and 0, respectively. A typical (consumption) **bundle** for agent i is a pair $z_i = (x_i, t_i) \in \{0,1\} \times \mathbb{R}$, where x_i is the consumption of the object and t_i is the payment for agent i.

Each agent has a **utility function** $u_i : \{0,1\} \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ such that (i) $u_i(0,0) = 0$ and (ii) there is a budget $b_i \in \mathbb{R}_{++} \cup \{\infty\}$ such that for each $x_i \in \{0,1\}$ and each $t_i \in \mathbb{R}$, if $t_i \leq b_i$, then $u_i(x_i, t_i) \neq -\infty$, and otherwise, $u_i(x_i, t_i) = -\infty$. Note that we allow a budget to vary depending on a utility function. For the sake of convenience, if $t_i = \infty$, then we let $u_i(x_i, t_i) = -\infty$. We denote by $\mathcal{U}$ a typical class of utility functions, and call it a **domain**.

Figure 1 illustrates a utility function $u_i \in \mathcal{U}$. Each horizontal axis in the figure represents the set of bundles for the corresponding consumption level, and the payment level is indicated by the distance from the vertical axis. For example, z_i in the figure denotes the bundle with no object and no payment, while z_i' denotes the bundle where agent i receives the object and pays 3. Given two bundles $z_i, z_i' \in \{0,1\} \times \mathbb{R}$, if $u_i(z_i) = u_i(z_i')$, then an indifference curve is drawn connecting z_i and z_i'. This curve represents an indifference relation between the two bundles. In Figure 1, the middle indifference curve shows that $(0,0)$ and $(1,3)$ provide the same utility level.

We make the following assumptions about utility functions:

1. *Finiteness:* For each $x_i, x_i' \in \{0,1\}$ and each $t_i \in \mathbb{R}$, if $u_i(x_i, t_i) \geq u_i(x_i', b_i)$, then there is $t_i' \in \mathbb{R}$ such that $u_i(x_i', t_i') = u_i(x_i, t_i)$.

2. *Money monotonicity:* For each $x_i \in \{0,1\}$ and each $t_i, t_i' \in \mathbb{R}$, if $t_i < t_i' \leq b_i$, then $u_i(x_i, t_i) > u_i(x_i, t_i')$.

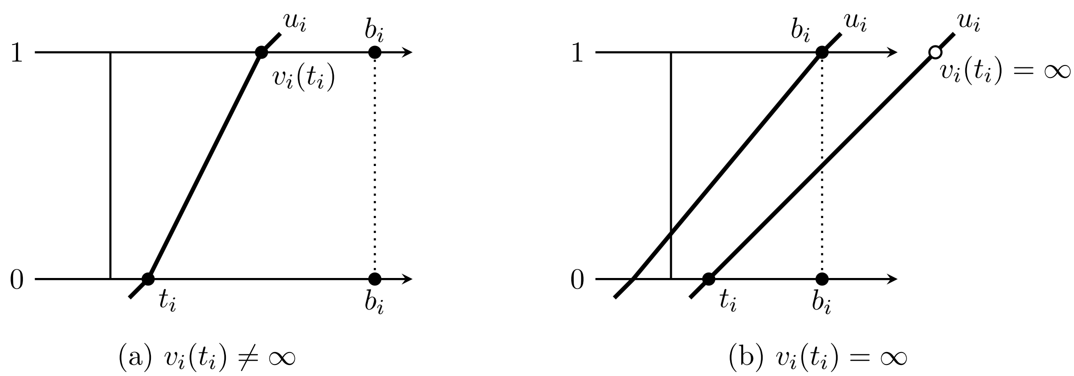


Figure 2: Illustration of a valuation

3. *Object desirability:* For each $t_i \in \mathbb{R}$ with $t_i \leq b_i$, $u_i(1, t_i) > u_i(0, t_i)$.

Let $\mathcal{U}^C$ be the set of all utility functions satisfying the above three properties, and call it the **classical domain**. Throughout the paper, we consider a domain included by the classical domain, that is, $\mathcal{U} \subseteq \mathcal{U}^C$.

Given $u_i \in \mathcal{U}^C$ and $t_i \in \mathbb{R}$, we define the **valuation for** u_i **from** t_i by $v_i(t_i)$ such that $u_i(1, v_i(t_i)) = u_i(0, t_i)$ if $u_i(0, t_i) \geq u_i(1, b_i)$, and $v_i(t_i) = \infty$ otherwise. Figure 2 illustrates a valuation. Note that by finiteness, if $b_i = \infty$, then $v_i(t_i) \neq \infty$. Similarly, given $u_i \in \mathcal{U}^C$ and $t_i \in \mathbb{R}$, we define the **compensation for** u_i **from** t_i by $c_i(t_i)$ such that $u_i(0, c_i(t_i)) = u_i(1, t_i)$ if $u_i(1, t_i) \geq u_i(0, b_i)$, and $c_i(t_i) = \infty$ otherwise. Note that by finiteness and money monotonicity, $v_i(t_i)$ and $c_i(t_i)$ are uniquely determined. Moreover, by object desirability, if $t_i \leq b_i$, $c_i(t_i) < t_i < v_i(t_i)$.

Given $u_i \in \mathcal{U}^C$, we denote the valuation for u_i from 0 by $v_i = v_i(0)$ for simplicity. We call $\min\{v_i, b_i\}$ a **truncated valuation for** u_i. If $v_i = \infty$, that is, $u_i(1, b_i) > u_i(0, 0)$, we say that *agent i's budget constraint is binding.* In this case, she is willing to pay up to the limit of her budget to obtain the object.

A **utility profile** is an n-tuple of agents' utility functions $u = (u_1, \ldots, u_n) \in \mathcal{U}^n$. Given $i \in N$ and $N' \subseteq N$, let $u_{-i} = (u_j)_{j \neq i}$ and $u_{-N'} = (u_j)_{j \in N \setminus N'}$. Given a utility profile $u \in \mathcal{U}^n$, let

$$N(u) = \left\{ i \in N : \min\{v_i, b_i\} \geq \max_{j \neq i} \min\{v_j, b_j\} \right\}$$

be the set of the agents who have the highest truncated valuation, and let

$$N^{\infty}(u) = \left\{ i \in N : \min\{v_i, b_i\} \geq \max_{j \neq i} \min\{v_j, b_j\} \text{ and } v_i = \infty \right\}.$$

be the set of the agents in $N(u)$ whose budget constraints are binding.

A feasible **object assignment** is an n-tuple $x = (x_1, \ldots, x_n)$ such that $\sum_{i \in N} x_i \leq 1$. Let X be the set of all feasible object assignments, that is, $X = \{(x_1, \ldots, x_n) \in \{0,1\}^n :$

$\sum_{i \in N} x_i \leq 1\}$. An **allocation** is a pair of a feasible object assignment and a vector of payments, $z = ((x_1, x_2, \ldots, x_n), (t_1, t_2, \ldots, t_n)) \in X \times \mathbb{R}^n$. We denote the set of all allocations by $Z = X \times \mathbb{R}^n$. Given $z \in Z$ and $i \in N$, $z_i = (x_i, t_i)$ denotes the bundle of agent i. Given $i \in N$ and $N' \subseteq N$, let $z_{-i} = (z_j)_{j \neq i}$ and $z_{-N'} = (z_j)_{j \in N \setminus N'}$.

A **rule** is a mapping $f = (x, t) \colon \mathcal{U}^n \to Z$. Given a rule f and a utility profile $u \in \mathcal{U}^n$, agent i's bundle under f at u is denoted by $f_i(u) = (x_i(u), t_i(u))$. Given $i \in N$ and $u_{-i} \in \mathcal{U}^{n-1}$, let $\mathcal{U}^W(u_{-i}) = \{u_i \in \mathcal{U} : x_i(u_i, u_{-i}) = 1\}$ be the set of i's utility functions for which she wins the object, and let $\mathcal{U}^L(u_{-i}) = \mathcal{U} \setminus \mathcal{U}^W(u_{-i})$.

We introduce the following basic properties:

- **Individual rationality**: For each $u \in \mathcal{U}^n$ and each $i \in N$, $u_i(f_i(u)) \geq 0$.
- **No subsidy for losers**: For each $u \in \mathcal{U}^n$ and each $i \in N$, if $x_i(u) = 0$, then $t_i(u) \geq 0$.
- **Strategy-proofness**: For each $u \in \mathcal{U}^n$, each $i \in N$ and each $u'_i \in \mathcal{U}$, $u_i(f_i(u)) \geq u_i(f_i(u'_i, u_{-i}))$.

3.1 Efficiency properties

Given $u \in \mathcal{U}^n$, an allocation $z \in Z$ is **efficient for** u if there is no $z' \in Z$ such that (i) for each $i \in N$, $u_i(z'_i) \geq u_i(z_i)$, (ii) $\sum_{i \in N} t'_i \geq \sum_{i \in N} t_i$, and (iii) at least one inequality in (i) and (ii) holds strictly. We say that z' **dominates** z **for** u if z' satisfies (i), (ii), and (iii) above. We similarly define *efficiency* as a property imposed on a rule.

- **Efficiency**: For each $u \in \mathcal{U}^n$, $f(u)$ is efficient for u.

As Dobzinski et al. (2012) have shown, if budget constraints exist, achieving *efficiency* and *strategy-proofness* in addition to *individual rationality* and *no subsidy for losers* is impossible.[8] Thus, we aim for certain degrees of partial *efficiency*. To do so, we decompose efficiency into two properties.

Given $u \in \mathcal{U}^n$, let $Z^{IR}(u) \subseteq Z$ be the set of individually rational allocations, that is, for each $z \in Z^{IR}(u)$ and each $i \in N$, $u_i(z_i) \geq 0$. Since we consider only *individually rational* rules throughout the paper, we may restrict our attention to the set of individually rational allocations $Z^{IR}(u)$.[9]

We show that *efficiency* can be decomposed into two properties.

Proposition 1. *Let $u \in \mathcal{U}^n$ and $z \in Z^{IR}(u)$. Then, z is efficient for u if and only if*

[8] Dobzinski et al. (2012) consider the situation where income effects do not exist. However, we show that the parallel impossibility result holds even if income effects exist. See Corollary 1.

[9] Our discussion is also true even if we replace $Z^{IR}(u)$ with the set of allocations such that for each $i \in N$, $t_i \leq b_i$. We say that such allocations satisfy **within budget**. Note that *within budget* is weaker than *individual rationality*.

(i) $\sum_{i \in N} x_i = 1$, *and*

(ii) *there is no* $z' \in Z$ *such that* $\sum_{i \in N} x'_i = \sum_{i \in N} x_i$ *and it dominates* z *for* u.

For the properties in Proposition 1, we call (i) **no wastage** and (ii) **constrained efficiency**. We similarly define these properties as those imposed on a rule.

- **No wastage**: For each $u \in \mathcal{U}^n$, $\sum_{i \in N} x_i(u) = 1$.
- **Constrained efficiency**: For each $u \in \mathcal{U}^n$, $f(u)$ is constrained efficient for u.

In this study, we focus on the rules that satisfy *constrained efficiency* along with *individual rationality*, *no subsidy for losers*, and *strategy-proofness*.

3.2 Fairness properties

We define a weak *fairness* property introduced by Sakai (2013a). It says that if two agents with identical utility functions differ in outcome (one being a winner, the other a loser), then the loser does not prefer the winner's outcome to her own.

- **Weak envy-freeness for equals**: For each $u \in \mathcal{U}^n$ and each $i, j \in N$, if $u_i = u_j$, $x_i(u) = 0$, and $x_j(u) = 1$, then $u_i(f_i(u)) \geq u_i(f_j(u))$.

Weak envy-freeness for equals is so weak that it is implied by many *fairness* properties. All of the properties below are stronger than *weak envy-freeness for equals*.

- **Equal treatment of equals**: For each $u \in \mathcal{U}^n$ and each $i, j \in N$, if $u_i = u_j$, then $u_i(f_i(u)) = u_i(f_j(R))$.
- **Envy-freeness**: For each $u \in \mathcal{U}^n$ and each $i, j \in N$, $u_i(f_i(u)) \geq u_i(f_j(u))$.
- **Anonymity in welfare**: For each $u, u' \in \mathcal{U}^n$ and each $i, j \in N$, if $u_i = u'_j$, $u_j = u'_i$, and $u_{-\{i,j\}} = u'_{-\{i,j\}}$, then $u_i(f_i(u)) = u_i(f_j(u'))$.

4 Truncated Vickrey rules with endogenous reserve prices

In this section, we introduce *truncated Vickrey rules with endogenous reserve prices*. First we define *(generalized) Vickrey rules* (Vickrey 1961; Saitoh and Serizawa 2008; Sakai 2008).[10]

[10] In their definitions, the object is always assigned to some agent with the maximum valuation. However, in our definition, if ties exist, the object could not be assigned to any agent. In this sense, our definition does not precisely correspond to theirs.

Definition 1. A rule f on $\mathcal{U}^n$ is a **(generalized) Vickrey rule** if for each $u \in \mathcal{U}^n$ and each $i \in N$,

$$x_i(u) = \begin{cases} 1 & \text{if } v_i > \max_{j \neq i} v_j \\ 0 & \text{if } v_i < \max_{j \neq i} v_j \end{cases},$$

and

$$t_i(u) = \begin{cases} \max_{j \neq i} v_j & \text{if } x_i(u) = 1 \\ 0 & \text{if } x_i(u) = 0 \end{cases}.$$

A Vickrey rule may violate *individual rationality*. To see this, let f be a Vickrey rule, and let $u \in \mathcal{U}^n$ and $i \in N$ be such that $v_i > \max_{j \neq i} v_j > b_i$. Then, by definition, $f_i(u) = (1, \max_{j \neq i} v_j)$. However, since $t_i(u)$ exceeds her budget b_i, $u_i(f_i(u)) = -\infty$, which is a violation of *individual rationality*. To overcome this problem, Le (2018) introduces a *truncated Vickrey rule*.

Definition 2. A rule f on $\mathcal{U}^n$ is a **truncated Vickrey rule** if for each $u \in \mathcal{U}^n$ and each $i \in N$,

$$x_i(u) = \begin{cases} 1 & \text{if } \min\{v_i, b_i\} > \max_{j \neq i} \min\{v_j, b_j\} \\ 0 & \text{if } \min\{v_i, b_i\} < \max_{j \neq i} \min\{v_j, b_j\} \end{cases},$$

and

$$t_i(u) = \begin{cases} \max_{j \neq i} \min\{v_j, b_j\} & \text{if } x_i(u) = 1 \\ 0 & \text{if } x_i(u) = 0 \end{cases}.$$

Any truncated Vickrey rule f satisfies *individual rationality* because for each $u \in \mathcal{U}^n$ and each $i \in N$, if $x_i(u) = 1$, then $t_i(u) \leq \min\{v_i, b_i\}$. Note that if a domain contains no budget constraint, that is, $b_i = \infty$ for each $u \in \mathcal{U}^n$ and each $i \in N$, then a truncated Vickrey rule coincides with a Vickrey rule on the domain.

Next, we introduce *truncated Vickrey rules with endogenous reserve prices*, which are a generalization of truncated Vickrey rules.

Given $i \in N$, i's **(reserve) price function** is $r_i : \mathcal{U}^{n-1} \to \mathbb{R} \cup \{\infty\}$, whose outputs are independent of i's utility functions. Let $\mathcal{R}$ be the set of all price functions. A **(reserve) price function profile** $r = (r_1, \ldots, r_n) \in \mathcal{R}^n$ is an n-tuple of price functions for each agent.

Definition 3. Given a price function profile $r \in \mathcal{R}^n$, a rule f on $\mathcal{U}^n$ is a **truncated Vickrey rule with endogenous reserve prices** r if for each $u \in \mathcal{U}^n$ and each $i \in N$,

$$x_i(u) = \begin{cases} 1 & \text{if } \min\{v_i, b_i\} > \max\{\max_{j \neq i} \min\{v_j, b_j\}, r_i(u_{-i})\} \\ 0 & \text{if } \min\{v_i, b_i\} < \max\{\max_{j \neq i} \min\{v_j, b_j\}, r_i(u_{-i})\} \end{cases}, \tag{V-i}$$

and

$$t_i(u) = \begin{cases} \max\{\max_{j \neq i} \min\{v_j, b_j\}, r_i(u_{-i})\} & \text{if } x_i(u) = 1 \\ 0 & \text{if } x_i(u) = 0 \end{cases}. \tag{V-ii}$$

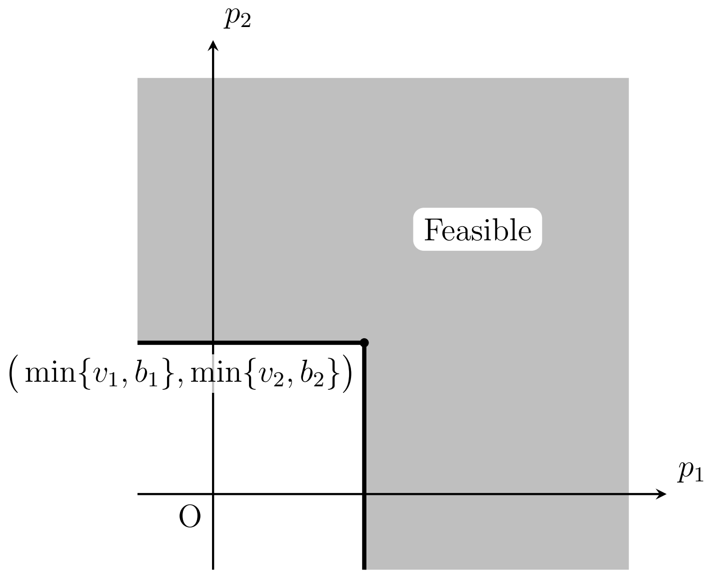


Figure 3: Illustration of feasible prices

Finally, we define general rules that include all the rules mentioned above as special cases.

Definition 4 (Shinozaki 2024)**.** Given a price function profile $p \in \mathcal{R}^n$, a rule f on $\mathcal{U}^n$ is a **threshold-price rule with** p if for each $u \in \mathcal{U}^n$ and each $i \in N$,

$$x_i(u) = \begin{cases} 1 & \text{if } \min\{v_i, b_i\} > p_i(u_{-i}) \\ 0 & \text{if } \min\{v_i, b_i\} < p_i(u_{-i}) \end{cases}, \tag{P-i}$$

and

$$t_i(u) = \begin{cases} p_i(u_{-i}) & \text{if } x_i(u) = 1 \\ 0 & \text{if } x_i(u) = 0 \end{cases}. \tag{P-ii}$$

Given $p \in \mathcal{R}^n$, $u \in \mathcal{U}^n$, and $i, j \in N$ with $i \neq j$, if $\min\{v_i, b_i\} > p_i(u_{-i})$ and $\min\{v_j, b_j\} > p_j(u_{-j})$, then $x_i(u) = x_j(u) = 1$. However, this violates feasibility of object assignments. Hence, we exclude such price function profiles. Formally, we restrict our focus to price function profiles p that satisfies the following condition for each $u \in \mathcal{U}^n$:

$$|\{i \in N : \min\{v_i, b_i\} > p_i(u_{-i})\}| \leq 1.$$

Figure 3 illustrates the set of feasible price profiles when $n = 2$. The set of feasible price profiles for (u_1, u_2) is represented by the gray area in the figure. For any price profile (p_1, p_2) in the area, at least one price is no less than the truncated valuation. Hence, such prices ensures feasibility of the object assignments.

5 Results

This section consists of five subsections.

In the first subsection, we investigate *strategy-proof* rules. We focus on threshold-price rules, and so on truncated Vickrey rules with endogenous reserve prices. In contrast to the case without budget constraints, some threshold-price rules may violate *strategy-proofness*. Consequently, we establish a necessary and sufficient condition for these rules to satisfy *strategy-proofness* (Proposition 2). Subsequently, we show that threshold-price rules are the only rules satisfying *individual rationality*, *no subsidy for losers*, and *strategy-proofness* (Theorem 1).

In the second subsection, we investigate *constrained efficient* rules. We first show a necessary and sufficient condition for truncated Vickrey rules with endogenous reserve prices to satisfy *constrained efficiency* (Proposition 3). Next, we characterize these rules by *constrained efficiency*, *individual rationality*, *no subsidy*, and *strategy-proofness* (Theorem 2). Finally, we show that these rules always violate *no wastage*, which shows the incompatibility between *efficiency* and *strategy-proofness* (Proposition 4, Corollary 1).

In the third subsection, we investigate the relationship between *constrained efficiency* and *fairness* properties. We first show a necessary and sufficient condition for truncated Vickrey rules with endogenous reserve prices to satisfy *weak envy-freeness for equals* (Proposition 5). Next, we show that *weak envy-freeness for equals* implies *constrained efficiency* if a rule satisfies *individual rationality*, *no subsidy for losers*, and *strategy-proofness* (Theorem 3). This result shows that the parallel characterization results hold even if we replace *constrained efficiency* with the *fairness* conditions; *weak envy-freeness for equals* *equal treatment of equals*, *envy-freeness*, and *anonymity in welfare* (Corollary 2, Proposition 6, Corollary 3).

In the fourth subsection, we provide a simple example that satisfies all the properties by using the results obtained in the first three subsections.

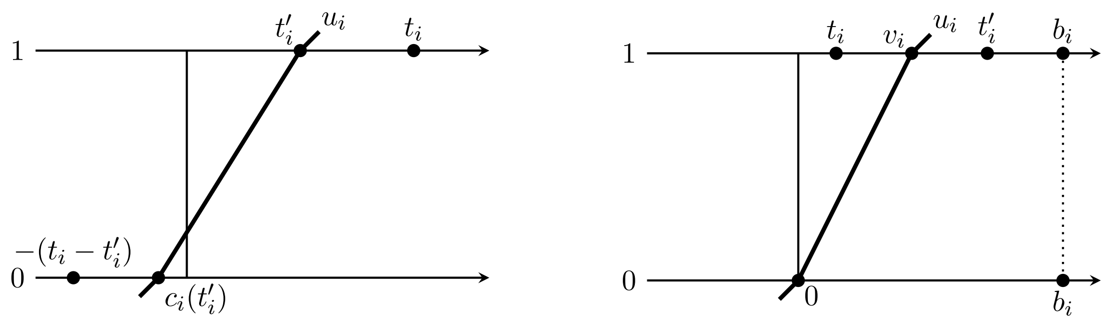
In the last subsection, we investigate the independence of the properties.

Before going to these subsections, we provide a *richness* condition on a domain.

Definition 5. A domain $\mathcal{U}$ is **rich** if the following two conditions are satisfied.

- **Small compensation**: For each $t_i > 0$ and each $t'_i < t_i$, there exists $u_i \in \mathcal{U}$ such that $-(t_i - t'_i) < c_i(t'_i) < 0$.
- **Density**: For each $t_i, t'_i \in \mathbb{R}_+$ with $t_i < t'_i$, there exists $u_i \in \mathcal{U}$ such that $t_i < \min\{v_i, b_i\} < t'_i$.

Small compensation says that for any two distinct payments, there exists a utility function such that the compensation does not exceed the change of the payment. Density says that for each non-negative two numbers, there exists a utility function whose truncated valuation is between the two numbers. Figure 4 illustrates these two conditions.



(a) Illustration of small compensation

(b) Illustration of density

Figure 4: Illustration of richness

The class of rich domains includes many economically meaningful domains. The following domains are examples:

- **Classical domain**: $\mathcal{U}^C$.
- **No budget constraint domain**: $\mathcal{U}^{NBC} = \{u_i \in \mathcal{U}^C : b_i = \infty\}$.
- **Positive income effect domain**:
  $\mathcal{U}^{PIE} = \{u_i \in \mathcal{U}^C : \forall t_i, t'_i \in \mathbb{R}, t_i < t'_i \leq c_i(b_i) \Rightarrow v_i(t_i) - t_i > v_i(t'_i) - t'_i\}$.
- **No income effect domain**:
  $\mathcal{U}^{NIE} = \{u_i \in \mathcal{U}^C : \forall t_i, t'_i \in \mathbb{R}, t_i < t'_i \leq c_i(b_i) \Rightarrow v_i(t_i) - t_i = v_i(t'_i) - t'_i\}$.
- **Quasi-linear domain**: $\mathcal{U}^Q = \mathcal{U}^{NBC} \cap \mathcal{U}^{NIE}$.

Moreover, a domain including a rich domain is also rich (e.g., $\mathcal{U} \supseteq \mathcal{U}^Q$). Note that the class of rich domains does not include the case where agents' budgets are public information.

5.1 Strategy-proofness

In Definition 4, we specify no tie-breaking rule, that is, for each $u \in \mathcal{U}^n$ and each $i \in N$, when $\min\{v_i, b_i\} = p_i(u_{-i})$, it is possible that either $x_i(u) = 1$ or $x_i(u) = 0$. For some tie-breaking rules, a threshold-price rule, and thus a truncated Vickrey rule with endogenous reserve prices, does not satisfy *strategy-proofness*.

Example 1. Assume $n = 2$. Let f be a truncated Vickrey rule such that for each $u \in \mathcal{U}^n$,

$$(x_1(u), x_2(u)) = \begin{cases} (1,0) & \text{if } \min\{v_1, b_1\} > \min\{v_2, b_2\} \\ (0,1) & \text{if } \min\{v_1, b_1\} \leq \min\{v_2, b_2\} \end{cases}.$$

This definition implies that agent 2 always has a higher priority than agent 1.

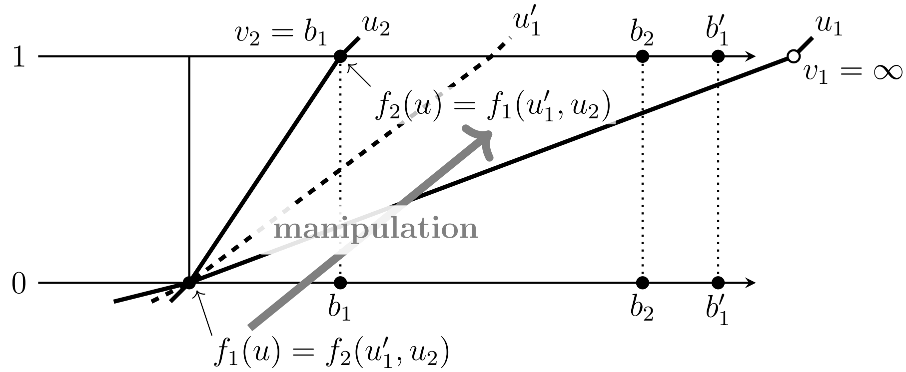


Figure 5: Violation of *strategy-proofness*

Figure 5 illustrates that the rule f does not satisfy *strategy-proofness*. Let $u \in \mathcal{U}^2$ be such that $v_1 = \infty$, $v_2 \neq \infty$, and $b_1 = v_2$. Then, $\min\{v_1, b_1\} = b_1 = v_2 = \min\{v_2, b_2\}$. Hence, by definition, $f_1(u) = (0, 0)$. Let $u_1' \in \mathcal{U}$ be such that $\min\{v_1', b_1'\} > v_2$. Then, by $\min\{v_1', b_1'\} > v_2 = \min\{v_2, b_2\}$, $f_1(u_1', u_2) = (1, v_2)$. Hence,

$$u_1(f_1(u_1', u_2)) \underset{f_1(u_1', u_2) = (1, v_2)}{=} u_1(1, v_2) \underset{v_2 = b_1}{=} u_1(1, b_1) \underset{v_1 = \infty}{>} u_1(0, 0) \underset{f_1(u) = (0,0)}{=} u_1(f_1(u)).$$

However, this is a contradiction to *strategy-proofness*. ■

In Example 1, if we set $x_1(u) = 1$, then $f_1(u) = (1, v_2)$, and so we can avoid beneficial misrepresentation of utility function u_1' for agent 1. Thus, *strategy-proofness* requires that an agent with $v_i = \infty$ has a high priority when a tie exists.

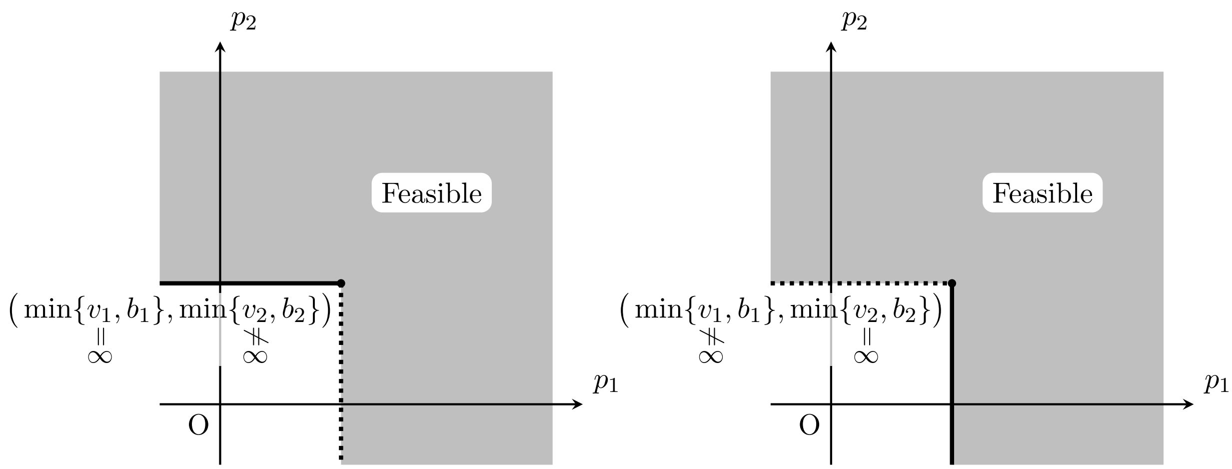
We formally define this tie-braking rule. We say that a threshold-price rule f with p has a **prioritized tie-breaking rule** if for each $u \in \mathcal{U}^n$ and each $i \in N$, if $\min\{v_i, b_i\} = p_i(u_{-i})$ and $v_i = \infty$, then $x_i(u) = 1$.

A prioritized tie-breaking rule affects the set of feasible price function profiles. As explained in Section 4, we must exclude prices which are less than truncated valuations for at least two agents. Furthermore, if *strategy-proofness* is required, then we must also exclude prices which are same as truncated valuations for at least two agents whose budget constraints are binding. Formally, we assume that any function profile p satisfies the following condition for each $u \in \mathcal{U}^n$:

$$|\{i \in N : \min\{v_i, b_i\} \geq p_i(u_{-i}) \text{ and } v_i > p_i(u_{-i})\}| \leq 1.$$

Figure 6 illustrates the set of feasible price profiles when $n = 2$. If $v_1 = \infty$ and $p_1 = \min\{v_1, b_1\}$, then a prioritized tie-breaking rule requires $x_1(u) = 1$. In this case, if $p_2 < \min\{v_1, b_1\}$, feasibility is violated. Therefore, such a price profile must be excluded, which is represented by the dotted line in Figure 6 (a). Similarly, Figure 6 (b) illustrates the set of feasible prices when $v_2 = \infty$.

The prioritized tie-breaking rule is shown to be a necessary and sufficient condition to satisfy *strategy-proofness*.



(a) $v_1 = \infty$ and $v_2 \neq \infty$

(b) $v_1 \neq \infty$ and $v_2 = \infty$

Figure 6: Illustration of feasible prices with a prioritized tie-breaking rule

Proposition 2. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ be rich. Let f on $\mathcal{U}^n$ be a threshold-price rule with $p \in \mathcal{R}^n$. Then, f satisfies strategy-proofness if and only if it has a prioritized tie-breaking rule.*

Our second result is a characterization of rules satisfying *individual rationality*, *no subsidy for losers*, and *strategy-proofness*.

Theorem 1. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ be rich. Then, a rule f on $\mathcal{U}^n$ satisfies individual rationality, no subsidy for losers, and strategy-proofness if and only if it is a threshold-price rule with $p \in \mathcal{R}^n$ that has a prioritized tie-breaking rule.*

Nisan (2007), Mukherjee (2014), Sprumont (2013), and Shinozaki (2024) show similar results when there is no hard budget constraint. Moreover, they do not allow income effects except for Shinozaki (2024). In their results, since there is no hard budget constraint, any tie-breaking rule ensures *strategy-proofness*. In fact, if a domain $\mathcal{U}$ satisfies that for each $u_i \in \mathcal{U}$, $b_i = \infty$, and so $v_i \neq \infty$, then any threshold-price rule on $\mathcal{U}$ has a prioritized tie-breaking rule. Hence, our result implies theirs while the converse is not true. Overall, our contribution is to clarify the structure of tie-breaking rules ensuring *strategy-proofness* in general environments.

5.2 Constrained efficiency

Similarly to *strategy-proofness*, for some tie-breaking rule, a truncated Vickrey rule with endogenous reserve prices may violate *constrained efficiency*.

Example 2. Assume $n = 2$. Let f be the same rule as that defined in Example 1. Figure 7 illustrates that f does not satisfy *constrained efficiency*. Let $u \in \mathcal{U}^2$ be such

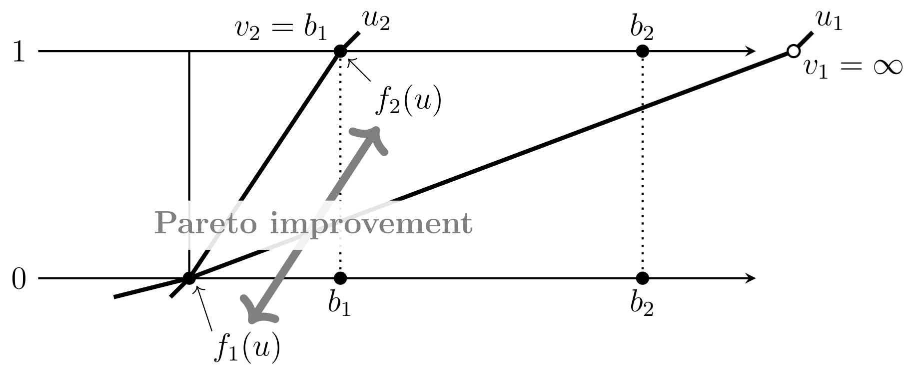


Figure 7: Violation of *constrained efficiency*

that $v_1 = \infty$, $v_2 \neq \infty$, and $b_1 = v_2$. Then, $\min\{v_1, b_1\} = b_1 = v_2 = \min\{v_2, b_2\}$, and so by definition, $f_1(u) = (0,0)$ and $f_2(u) = (1, b_1)$. Hence,

$$u_1(f_2(u)) \underset{f_2(u)=(1,b_1)}{=} u_1(1, b_1) \underset{v_1=\infty}{>} u_1(0,0) \underset{f_1(u)=(0,0)}{=} u_1(f_1(u)),$$

and

$$u_2(f_1(u)) \underset{f_1(u)=(0,0)}{=} u_2(0,0) \underset{v_2\neq\infty}{=} u_2(1, v_2) \underset{b_1=v_2}{=} u_2(1, b_1) \underset{f_2(u)=(1,b_1)}{=} u_2(f_2(u)).$$

However, these expressions contradict *constrained efficiency*. ■

In Example 2, if we exclude the agent 2 from the candidates who win the objects, then we can avoid the reallocation which undermines *constrained efficiency*.

We formally define this tie-breaking rule. A truncated Vickrey rule with endogenous reserve prices r has an **exclusive tie-breaking rule** if for each $u \in \mathcal{U}^n$ and each $i \in N$, if $\min\{v_i, b_i\} = \max\{\max_{j\neq i} \min\{v_j, b_j\}, r_i(u_{-i})\}$, $v_i \neq \infty$, and $N^\infty(u) \neq \emptyset$, then $x_i(u) = 0$.

First, we show that the exclusive tie-breaking rule is a necessary and sufficient condition for truncated Vickrey rules with endogenous reserve prices to satisfy *constrained efficiency*.

Proposition 3. *Let $\mathcal{U} \subseteq \mathcal{U}^C$. Let f on $\mathcal{U}^n$ be a truncated Vickrey rule with endogenous reserve prices $r \in \mathcal{R}^n$. Then, f satisfies constrained efficiency if and only if it has an exclusive tie-breaking rule.*

Next, we characterize truncated Vickrey rules with endogenous reserve prices by *constrained efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*.

Theorem 2. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ be rich. Then, a rule f on $\mathcal{U}^n$ satisfies constrained efficiency, individual rationality, no subsidy for losers, and strategy-proofness if and only if it is a truncated Vickrey rule with endogenous reserve prices $r \in \mathcal{R}^n$ that has an exclusive and prioritized tie-breaking rule.*

The final result shows that *no wastage* is incompatible with the four properties.

Proposition 4. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ be rich and contain u_0 such that $v_0 = \infty$. Let f on $\mathcal{U}^n$ be a truncated Vickrey rule with endogenous reserve prices $r \in \mathcal{R}^n$ that has an exclusive and prioritized tie-breaking rule. Then, f violates no wastage.*

Let $\mathcal{U}^{NBB}$ be the domain of utility functions without binding budget constraints, that is,

$$\mathcal{U}^{NBB} = \left\{ u_i \in \mathcal{U}^C : v_i \neq \infty \right\}.$$

Then, Proposition 4 shows that $\mathcal{U}^{NBB}$ is the unique maximal domain with respect to the existence of rules satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. Note that the existence of such rules follows from Saitoh and Serizawa (2008), and Sakai (2008).

Corollary 1. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ be rich. Then, $\mathcal{U} \subseteq \mathcal{U}^{NBB}$ if and only if there is a rule on $\mathcal{U}^n$ that satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness.*

The impossibility result of Dobzinski et al. (2012) does not imply ours. In their work, they consider the no income effect domain $\mathcal{U}^{NIE}$, and show the impossibility result on $\mathcal{U}^{NIE}$. While their work does not allow income effects, the impossibility result in Corollary 1 holds on any domain $\mathcal{U} \nsubseteq \mathcal{U}^{NBB}$ which may allow income effects. Furthermore, such domains include $\mathcal{U}^{NIE}$ because $\mathcal{U}^{NIE} \nsubseteq \mathcal{U}^{NBB}$. Hence, our result is more general than the result of Dobzinski et al. (2012).

5.3 Fairness

For some reserve prices, a truncated Vickrey rule with endogenous reserve prices may not satisfy *weak envy-freeness for equals*.

Example 3. Assume $n = 2$. Let f be the same rule as that defined in Example 1. Note that this rule f is the truncated Vickrey rule with endogenous reserve prices r such that for each $u \in \mathcal{U}^n$ and each $i, j \in N$, $r_i(u_j) \leq \min\{v_j, b_j\}$.

Figure 8 illustrates that f does not satisfy *weak envy-freeness for equals*. Let $u \in \mathcal{U}^2$ be such that $u_1 = u_2$ and $v_1 = v_2 = \infty$. By $\min\{v_1, b_1\} = b_1 = b_2 = \min\{v_2, b_2\}$, $f_1(u) = (0, 0)$ and $f_2(u) = (1, b_1)$. Hence,

$$u_1(f_2(u)) \underset{f_2(u)=(1,b_1)}{=} u_1(1, b_1) \underset{v_1=\infty}{>} u_1(0, 0) \underset{f_1(u)=(0,0)}{=} u_1(f_1(u)).$$

However, this is a contradiction to *weak envy-freeness for equals*. ■

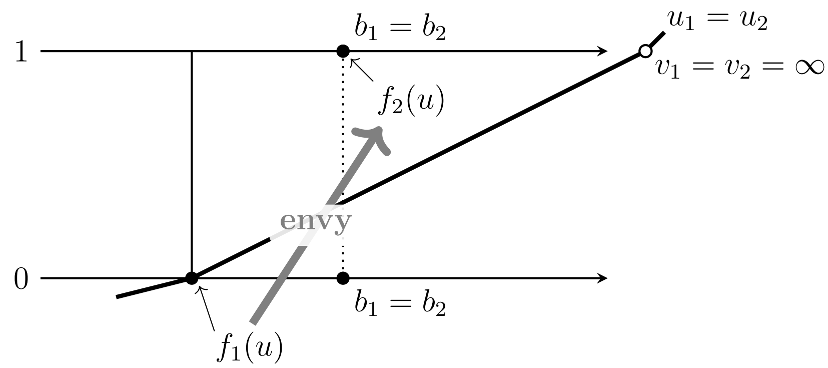


Figure 8: Violation of *Weak envy-freeness for equals*

In Example 3, since agent 2's reserve price is no more than her payment, that is, $r_2(u_1) \leq \min\{v_1, b_1\} = t_2(u)$, the truncated Vickrey rule realizes so low payment for agent 2 that agent 1 envies agent 2. To handle this problem, we must increase the agent 2's reserve price, that is, must set $r_2(u_1) > \min\{v_1, b_1\}$. Then, since agent 2's payment increases, agent 1 will no longer envy agent 2.

Formally, we define this condition. We say that a price function profile r is **non-negligible** if for each $i \in N$ and each $u_{-i} \in \mathcal{U}^{n-1}$, if there is $j \in N \setminus \{i\}$ such that $\min\{v_j, b_j\} = \max_{k \neq i} \min\{v_k, b_k\}$ and $v_j = \infty$, then $r_i(u_{-i}) > \max_{k \neq i} \min\{v_k, b_k\}$.

We show that the non-negligibility condition is a necessary and sufficient condition for truncated Vickrey rules with endogenous reserve prices that has a prioritized tie-breaking rule to satisfy *weak envy-freeness for equals*, *equal treatment for equals*, and *envy-freeness.*

Proposition 5. *Let $\mathcal{U} \subseteq \mathcal{U}^C$. Let f on $\mathcal{U}^n$ be a truncated Vickrey rule with endogenous reserve prices $r \in \mathcal{R}^n$ that has a prioritized tie-breaking rule. Then, the following statements are equivalent:*

(i) *The rule f satisfies weak envy-freeness for equals.*

(ii) *The rule f satisfies equal treatment of equals.*

(iii) *The rule f satisfies envy-freeness.*

(iv) *The reserve price profile r is non-negligible.*

We show that if *individual rationality*, *no subsidy for losers*, and *strategy-proofness* is satisfied, then *weak envy-freeness for equals* implies *constrained efficiency.*

Theorem 3. *Let $\mathcal{U} \subseteq \mathcal{U}^C$. If a rule f on $\mathcal{U}^n$ satisfies weak envy-freeness for equals, individual rationality, no subsidy for losers, and strategy-proofness, then it satisfies constrained efficiency.*

By Theorems 2 and 3, and Proposition 5, we get the following characterization results.

Corollary 2. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ be rich. Let f be a rule on $\mathcal{U}^n$. Then, the following statements are equivalent:*

(i) *The rule f satisfies weak envy-freeness for equals, individual rationality, no subsidy for losers, and strategy-proofness.*

(ii) *The rule f satisfies equal treatment of equals, individual rationality, no subsidy for losers, and strategy-proofness.*

(iii) *The rule f satisfies envy-freeness, individual rationality, no subsidy for losers, and strategy-proofness.*

(iv) *The rule f is a truncated Vickrey rule with non-negligible endogenous reserve prices r that has a prioritized tie-breaking rule.*

As for *anonymity in welfare*, we need an additional condition on reserve prices. A price function profile $r \in \mathcal{R}$ is **upper anonymous** if for each $u, u' \in \mathcal{U}^n$ and $i, j \in N$ such that $u_i = u'_j$, $u_j = u'_i$, and $u_{-\{i,j\}} = u'_{-\{i,j\}}$,

$$\max\{\max_{k \neq i} \min\{v_k, b_k\}, r_i(u_{-i})\} = \max\{\max_{k \neq j} \min\{v'_k, b'_k\}, r_j(u'_{-j})\}.$$

This condition and non-negligibility are shown to be necessary and sufficient for truncated Vickrey rules with endogenous reserve prices to satisfy *anonymity in welfare*.

Proposition 6. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ be rich. Let f on $\mathcal{U}^n$ be a truncated Vickrey rule with endogenous reserve prices $r \in \mathcal{R}^n$ that has a prioritized tie-breaking rule. Then, f satisfies anonymity in welfare if and only if r is upper anonymous and non-negligible.*

By Theorems 2 and 3, and Proposition 6, we get the following characterization result.

Corollary 3. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ be rich. Then, a rule f on $\mathcal{U}^n$ satisfies anonymity in welfare, individual rationality, no subsidy for losers, and strategy-proofness if and only if it is a truncated Vickrey rule with upper anonymous and non-negligible endogenous reserve prices $r \in \mathcal{R}^n$ that has a prioritized tie-breaking rule.*

5.4 Simple example

We provide a simple rule that satisfies all the properties presented in this paper other than *no wastage.*

Example 4. Let f be a truncated Vickrey rule with endogenous reserve prices $r \in \mathcal{R}^n$ such that for each $u \in \mathcal{U}^n$ and each $i \in N$,

$$f_i(u) = \begin{cases} (1, \max_{j \neq i} \min\{v_j, b_j\} + \varepsilon) & \text{if } \min\{v_i, b_i\} \geq \max_{j \neq i} \min\{v_j, b_j\} + \varepsilon \\ (0, 0) & \text{if } \min\{v_i, b_i\} < \max_{j \neq i} \min\{v_j, b_j\} + \varepsilon \end{cases},$$

where $\varepsilon > 0$. Then, f has a prioritized tie-breaking rule, and r satisfies upper anonymity and non-negligibility. Hence, by Corollary 3, f satisfies *anonymity in welfare*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. Moreover, f satisfies *weak envy-freeness for equals*, *equal treatment of equals*, and *envy-freeness* because *anonymity in welfare* is stronger than these properties. Finally, by Theorem 3, f satisfies *constrained efficiency*. ■

In the aforementioned rule, if the agent with the highest truncated valuation can pay an additional cost of ε above the second highest truncated valuation, then she can receive the object. Conversely, if the agent with the highest truncated valuation cannot pay this additional cost, then no one receives the object, resulting in a violation of *no wastage*. However, the smaller ε is, the lower the likelihood of violating *no wastage*. Therefore, by setting ε to a very small value, we can get an *almost efficient* rule.

5.5 Independence

We consider the independence of the properties. We assume $\mathcal{U} \subseteq \mathcal{U}^C$ is rich.

Example 5 (Dropping *constrained efficiency*)**.** Let f on $\mathcal{U}^n$ be such for each $u \in \mathcal{U}^n$,

$$f_1(u) = \begin{cases} (1, \max_{j\neq 1}\min\{v_j, b_j\}) & \text{if } \min\{v_1, b_1\} \geq \max_{j\neq 1}\min\{v_j, b_j\} \\ (0,0) & \text{if } \min\{v_1, b_1\} < \max_{j\neq 1}\min\{v_j, b_j\} \end{cases},$$

and for each $i \in N \setminus \{1\}$,

$$f_i(u) = \begin{cases} (1, \max_{j\neq i}\min\{v_j, b_j\} + \varepsilon) & \text{if } \min\{v_i, b_i\} \geq \max_{j\neq i}\min\{v_j, b_j\} + \varepsilon \\ (0,0) & \text{if } \min\{v_i, b_i\} < \max_{j\neq i}\min\{v_j, b_j\} + \varepsilon \end{cases},$$

where $\varepsilon > 0$.

Then, by Theorem 1, f satisfies *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. However, since f does not have an exclusive tie-breaking rule[11], by Proposition 3, it violates *constrained efficiency*. Furthermore, by Theorem 3, f also violates *weak envy freeness for equals*, *equal treatment of equals*, *envy-freeness*, and *anonymity in welfare*. ■

Example 6 (Dropping *individual rationality* or *no subsidy for losers*)**.** Let f on $\mathcal{U}^n$ be such that for each $u \in \mathcal{U}^n$ and each $i \in N$,

$$f_i(u) = \begin{cases} (1, \max_{j\neq i}\min\{v_j(\delta), b_j\} + \varepsilon) & \text{if } \min\{v_i(\delta), b_i\} \geq \max_{j\neq i}\min\{v_j(\delta), b_j\} + \varepsilon \\ (0,\delta) & \text{if } \min\{v_i(\delta), b_i\} < \max_{j\neq i}\min\{v_j(\delta), b_j\} + \varepsilon \end{cases},$$

[11]To see this, let $u \in \mathcal{U}^n$ be such that for each $i \in N$, $\min\{v_i, b_i\} = \min\{v_1, b_1\}$, and $v_i \neq \infty$ if $i = 1$ and $v_i = \infty$ if $i \neq 1$. By the definition of f, $x_1(u) = 1$. Moreover, by the definition of u, $\min\{v_1, b_1\} = \max_{j\neq 1}\min\{v_j, b_j\}$, $v_1 \neq \infty$, and $N^\infty(u) = N \setminus \{1\}$. However, by $x_1(u) = 1$, these expressions violates that f has an exclusive tie-breaking rule.

where $\varepsilon > 0$ and $\delta \in \mathbb{R}$.

Then, f satisfies *anonymity in welfare*, *envy-freeness*, *equal treatment of equals*, *constrained efficiency*, and *strategy-proofness*. Moreover, f violates *individual rationality* if $\delta > 0$, and violates *no subsidy for losers* if $\delta < 0$. ■

Example 7 (Dropping *strategy-proofness*)**.** Let f on $\mathcal{U}^n$ be such that for each $u \in \mathcal{U}^n$ and each $i \in N$,

$$f_i(u) = \begin{cases} (1, \min\{v_i, b_i\}) & \text{if } \min\{v_i, b_i\} > \max_{j \neq i} \min\{v_j, b_j\} \\ (0,0) & \text{if } \min\{v_i, b_i\} \leq \max_{j \neq i} \min\{v_j, b_j\} \end{cases}.$$

Then, f satisfies *individual rationality* and *no subsidy for losers*. Hence, by Theorem 1, f violates *strategy-proofness*. Moreover, f satisfies *constrained efficiency*.[12] Finally, *anonymity in welfare* is satisfied, so is *weak envy-freeness for equals*, *equal treatment of equals*, and *envy-freeness*. ■

The following two examples show that even if *individual rationality*, *no subsidy for losers*, and *strategy-proofness* are satisfied, *constrained efficiency* does not imply *weak envy-freeness for equals*, and *weak envy-freeness for equals* also does not imply *anonymity in welfare*.

Example 8 (*Constrained efficiency* $\not\Rightarrow$ *weak envy-freeness for equals*)**.** Let f on $\mathcal{U}^n$ be such for each $u \in \mathcal{U}^n$,

$$f_1(u) = \begin{cases} (1, \max_{j \neq 1} \min\{v_j, b_j\}) & \text{if } \min\{v_1, b_1\} \geq \max_{j \neq 1} \min\{v_j, b_j\} \text{ and } v_1 > \max_{j \neq 1} \min\{v_j, b_j\} \\ (0,0) & \text{if } \min\{v_1, b_1\} < \max_{j \neq 1} \min\{v_j, b_j\} \text{ or } v_1 \leq \max_{j \neq 1} \min\{v_j, b_j\} \end{cases},$$

and for each $i \in N \setminus \{1\}$,

$$f_i(u) = \begin{cases} (1, \max_{j \neq i} \min\{v_j, b_j\} + \varepsilon) & \text{if } \min\{v_i, b_i\} \geq \max_{j \neq i} \min\{v_j, b_j\} + \varepsilon \\ (0,0) & \text{if } \min\{v_i, b_i\} < \max_{j \neq i} \min\{v_j, b_j\} + \varepsilon \end{cases},$$

where $\varepsilon > 0$.

Then, f is a truncated Vickrey rule with r that has an exclusive and prioritized tie-breaking rule, where r satisfies $r_1(u_{-1}) = \max_{j \neq 1}\{v_j, b_j\}$ and $r_i(u_{-i}) = \max_{j \neq i} \min\{v_j, b_j\} + \varepsilon$ for $i \neq 1$. Hence, by Theorem 2, f satisfies *constrained efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. On the other hand, by $r_1(u_{-1}) = \max_{j \neq 1} \min\{v_j, b_j\}$, r is not non-negligible. Hence, by Proposition 5, *weak envy-freeness for equals* is violated, so is *equal treatment of equals*, *envy-freeness*, and *anonymity in welfare*. ■

[12] If f violates *constrained efficiency*, then by Lemma 1 in Appendix C, there are $u \in \mathcal{U}^n$ and $i, j \in N$ such that $x_i(u) = 1$, $x_j(u) = 0$, and $\min\{c_i(t_i(u)), b_i\} + \min\{v_j(t_j(u)), b_j\} \geq t_i(u) + t_j(u)$. By $c_i(t_i(u)) \leq 0$, $t_i(u) = \min\{v_i, b_i\}$ and $t_j(u) = 0$, we get $\min\{v_j, b_j\} \geq \min\{v_i, b_i\}$. However, this contradicts $x_i(u) = 1$.

Example 9 (*Weak envy-freeness for equals $\not\Rightarrow$ anonymity in welfare*). Let f on $\mathcal{U}^n$ be such that for each $u \in \mathcal{U}^n$ and each $i \in N$,

$$f_i(u) = \begin{cases} (1, \max_{j \neq i} \min\{v_j, b_j\} + \varepsilon_i) & \text{if } \min\{v_i, b_i\} \geq \max_{j \neq i} \min\{v_j, b_j\} + \varepsilon_i \\ (0, 0) & \text{if } \min\{v_i, b_i\} < \max_{j \neq i} \min\{v_j, b_j\} + \varepsilon_i \end{cases},$$

where $\varepsilon_1 > \cdots > \varepsilon_n > 0$.

Then, f is a truncated Vickrey rule with non-negligible reserve prices r that has an exclusive and prioritized tie-breaking rule. Hence, by Theorem 2 and Corollary 2, f satisfies *constrained efficiency*, *weak envy-freeness for equals*, *equal treatment of equals*, *envy-freeness*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. However, since r does not satisfy upper anonymity, by Proposition 6, f violates *anonymity in welfare*. ■

6 Conclusion

We consider the single-object allocation problem with hard budget constraints and income effects, and show that truncated Vickrey rules with endogenous reserve prices can be characterized by *constrained efficiency* or *weak envy-freeness for equals*, along with *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. Whether truncated Vickrey rules with endogenous reserve prices satisfy *constrained efficiency*,*weak envy-freeness for equals*, or *strategy-proofness* critically depends on the structure of the reserve prices and the tie-breaking rule. Our results suggest that commonly used fixed tie-breaking rule and fixed reserve prices in real-life auctions may not guarantee these desirable properties. This paper, therefore, highlights the importance of appropriately operating tie-breaking rules and reserve prices. Furthermore, as one solution, we demonstrate that a simple rule, a truncated Vickrey rule with a small extra cost, can attain these desirable properties.

Appendices: Proofs

Appendix A: Decomposition of efficiency (Section 3.1)

Proposition 1. *Let $u \in \mathcal{U}^n$ and $z \in Z^{IR}(u)$. Then, z is efficient for u if and only if*

(i) $\sum_{i \in N} x_i = 1$, *and*

(ii) *there is no $z' \in Z$ such that $\sum_{i \in N} x'_i = \sum_{i \in N} x_i$ and it dominates z for u.*

Proof. Only if: Assume z is efficient for u. Since (ii) is obvious, we show (i).

Suppose $\sum_{i \in N} x_i = 0$. Let $i \in N$. By $\sum_{j \in N} x_j = 0$, $x_i = 0$. By $z \in Z^{IR}(u)$, $t_i \leq b_i$, and so by object desirability, $u_i(1, t_i) > u_i(0, t_i) = u_i(z_i)$. Hence, $((1, t_i), z_{-i})$ dominates z for u, a contradiction.

IF: Assume (i) and (ii) hold. Suppose z is not *efficient* for u. Then, there is $z' \in Z$ such that (a) for each $i \in N$, $u_i(z'_i) \geq u_i(z_i)$, (b) $\sum_{i\in N} t'_i \geq \sum_{i\in N} t_i$, and (c) at least one inequality holds strictly.

By (ii), $\sum_{i\in N} x'_i \neq \sum_{i\in N} x_i$. By (i), $\sum_{i\in N} x_i = 1$, and so $\sum_{i\in N} x'_i = 0$. By $\sum_{i\in N} x_i = 1$, there is $i \in N$ such that $x_i = 1$. By $\sum_{j\in N} x'_j = 0$, $x'_i = 0$. By $t_i \leq b_i$, $u_i(1, t_i) > u_i(0, t_i) \geq u_i(0, b_i)$, so that $u_i(0, c_i(t_i)) = u_i(z_i)$ and $c_i(t_i) < t_i$. By $x'_i = 0$ and $u_i(z'_i) \geq u_i(z_i) = u_i(0, c_i(t_i))$, $t'_i \leq c_i(t_i)$. For each $j \in N \setminus \{i\}$, by $x'_j = x_j = 0$ and $u_j(z'_j) \geq u_j(z_i)$, $t'_j \leq t_j$. Thus, by $t'_i \leq c_i(t_i) < t_i$, $\sum_{j\in N} t'_j \leq c_i(t_i) + \sum_{j\neq i} t_j < \sum_{j\in N} t_j$. However, this contradicts (b). □

Appendix B: Strategy-proofness (Section 5.1)

Proposition 2. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ be rich. Let f on $\mathcal{U}^n$ be a threshold-price rule with $p \in \mathcal{R}^n$. Then, f satisfies strategy-proofness if and only if it has a prioritized tie-breaking rule.*

Proof. ONLY IF: Assume f satisfies *strategy-proofness*. Let $u \in \mathcal{U}^n$ and $i \in N$ be such that $\min\{v_i, b_i\} = p_i(u_{-i})$ and $v_i = \infty$. Suppose $x_i(u) = 0$. By (P-ii), $f_i(u) = (0, 0)$. Let $u'_i \in \mathcal{U}$ be such that $\min\{v'_i, b'_i\} > p_i(u_{-i})$.[13] By (P-i) and (P-ii), $f_i(u'_i, u_{-i}) = (1, p_i(u_{-i}))$. By $\min\{v_i, b_i\} = p_i(u_{-i})$ and $v_i = \infty$, $u_i(1, p_i(u_{-i})) > u_i(0, 0)$. Thus, by $f_i(u) = (0, 0)$ and $f_i(u'_i, u_{-i}) = (1, p_i(u_{-i}))$, $u_i(f_i(u'_i, u_{-i})) > u_i(f_i(u))$. However, this contradicts *strategy-proofness*.

IF: Assume f has a prioritized tie-breaking rule. Suppose f violates *strategy-proofness*. Then, there are $u \in \mathcal{U}^n$, $i \in N$, and $u'_i \in \mathcal{U}$ such that $u_i(f_i(u'_i, u_{-i})) > u_i(f_i(u))$. By $u_i(f_i(u'_i, u_{-i})) > u_i(f_i(u))$ and (P-ii), $x_i(u) \neq x_i(u'_i, u_{-i})$. If $x_i(u'_i, u_{-i}) = 0$, then $u_i(0, 0) > u_i(f_i(u)) =$, and so by $u_i(1, \min\{v_i, b_i\}) \geq u_i(0, 0)$ and $x_i(u) = 1$, $\min\{v_i, b_i\} < t_i(u) = p_i(u_{-i})$. However, this contradicts (P-i). Hence, $x_i(u'_i, u_{-i}) = 1$, and so $x_i(u) = 0$. By $u_i(f_i(u'_i, u_{-i})) > u_i(f_i(u))$, $u_i(1, p_i(u_{-i})) > u_i(0, 0)$. By $x_i(u) = 0$ and (P-i), we get $(*)$ $\min\{v_i, b_i\} \leq p_i(u_{-i})$. By $u_i(1, p_i(u_{-i})) > u_i(0, 0)$, we get $(**)$ $p_i(u_{-i}) \leq \min\{v_i, b_i\}$ Hence, by $(*)$ and $(**)$, $\min\{v_i, b_i\} = p_i(u_{-i})$. By $u_i(1, p_i(u_{-i})) > u_i(0, 0)$ and $p_i(u_{-i}) = \min\{v_i, b_i\}$, $u_i(1, \min\{v_i, b_i\}) > u_i(0, 0)$, which implies $v_i = \infty$. However, by $x_i(u) = 0$ and $\min\{v_i, b_i\} = p_i(u_{-i})$, this contradicts that f has a prioritized tie-breaking rule. □

Theorem 1. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ be rich. Then, a rule f on $\mathcal{U}^n$ satisfies individual rationality, no subsidy for losers, and strategy-proofness if and only if it is a threshold-price rule with $p \in \mathcal{R}^n$ that has a prioritized tie-breaking rule.*

Proof. ONLY IF: Assume f satisfies *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. Let $i \in N$.

First, we show that for each $u_i, u'_i \in \mathcal{U}$ and $u_{-i} \in \mathcal{U}^{n-1}$, if $x_i(u_i, u_{-i}) = x_i(u'_i, u_{-i})$, then $t_i(u_i, u_{-i}) = t_i(u'_i, u_{-i})$. Let $u_i, u'_i \in \mathcal{U}$ and $u_{-i} \in \mathcal{U}^{n-1}$ be such that $x_i(u_i, u_{-i}) =$

[13] By density, u'_i exists.

$x_i(u_i', u_{-i})$. Suppose, without loss of generality, that $t_i(u_i, u_{-i}) < t_i(u_i', u_{-i})$. By *individual rationality*, $t_i(u_i', u_{-i}) \leq b_i'$. Hence, by $x_i(u_i, u_{-i}) = x_i(u_i', u_{-i})$ and $t_i(u_i, u_{-i}) < t_i(u_i', u_{-i}) \leq b_i'$, $u_i'(f_i(u_i, u_{-i})) > u_i'(f_i(u_i', u_{-i}))$. However, this contradicts *strategy-proofness*. Thus, $t_i(u_i, u_{-i}) = t_i(u_i', u_{-i})$.

Let $p_i : \mathcal{U}^{n-1} \to \mathbb{R} \cup \{\infty\}$ be such that for each $u_{-i} \in \mathcal{U}^{n-1}$,

$$p_i(u_{-i}) = \begin{cases} t_i(u_i, u_{-i}) & \text{if there exists } u_i \in \mathcal{U}^W(u_{-i}) \\ \infty & \text{if } \mathcal{U}^W(u_{-i}) = \emptyset \end{cases}.$$

Note that, by the statement shown in the previous paragraph, p_i is uniquely defined. Moreover, for each $u \in \mathcal{U}^n$, if $x_i(u) = 0$, then by *individual rationality* and *no subsidy for losers*, $t_i(u) = 0$. Thus, (P-ii) holds.

By Proposition 2, it suffices to show (P-i). Let $u \in \mathcal{U}^n$. First assume $\min\{v_i, b_i\} < p_i(u_{-i})$. Suppose $x_i(u) = 1$. Then, by (P-ii), $f_i(u) = (1, p_i(u_{-i}))$. By $\min\{v_i, b_i\} < p_i(u_{-i}) = t_i(u)$, $0 > u_i(f_i(u))$. However, this contradicts *individual rationality*, and so $x_i(u) = 0$. Next assume $\min\{v_i, b_i\} > p_i(u_{-i})$. Suppose $x_i(u) = 0$. Then, by (P-ii), $f_i(u) = (0, 0)$. If $\mathcal{U}^W(u_{-i}) = \emptyset$, then by definition, $p_i(u_{-i}) = \infty$. However, this contradicts $\min\{v_i, b_i\} > p_i(u_{-i})$, and so $\mathcal{U}^W(u_{-i}) \neq \emptyset$. Let $u_i' \in \mathcal{U}^W(u_{-i})$. Then, by (P-ii), $f_i(u_i', u_{-i}) = (1, p_i(u_{-i}))$. By $\min\{v_i, b_i\} > p_i(u_{-i})$, $f_i(u_i', u_{-i}) = (1, p_i(u_{-i}))$, and $f_i(u) = (0, 0)$, we get $u_i(f_i(u_i', u_{-i})) > u_i(f_i(u))$. However, this contradicts *strategy-proofness*, and so $x_i(u) = 1$.

If: Assume that f is a threshold-price rule with $p \in \mathcal{R}^n$ that has a prioritized tie-breaking rule. By (P-ii), f satisfies *no subsidy for losers*. By Proposition 2, f satisfies *strategy-proofness*. Thus, we show *individual rationality*.

Let $u \in \mathcal{U}^n$ and $i \in N$. If $x_i(u) = 0$, then by (P-ii), $f_i(u) = (0, 0)$, and so $u_i(f_i(u)) = 0$. If $x_i(u) = 1$, then by (P-ii), $f_i(u) = (1, p_i(u_{-i}))$. By $x_i(u) = 1$ and (P-i), $\min\{v_i, b_i\} \geq p_i(u_{-i})$, and so $u_i(f_i(u)) \geq 0$. □

Appendix C: Constrained efficiency (Section 5.2)

The following result shows a necessary and sufficient condition for an allocation to satisfy *constrained efficiency*, which is useful for the proofs of the results pertaining *constrained efficiency*.

Lemma 1. *Let $u \in \mathcal{U}^n$ and $z \in Z^{IR}(u)$. Then, z is constrained efficient for u if and only if there are no $i, j \in N$ such that* (i) $x_i = 1$ *and* $x_j = 0$, (ii) $\min\{c_i(t_i), b_i\} + \min\{v_j(t_j), b_j\} \geq t_i + t_j$, *and* (iii) $c_i(t_i) + v_j(t_j) > t_i + t_j$.

Proof. Only if: We show the contrapositive. Suppose there are $i, j \in N$ satisfying (i), (ii), and (iii).

Let $z' \in Z$ be such that $z_i' = (0, \min\{c_i(t_i), b_i\})$, $z_j' = (1, \min\{v_j(t_j), b_j\})$, and for each $k \in N \setminus \{i, j\}$, $z_k' = z_k$. By (i), $\sum_{k \in N} x_k' = \sum_{k \in N} x_k = 1$. By $x_i = 1$, $u_i(z_i') =$

$u_i(0, \min\{c_i(t_i), b_i\}) \geq u_i(z_i)$. By $x_j = 0$, $u_j(z'_j) = u_j(1, \min\{v_j(t_j), b_j\}) \geq u_j(z_j)$. If $c_i(t_i) = \infty$, then $u_i(z'_i) > u_i(z_i)$. If $v_j(t_j) = \infty$, then $u_j(z'_j) > u_j(z_j)$. If $c_i(t_i) \neq \infty$ and $v_j(t_j) \neq \infty$, then by (iii), $t'_i + t'_j = c_i(t_i) + v_j(t_j) > t_i + t_j$, and so $\sum_{k \in N} t'_k > \sum_{k \in N} t_k$. Thus, z' dominates z for u in any case, and so it is not *constrained efficient* for u.

IF: Assume there are no $i, j \in N$ satisfying (i), (ii), and (iii). Suppose z is not *constrained efficient*. Then, there is $z' \in Z$ such that (a) $\sum_{i \in N} x'_i = \sum_{i \in N} x_i$, (b) for each $i \in N$, $u_i(z'_i) \geq u_i(z_i)$, (c) $\sum_{i \in N} t'_i \geq \sum_{i \in N} t_i$, and (d) at least one inequality in (b) and (c) holds strictly.

If $x = x'$, then by (b), for each $i \in N$, $t'_i \leq t_i$, and so by (c), $t'_i = t_i$. Hence, for each $i \in N$, $u_i(z'_i) = u_i(z_i)$ and $\sum_{i \in N} t'_i = \sum_{i \in N} t_i$. However, these equalities contradicts (d). Thus, $x \neq x'$.

By (a) and $x \neq x'$, there are $i, j \in N$ such that $(x_i, x_j) = (1, 0)$ and $(x'_i, x'_j) = (0, 1)$. By (b),

$$\begin{cases} \min\{c_i(t_i), b_i\} \geq t'_i, \\ \min\{v_j(t_j), b_j\} \geq t'_j, \\ t_k \geq t'_k & \text{for each } k \in N \setminus \{i, j\}. \end{cases} \tag{1}$$

These inequalities and (c) imply

$$\min\{c_i(t_i), b_i\} + \min\{v_j(t_j), b_j\} + \sum_{k \in N \setminus \{i,j\}} t_k \geq \sum_{k \in N} t'_k \geq \sum_{k \in N} t_k. \tag{2}$$

If at least one of inequalities in (2) holds strictly, then

$$\min\{c_i(t_i), b_i\} + \min\{v_j(t_j), b_j\} > t_i + t_j.$$

Thus, i, j satisfy (i), (ii), and (iii), but this is a contradiction. Hence,

$$\min\{c_i(t_i), b_i\} + \min\{v_j(t_j), b_j\} + \sum_{k \in N \setminus \{i,j\}} t_k = \sum_{k \in N} t'_k, \tag{3}$$

$$\sum_{k \in N} t'_k = \sum_{k \in N} t_k. \tag{4}$$

By (3), any expression in (1) holds with equality, that is,

$$\begin{cases} \min\{c_i(t_i), b_i\} = t'_i, \\ \min\{v_j(t_j), b_j\} = t'_j, \\ t_k = t'_k & \text{for each } k \in N \setminus \{i, j\}. \end{cases} \tag{5}$$

By (4), (5), and (d), $u_i(z'_i) > u_i(z_i)$ or $u_j(z'_j) > u_i(z_j)$. If $u_i(z'_i) > u_i(z_i)$, then by $t'_i = \min\{c_i(t_i), b_i\}$, $c_i(t_i) = \infty$. If $u_j(z'_j) > u_j(z_j)$, then by $t'_j = \min\{v_j(t_j), b_j\}$, $v_j(t_j) = \infty$. Thus, in any case, $c_i(t_i) + v_j(t_j) > t_i + t_j$, and so i, j satisfy (i), (ii), and (iii). However, this is a contradiction. □

Proposition 3. *Let $\mathcal{U} \subseteq \mathcal{U}^C$. Let f on $\mathcal{U}^n$ be a truncated Vickrey rule with endogenous reserve prices $r \in \mathcal{R}^n$. Then, f satisfies constrained efficiency if and only if it has an exclusive tie-breaking rule.*

Proof. Given $i \in N$ and $u_{-i} \in \mathcal{U}^{n-1}$, let $r_i^*(u_{-i}) = \max\{\max_{j \neq i} \min\{v_j, b_j\}, r_i(u_{-i})\}$.

ONLY IF: Assume f satisfies *constrained efficiency*. Let $u \in \mathcal{U}^n$ and $i \in N$ be such that $\min\{v_i, b_i\} = r_i^*(u_{-i})$, $v_i \neq \infty$, and $N^\infty(u) \neq \emptyset$. Suppose $x_i(u) = 1$.

By $\min\{v_i, b_i\} = r_i^*(u_{-i})$ and $v_i \neq \infty$, $i \in N(u) \setminus N^\infty(u)$. By $N^\infty(u) \neq \emptyset$, there is $j \in N^\infty(u)$. By $i \in N(u) \setminus N^\infty(u)$, $i \neq j$. By $i, j \in N(u)$, $\min\{v_i, b_i\} = \min\{v_j, b_j\}$. By $\min\{v_i, b_i\} = \min\{v_j, b_j\}$ and $i \neq j$, $\min\{v_i, b_i\} \geq r_i^*(u_{-i}) \geq \min\{v_j, b_j\} = \min\{v_i, b_i\}$, and so, $r_i^*(u_{-i}) = \min\{v_i, b_i\} = \min\{v_j, b_j\}$. Thus, by (V-ii), $t_i(u) = \min\{v_i, b_i\} = \min\{v_j, b_j\}$.

By $t_i(u) = \min\{v_i, b_i\}$ and $v_i \neq \infty$, $u_i(f_i(u)) = u_i(0, 0)$, and so $c_i(t_i(u)) = 0$. Thus, by $b_i > 0$,

$$\min\{c_i(t_i(u)), b_i\} = c_i(t_i(u)) = 0. \tag{6}$$

By $x_j(u) = 0$ and (V-ii), $t_j(u) = 0$, and so $v_j(t_j(u)) = v_j$. Thus, by $\min\{v_j, b_j\} = t_i(u)$,

$$\min\{v_j(t_j(u)), b_j\} = t_i(u). \tag{7}$$

By (6), (7), and $t_j(u) = 0$,

$$\min\{c_i(t_i(u)), b_i\} + \min\{v_j(t_j(u)) + b_j\} = t_i(u) + t_j(u).$$

Moreover, by $v_j(t_j(u)) = v_j = \infty$,

$$c_i(t_i(u)) + v_j(t_j(u)) > t_i(u) + t_j(u).$$

Thus, i, j satisfy all the conditions in Lemma 1. However, it contradicts *constrained efficiency*.

IF: Assume f has an exclusive tie-breaking rule. Suppose f violates *constrained efficiency*. Then, by Lemma 1, there are $u \in \mathcal{U}^n$ and $i, j \in N$ such that (i) $x_i(u) = 1$ and $x_j(u) = 0$, (ii) $\min\{c_i(t_i(u)), b_i\} + \min\{v_j(t_j(u)), b_j\} \geq t_i(u) + t_j(u)$, and (iii) $c_i(t_i(u)) + v_j(t_j(u)) > t_i(u) + t_j(u)$.

By *individual rationality*, $c_i(t_i(u)) \leq 0$. If $c_i(t_i(u)) < 0$, then by (ii) and $t_j(u) = 0$, $\min\{v_j, b_j\} > t_i(u)$. However, by $t_i(u) = r_i^*(u_{-i}) \geq \min\{v_j, b_j\}$, this is a contradiction, and so $c_i(t_i(u)) = 0$. Moreover, by $c_i(t_i(u)) = 0$, $u_i(f_i(u)) = u_i(0, 0)$, which implies $v_i \neq \infty$ and $\min\{v_i, b_i\} = r_i^*(u_{-i})$.

By $\min\{c_i(t_i(u)), b_i\} = c_i(t_i(u)) = 0$, $t_j(u) = 0$, and $t_i(u) = r_i^*(u_{-i})$, (ii) implies $\min\{v_j, b_j\} \geq r_i^*(u_{-i})$ and (iii) implies $v_j > r_i^*(u_{-i})$. Thus, by $r_i^*(u_{-i}) \geq \min\{v_j, b_j\}$, $v_j > r_i^*(u_{-i}) = \min\{v_j, b_j\}$. By $\min\{v_i, b_i\} = r_i^*(u_{-i}) = \min\{v_j, b_j\}$, $j \in N(u)$. By $v_j > \min\{v_j, b_j\}$, $v_j = \infty$, and so $j \in N^\infty(u)$. However, by $\min\{v_i, b_i\} = r_i^*(u_{-i})$, $v_i \neq \infty$, and $x_i(u) = 1$, this contradicts that f has an exclusive tie-breaking rule. $\square$

Theorem 2. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ be rich. Then, a rule f on $\mathcal{U}^n$ satisfies constrained efficiency, individual rationality, no subsidy for losers, and strategy-proofness if and only if it is a truncated Vickrey rule with endogenous reserve prices $r \in \mathcal{R}^n$ that has an exclusive and prioritized tie-breaking rule.*

Proof. Since "if" part follows from Theorem 1 and Proposition 3, we only show "only if" part. Assume f satisfies *constrained efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. By Theorem 1, f is a threshold-price rule with $r \in \mathcal{R}^n$ that has prioritized tie-breaking rule.

By Proposition 3, to prove the result, it suffices to show that f is a truncated Vickrey rule with endogenous reserve prices r, that is, for each $i \in N$ and each $u_{-i} \in \mathcal{U}^{n-1}$, $r_i(u_{-i}) \geq \max_{j \neq i} \min\{v_j, b_j\}$.

Suppose there are $i \in N$ and $u_{-i} \in \mathcal{U}^{n-1}$ such that $r_i(u_{-i}) < \max_{j \neq i} \min\{v_j, b_j\}$. Let $j \in N \setminus \{i\}$ be such that $\min\{v_j, b_j\} = \max_{k \neq i} \min\{v_k, b_k\}$. Let $u_i \in \mathcal{U}$ be such that $-(\min\{v_j, b_j\} - r_i(u_{-i})) < c_i(r_i(u_{-i})) < 0$.[14]

By $c_i(r_i(u_{-i})) < 0$, $u_i(1, r_i(u_{-i})) = u_i(0, c_i(r_i(u_{-i}))) > u_i(0, 0)$. By $u_i(1, r_i(u_{-i})) > u_i(0, 0)$, either $r_i(u_{-i}) < \min\{v_i, b_i\}$, or $r_i(u_{-i}) = \min\{v_i, b_i\}$ and $v_i = \infty$. Since f has a prioritized tie-breaking rule, $x_i(u) = 1$ in any case. By (P-ii), $t_i(u) = r_i(u_{-i})$.

By $x_i(u) = 1$, $x_j(u) = 0$, and so $t_j(u) = 0$. By $c_i(r_i(u_{-i})) < 0 < b_i$, $\min\{c_i(r_i(u_{-i})), b_i\} = c_i(r_i(u_{-i}))$, and so $-(\min\{v_j, b_j\} - r_i(u_{-i})) < \min\{c_i(r_i(u_{-i})), b_i\}$. Thus, by $t_i(u) = r_i(u_{-i})$ and $t_j(u) = 0$, $t_i(u) + t_j(u) < \min\{c_i(t_i(u)), b_i\} + \min\{v_j(t_j(u)), b_j\}$. However, by Lemma 1, this contradicts *constrained efficiency*. $\square$

Proposition 4. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ be rich and contain u_0 such that $v_0 = \infty$. Let f on $\mathcal{U}^n$ be a truncated Vickrey rule with endogenous reserve prices $r \in \mathcal{R}^n$ that has an exclusive and prioritized tie-breaking rule. Then, f violates no wastage.*

Proof. Given $i \in N$ and $u_{-i} \in \mathcal{U}^n$, let $r_i^*(u_{-i}) = \max\{\max_{j \neq i} \min\{v_i, b_i\}, r_i(u_{-i})\}$.

Let $u \in \mathcal{U}^n$ be such that for each $i \in N$, $u_i = u_0$. Let $i \in N$ be such that $x_i(u) = 0$. By $v_i = v_0 = \infty$, the prioritized tie-breaking rule requires $\min\{v_i, b_i\} < r_i^*(u_{-i})$. Let $u_i' \in \mathcal{U}$ be such that $\min\{v_i, b_i\} < \min\{v_i', b_i'\} < r_i^*(u_{-i})$,[15] and let $u' = (u_i', u_{-i})$.

By $u_{-i}' = u_{-i}$, $\min\{v_i', b_i'\} < r_i^*(u_{-i}')$, and so $x_i(u') = 0$. For each $j \in N \setminus \{i\}$, by $u_j' = u_0$ and $\min\{v_0, b_0\} = \min\{v_i, b_i\} < \min\{v_i', b_i'\}$, we get $\min\{v_j', b_j'\} < \min\{v_i', b_i'\} \leq r_j^*(u_{-j}')$, and so $x_j(u') = 0$. Hence, $\sum_{j \in N} x_j(u') = 0$, that is, f violates *no wastage*. $\square$

Appendix D: Fairness (Section 5.3)

Proposition 5. *Let $\mathcal{U} \subseteq \mathcal{U}^C$. Let f on $\mathcal{U}^n$ be a truncated Vickrey rule with endogenous reserve prices $r \in \mathcal{R}^n$ that has a prioritized tie-breaking rule. Then, the following statements are equivalent:*

[14] By small compensation, u_i exists.

[15] By density, u_i' exists.

(i) *The rule f satisfies weak envy-freeness for equals.*

(ii) *The rule f satisfies equal treatment for equals.*

(iii) *The rule f satisfies envy-freeness.*

(iv) *The reserve price profile r is non-negligible.*

Proof. We show (i) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (iii). Since (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) is obvious, these two statements complete the proof. Given $i \in N$ and $u_{-i} \in \mathcal{U}^{n-1}$, let $r_i^*(u_{-i}) = \max\{\max_{j\neq i}\min\{v_j, b_j\}, r_i(u_{-i})\}$.

(i) $\Rightarrow$ (iv): Assume f satisfies *weak envy-freeness for equals*. Let $i \in N$ and $u_{-i} \in \mathcal{U}^{n-1}$, and let $j \in N$ be such that $\min\{v_j, b_j\} = \max_{k\neq i}\min\{v_k, b_k\}$ and $v_j = \infty$. Suppose $r_i(u_{-i}) \leq \max_{k\neq i}\min\{v_k, b_k\}$.

By $\min\{v_j, b_j\} = \max_{k\neq i}\min\{v_k, b_k\} \geq r_i(u_{-i})$, $r_i^*(u_{-i}) = \min\{v_j, b_j\}$. Let $u_i \in \mathcal{U}$ be such that $u_i = u_j$. By $\min\{v_j, b_j\} = r_i^*(u_{-i})$, $v_j = \infty$, and $u_i = u_j$, we get $\min\{v_i, b_i\} = r_i^*(u_{-i})$ and $v_i = \infty$. Thus, a prioritized tie-breaking rule requires $x_i(u) = 1$. By $x_i(u) = 1$, $r_i^*(u_{-i}) = \min\{v_j, b_j\}$, and (V-ii), we get $f_i(u) = (1, \min\{v_j, b_j\})$. By $x_i(u) = 1$, $x_j(u) = 0$, and hence by (V-ii), $f_j(u) = (0, 0)$. By $v_j = \infty$, $u_j(1, \min\{v_j, b_j\}) > u_j(0, 0)$. Thus, by $f_i(u) = (1, \min\{v_j, b_j\})$ and $f_j(u) = (0, 0)$, $u_j(f_i(u)) > u_j(f_j(u))$. However, by $u_i = u_j$, this contradicts *weak envy-freeness for equals*.

(iv) $\Rightarrow$ (iii): Assume r is non-negligible. Let $u \in \mathcal{U}^n$ and $i, j \in N$. If $x_i(u) = x_j(u) = 0$, by $f_i(u) = f_j(u) = (0, 0)$, $u_i(f_i(u)) = u_i(f_j(u))$. If $x_i(u) = 1$ and $x_i(u) = 0$, then by $f_j(u) = (0, 0)$ and *individual rationality*, $u_i(f_i(u)) \geq u_i(f_j(u))$. Thus, assume $x_i(u) = 0$ and $x_j(u) = 1$. By $x_j(u) = 1$, $f_j(u) = (1, r_j^*(u_{-j}))$. By $i \neq j$, $\min\{v_i, b_i\} \leq r_j^*(u_{-j})$.

We consider two cases. First, assume $\min\{v_i, b_i\} < r_j^*(u_{-j})$. Then, by $f_i(u) = (0, 0)$ and $f_j(u) = (1, r_j^*(u_{-j}))$, $u_i(f_i(u)) > u_i(f_j(u))$. Next, assume $\min\{v_i, b_i\} = r_j^*(u_{-j})$. By $\min\{v_i, b_i\} = r_j^*(u_{-j})$, $\min\{v_i, b_i\} = \max_{k\neq j}\min\{v_k, b_k\} \geq r_j(u_{-j})$. Hence, by non-negligibility, $v_i \neq \infty$, and so $u_i(0, 0) = u_i(1, \min\{v_i, b_i\})$. Thus, by $f_i(u) = (0, 0)$, $f_j(u) = (1, r_j^*(u_{-j}))$, and $r_j^*(u_{-j}) = \min\{v_i, b_i\}$, we get $u_i(f_i(u)) = u_i(f_j(u))$. $\square$

Theorem 3. *Let $\mathcal{U} \subseteq \mathcal{U}^C$. If a rule f on $\mathcal{U}^n$ satisfies weak envy-freeness for equals, individual rationality, no subsidy for losers, and strategy-proofness, then it satisfies constrained efficiency.*

Proof. Assume f satisfies *weak envy-freeness for equals*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. Suppose f violates *constrained efficiency*. Then, by Lemma 1, there are $u \in \mathcal{U}^n$ and $i, j \in N$ such that (i) $x_i(u) = 1$ and $x_j(u) = 0$, (ii) $\min\{c_i(t_i(u)), b_i\} + \min\{v_j(t_j(u)), b_j\} \geq t_i(u) + t_j(u)$, and (iii) $c_i(t_i(u)) + v_j(t_j(u)) > t_i(u) + t_j(u)$.

First, we show $\min\{v_j, b_j\} \geq t_i(u)$ and $v_j > t_i(u)$. By *individual rationality* and *no subsidy for losers*, $t_j(u) = 0$. By *individual rationlity* and $b_i > 0$, $u_i(f_i(u)) \geq 0 > u_i(0, b_i)$, and so $u_i(0, c_i(t_i(u))) = u_i(f_i(u))$. By $u_i(0, c_i(t_i(u))) = u_i(f_i(u)) \geq 0$, $c_i(t_i(u)) \leq 0$. Hence, $0 \geq c_i(t_i(u)) = \min\{c_i(t_i(u)), b_i\}$. By $t_j(u) = 0$ and $0 \geq \min\{c_i(t_i(u)), b_i\}$, (ii) implies $\min\{v_j, b_j\} \geq t_i(u)$. By $t_j(u) = 0$ and $0 \geq c_i(t_i(u))$, (iii) implies $v_j > t_i(u)$.

Next, we derive a contradiction. By $\min\{v_j, b_j\} \geq t_i(u)$ and $v_j > t_i(u)$, $u_j(f_i(u)) > 0$. Let $u'_i = u_j$ and $u' = (u'_i, u_{-i})$. By $u'_i = u_j$, $u'_i(f_i(u)) > 0$. If $x_i(u') = 0$, then by *individual rationality* and *no subsidy*, $f_i(u') = (0, 0)$, and so $u'_i(f_i(u)) > 0 = u'_i(f_i(u'))$. However, this contradicts *strategy-proofness*. Hence, $x_i(u') = 1$. By $x_i(u) = x_i(u') = 1$ and *strategy-proofness*, $t_i(u') = t_i(u)$. By $x_i(u') = 1$, $x_j(u') = 0$, and so by *individual rationality* and *no subsidy for losers*, $f_j(u') = (0, 0)$. By $f_i(u') = f_i(u)$ and $f_j(u') = (0, 0)$, $u_j(f_i(u)) > 0$ implies $u_j(f_i(u')) > u_j(f_j(u'))$. However, by $u'_i = u_j$, this contradicts *weak envy-freeness for equals*. □

Proposition 6. *Let $\mathcal{U} \subseteq \mathcal{U}^C$ be rich. Let f on $\mathcal{U}^n$ be a truncated Vickrey rule with endogenous reserve prices $r \in \mathcal{R}^n$ that has a prioritized tie-breaking rule. Then, f satisfies anonymity in welfare if and only if r is upper anonymous and non-negligible.*

Proof. Given $i \in N$ and $u_{-i} \in \mathcal{U}^{n-1}$, let $r^*_i(u_{-i}) = \max\{\max_{j \neq i} \min\{v_j, b_j\}, r_i(u_{-i})\}$.

ONLY IF: Assume f satisfies *anonymity in welfare*. By Proposition 5, r is non-negligible. Thus, we show r is upper anonymous. Let $u, u' \in \mathcal{U}^n$ and $i, j \in N$ be such that $u_i = u'_j$, $u_j = u'_i$, and $u_{-\{i,j\}} = u'_{-\{i,j\}}$. Suppose $r^*_i(u_{-i}) \neq r^*_j(u'_{-j})$. Without loss of generality, assume $r^*_i(u_{-i}) > r^*_j(u'_{-j})$.

Let $\tilde{u}_i = \tilde{u}'_j \in \mathcal{U}$ be such that $r^*_i(u_{-i}) > \min\{\tilde{v}_i, \tilde{b}_i\} = \min\{\tilde{v}'_j, \tilde{b}'_j\} > r^*_j(u'_{-j})$.[16] Then, by (V-i) and (V-ii), $f_i(\tilde{u}_i, u_j, u_{-\{i,j\}}) = (0, 0)$ and $f_j(\tilde{u}'_j, u'_i, u_{-\{i,j\}}) = (1, r^*_j(u'_{-j}))$. By $\min\{\tilde{v}_i, \tilde{b}_i\} > r^*_j(u'_{-j})$, $\tilde{u}_i(1, r^*_j(u'_{-j})) > \tilde{u}_i(0, 0)$, and so $\tilde{u}_i(f_j(\tilde{u}'_j, u'_i, u_{-\{i,j\}})) > \tilde{u}_i(f_i(\tilde{u}_i, u_j, u_{-\{i,j\}}))$. However, by $\tilde{u}_i = \tilde{u}'_j$ and $u_j = u'_i$, this contradicts *anonymity in welfare*.

IF: Assume r is upper anonymous and non-negligible. Let $u, u' \in \mathcal{U}^n$ and $i, j \in N$ be such that $u_i = u'_j$, $u_j = u'_i$, and $u_{-\{i,j\}} = u'_{-\{i,j\}}$. By upper anonymity, $r^*_i(u_{-i}) = r^*_j(u'_{-j})$. If $x_i(u) = x_j(u')$, then by $r^*_i(u_{-i}) = r^*_j(u'_{-j})$, $f_i(u) = f_j(u')$, and so $u_i(f_i(u)) = u_i(f_j(u'))$. Assume $x_i(u) \neq x_j(u')$. Without loss of generality, assume $x_i(u) = 1$ and $x_j(u') = 0$. By $x_i(u) = 1$, $\min\{v_i, b_i\} \geq r^*_i(u_{-i})$, and by $x_j(u') = 0$, $\min\{v'_j, b'_j\} \leq r^*_j(u'_{-j})$. By $\min\{v_i, b_i\} = \min\{v'_j, b'_j\}$ and $r^*_i(u_{-i}) = r^*_j(u'_{-j})$, we get $\min\{v_i, b_i\} = r^*_i(u_{-i}) = r^*_j(u'_{-j}) = \min\{v'_j, b'_j\}$. By $x_i(u) = 1$ and $r^*_i(u_{-i}) = \min\{v_i, b_i\}$, $f_i(u) = (1, \min\{v_i, b_i\})$. By $x_j(u') = 0$, $f_j(u') = (0, 0)$. By $x_j(u') = 0$ and $\min\{v'_j, b'_j\} = r^*_j(u'_{-j})$, the prioritized tie-breaking rule requires $v'_j \neq \infty$. By $v_i = v'_j \neq \infty$, $u_i(1, \min\{v_i, b_i\}) = u_i(0, 0)$. Hence, by $f_i(u) = (1, \min\{v_i, b_i\})$ and $f_j(u') = (0, 0)$, we get $u_i(f_i(u)) = u_i(f_j(u'))$. □

[16] By density, $\tilde{u}_i$ exists.

References

[1] Adachi, T. (2014), Equity and the Vickrey allocation rule on general preference domains, *Social Choice and Welfare*, 42, 813–830.

[2] Aggarwal, G., S. Muthukrishnan, D. Pál, M. Pál (2009), General auction mechanism for search advertising, In: *the 18th International Conference on World Wide Web (WWW'09)*, Madrid, Spain, 241–250.

[3] Ashlagi, I., S. Serizawa (2012), Characterizing Vickrey allocation rule by anonymity, *Social Choice and Welfare*, 38, 53-542.

[4] Ausubel, L. (2004), An efficient dynamic auction for heterogenous commodities. *American Economic Review*, 96, 602–629.

[5] Baisa, B. (2017), Auction design without quasilinear preferences. *Theoretical Economics*, 12, 53–78.

[6] Baisa, B. (2020), Efficient multi-unit auctions for normal goods, *Theoretical Economics*, 15, 361–423.

[7] Basu, R., C Mukherjee (2023), Characterization of Vickrey auction with reserve price for multiple objects, *Review of Economic Design*, 27, 763–790.

[8] Basu, R., C Mukherjee (2024), Strategy-proof multidimensional mechanism design, *Mathematics of Operations Research*, forthcoming.

[9] Bulow, J., J. Levin, P. Milgrom (2009), Winning play in spectrum auctions. Technical report, *National Bureau of Economic Research*.

[10] Dobzinski, S., R. Lavi, N. Nisan (2012), Multi-unit auctions with budget limits. *Games and Economic Behavior*, 74, 486–503

[11] Dobzinski, S., R.P. Leme (2014), Efficiency guarantees in auctions with budgets. In: *the 41st International Colloquium Automata, Languages, and Programming (ICALP 2014)*, Copenhagen, Denmark, 392–404.

[12] Dütting, P., M. Henzinger, M. Starnberger (2015), Auctions with heterogeneous items and budget limits. *ACM Transactions on Economics and Computation*, 4, 4:1–4:17.

[13] Dütting, P., M. Henzinger, I. Weber (2015), An Expressive Mechanism for Auctions on the Web. *ACM Transactions on Economics and Computation*, 4, 1:1-1:34.

[14] Fiat, A., S. Leonardi, J. Saia, P. Sankowski (2011), Single valued combinatorial auctions with budgets. In: *the 12th ACM Conference on Electronic Commerce (EC'11)*, San Jose, California, 223–232.

[15] Holmström, B. (1979), Groves' scheme on restricted domains. *Econometrica*, 47, 1137–1144.

[16] Kazumura, T. and S. Serizawa (2016), Efficiency and strategy-proofness in object assignment problems with multi-demand preferences, *Social Choice and Welfare*, 47, 633–663.

[17] Kazumura, T., D. Mishra, S. Serizawa (2017), Mechanism design without quasilinearity. *ISER Discussion Paper*, No. 1005, Osaka University.

[18] Lavi, R., M. May (2012), A note on the incompatibility of strategy-proofness and Pareto-optimality in quasi-linear settings with public budgets. *Economic Letters*, 115, 100–103.

[19] Le, P. (2017), Mechanisms for combinatorial auctions with budget constraints. *Review of Economic Design*, 21, 1–31.

[20] Le, P. (2018), Pareto optimal budgeted combinatorial auctions. *Theoretical Economics*, 13, 831–868.

[21] Malik, M. and D. Mishra (2021), Pareto efficient combinatorial auctions: Dichotomous preferences without quasilinearity, *Journal of Economic Theory*, 191, 105128.

[22] Mukherjee, C. (2014), Fair and group strategy-proof good allocation with money, *Social Choice and Welfare*, 42, 289–311

[23] Mackenzie A., Y. Zhou (2022), Tract housing, the core, and pendulum auctions. *Graduate school of Economics Discussion Paper Series*, No. E-22-005, Kyoto University.

[24] Morimoto, S. and S. Serizawa (2015), Strategy-proofness and efficiency with non-quasi-linear preferences: A characterization of minimum price Walrasian rule, *Theoretical Economics*, 10, 445–487.

[25] Nisan, N. (2007), Introduction to mechanism design (for computer scientists). In: N. Nisan, T. Rougarden, E. Tardos, V. Vazirani (Eds.), *Algorithmic Game Theory*, Cambridge University Press, New York, USA, 209–241.

[26] Sakai, T. (2008), Second price auctions on general preference domains: two characterizations. *Economic Theory*, 37, 347–356.

[27] Sakai, T. (2013a), An equity characterization of second price auctions when preferences may not be quasilinear, *Review of Economic Design*, 17, 17–26.

[28] Sakai, T. (2013b), Axiomatizations of second price auctions with a reserve price, *International Journal of Economic Theory*, 9, 255-265.

[29] Saitoh, H., S. Serizawa (2008), Vickrey allocation rule with income effect. *Economic Theory*, 35, 391–401.

[30] Shinozaki, H., T. Kazumura, S. Serizawa (2022), Multi-unit object allocation problems with money for (non)decreasing incremental valuations: Impossibility and characterization theorems, *ISER Discussion Paper*, No. 1097.

[31] Shinozaki, H. (2023), Non-obvious manipulability and efficiency in package assignment problems with money for agents with income effects and hard budget constraints. *HIAS Discussion Paper Series*, No. HIAS-E-136, Hitotsubashi University.

[32] Shinozaki, H. (2024), Characterizing group strategy-proof rules in the object allocation problem with money. *Social Choice and Welfare*, forthcoming.

[33] Sprumont, Y. (2013), Constrained-optimal strategy-proof assignment: Beyond the Groves mechanisms, *Journal of Economic Theory*, 148, 1102–1121.

[34] Ting, H.F., X. Xiang (2012), Multi-unit auctions with budgets and non-uniform valuations. In: *the 23rd International Symposium on Algorithms and Computation (ISAAC 2012)*, Taipei, Taiwan, 669–678.

[35] Vickrey, W. (1961), Counterspeculation, auctions, and competitive sealed tenders, *Journal of Finance*, 16, 8–37.

[36] Wakabayashi, Y., R. Sakai, S. Serizawa (2025), A general characterization of the minimum price Walrasian rule with reserve price, *Games and Economic Behavior*, 150, 1–26.

[37] Yi, J. (2024), A note on the impossibility of multi-unit auctions with budget-constrained bidders. *Review of Economic Design*, 28, 331–345.

[38] Zhou, Y. and S. Serizawa (2018), Strategy-proofness and efficiency for non-quasi-linear and common-tiered-object preferences: Characterization of minimum price rule, *Game and Economic Behavior*, 109, 327–363.