Collective State Spaces

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Introduction

- We consider Dekel, Lipman, and Rustichini's (2001) (DLR) preferences:
  - For a set X, which is referred to as a *menu*,

$$U(X) = \sum_{s \in S} \pi(s) \max_{x \in X} u(x, s),$$

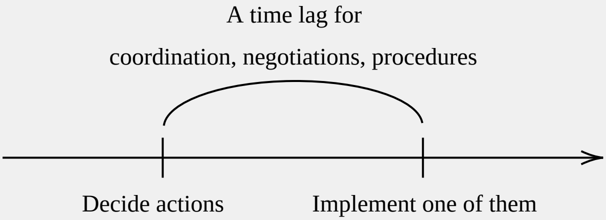
  where
    - S: a subjective state space,
    - π: a probability measure over S,
    - $u : A \times S \to \mathbb{R}$: a state dependent utility function.

- We **aggregate individual DLR preferences into social DLR preferences**.
  - No paper has tackled this aggregation problem yet.

Motivating Story

- Consider a meeting in a large company, which is held by
  - CEO (= society),
  - division heads (= individuals).
    - E.g., automobiles, social networking services (SNS), artificial intelligence (AI).
- They decide on the next action,
  - e.g., determining which another company to acquire.

Why Menu Preferences?



- The effectiveness of an action depends on the circumstances during its implementation.
  - But, the CEO and the devision heads do not know them when deciding an action.

⇒ They need multiple actions as candidates for the best option at the decision stage.

  - Multiple actions = a menu.

Why Subjective States? and Why Aggregation?

- Each division head only represents their division.
    - Automobile head: {gasoline engines, hydrogen engines}.
    - AI head: {Google, Apple}.

$\Rightarrow$ They hold different preferences over menus of actions.

- How should the CEO aggregate these preferences?
    - Especially, how should the CEO construct a comprehensive state space?

$$\begin{matrix} \{\text{gasoline engines, hydrogen engines}\} \\ \{\text{Google, Apple}\} \end{matrix} \implies ?$$

Model

- A: a finite set.
  - We refer to $a \in A$ as an *outcome*.
- $\Delta(A)$: the set of probability distributions over A.
  - We refer to $l = (l(a))_{a \in A} \in \Delta(A)$ as a *lottery* (or an option).
- $\mathcal{K}(\Delta(A))$: the set of nonempty and compact subsets in $\Delta(A)$, which is endowed with the Hausdorff topology.
  - We refer to $X \in \mathcal{K}(\Delta(A))$ as a *menu*.

Individual and Social Preferences

- $N = \{1, \ldots, n\}$: a set of individuals.
- Index 0 represents society.
- $\succsim_i$: a complete and transitive binary relation on the set of menus, $\mathcal{K}(\Delta(A))$.
  - $X \succsim_i Y$: Individual i evaluates that X is at least as good as Y.

Aggregation Problem

- $(\succsim_i)_{i\in N}$ and $\succsim_0$ admit the DLR representation:

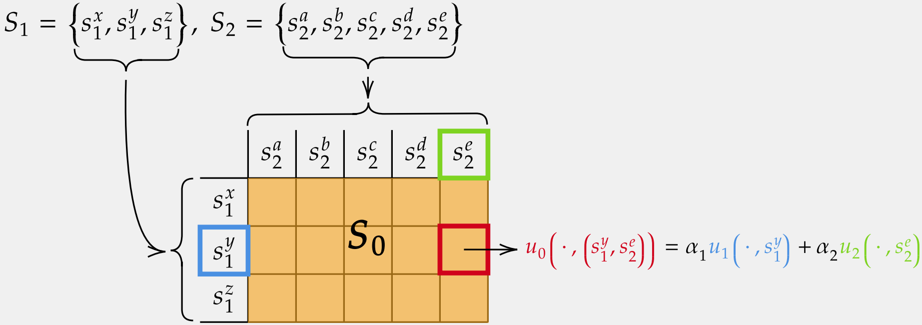
$$U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i).$$

  - S_i: a finite set.
  - π_i: a full support probability measure over S_i.
  - $u_i : \Delta(A) \times S_i \to \mathbb{R}$: a state dependent utility function.
    - Each $u_i(\cdot\,, s_i)$ is mixture-linear.

Question:

- How should society aggregate $(S_i, \pi_i, u_i)_{i\in N}$ into (S_0, π_0, u_0)?

Representation: Rough Preview



- The following axioms characterize this representation:
  1. **two restricted Pareto conditions**,
  2. **a violation of Pareto indifference**,
  3. a tentative technical axiom. (We are still working on this axiom.)

Outline of the Remaining Part

1. Preliminary clarifications on DLR preferences
2. A benchmark Pareto indifference
    2.1 An impossibility theorem
    2.2 Discussion
3. Our axioms
    3.1 Two axioms from the above discussion
    3.2 Two further axioms
4. Representation theorem
5. Proof

Features of DLR Preferences

DLR Representation: $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$.

- For all $X \supset Y$, $X \succsim_i Y$ must hold.

- $X \cup \{l\} \succ_i X$: "Individual i has a possibility to need option l."

$\iff$ There exists $s_i \in S_i$ such that $u_i(l, s_i) > u_i(l', s_i)$ for all $l' \in X$.

  - We do not know whether $u_i(l, s_i') \gtreqless u_i(l', s_i')$ under other $s_i' \in S_i$.

- $X \cup \{l\} \sim_i X$: "Individual i will never need option l."

$\iff$ For each $s_i \in S_i$, there exists $l_{s_i} \in X$ such that $u_i(l_{s_i}, s_i) \geq u_i(l, s_i)$.

Benchmark Pareto Indifference

Expanding Pareto Indifference

For all menus $X \in \mathcal{K}(\Delta(A))$ and all lotteries $l \in \Delta(A)$,

$$X \cup \{l\} \sim_i X \text{ for all } i \in N \Longrightarrow X \cup \{l\} \sim_0 X.$$

Interpretation:

- If no one needs option l, then neither does society.

Benchmark Theorem: Ex-post Dictatorship

DLR Representation: $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$.

Theorem

The DLR preference profile, $(\succsim_i)_{i \in N}$ and $\succsim_0$, satisfies Expanding Pareto Indifference if and only if for each $s_0 \in S_0$, there exist $i \in N$ and $s_i \in S_i$ such that $u_0(\cdot, s_0) = u_i(\cdot, s_i)$.

Interpretation:

- It says $S_0 \subset S_1 \cup \cdots \cup S_n$.
  - $\rightarrow$ Society plans to focus exclusively on one aspect.

Discussions about Expanding Pareto Indifference

Example:

- $N = \{1, 2\}$, $S_1 = \{s_1\}$, and $S_2 = \{s_2\}$.

  $\Rightarrow \; U_i(X) = \max_{l \in X} u_i(l, s_i)$.

- $u_1(l, s_1) > u_1(l'', s_1) >> u_1(l', s_1)$ and $u_2(l', s_2) > u_2(l'', s_2) >> u_2(l, s_2)$.

$$\Downarrow$$

- $\{l, l', l''\} \sim_i \{l, l'\}$ for $i = 1, 2$.
- However, $\{l, l', l''\} \succ_0 \{l, l'\}$ seems desirable.

  $\because$ Option l'' is highly regarded by everyone.

Lesson:

- If an ex-post disagreement will occur, society may need a compromise option.

Axiom 1: Weaker Pareto Indifference

Idea: If an option is surely Pareto dominated ex-post, society does not need it.

Pareto Indifference for Dominated Options

For all menus $X \in \mathcal{K}(\Delta(A))$ and all lotteries $\hat{l} \in \Delta(A)$, if

(1) $X \cup \{\hat{l}\} \sim_i X$ for some $i \in N$ and

(2) $\{\hat{l}, l\} \sim_j \{l\}$ for all $l \in X$ and all other individuals $j \in N \setminus \{i\}$,

then $X \cup \{\hat{l}\} \sim_0 X$.

- Under DLR preferences: $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$,
  - (1) $\iff$ In every $s_i \in S_i$, $\hat{l}$ is not the best among $X \cup \{\hat{l}\}$.
  - (2) $\iff$ In every $s_j \in S_j$, $\hat{l}$ is the worst among $X \cup \{\hat{l}\}$.

Axiom 2: Violation of Pareto Indifference

Idea: $\{l, l', l''\} \succ_0 \{l, l'\}$ if an ex-post disagreement between l and l' is sufficiently large.

Expansion toward Moderate Options

For all lotteries $\hat{l}, l_1, \ldots, l_n \in \Delta(A)$, if for each individual $i \in N$,

- $\{\hat{l}, l_j\} \sim_i \{\hat{l}\} \sim_i \{l_j\}$ for all $j \neq i$ and
- $\{\hat{l}, l_i\} \succ_i \{l_i\}$,

there exists $l^* := \sum_{i=1}^n \lambda_i l_i + (1 - \sum_{i=1}^n \lambda_i)\, \hat{l}$ ($(\lambda_i)_i \in (0,1)^n$ with $\sum_{i=1}^n \lambda_i < 1$) such that

$$\{l^*, l_1, \ldots, l_n\} \succ_0 \{l_1, \ldots, l_n\}.$$

$n = 2$ Case

Expansion toward Moderate Options (when $n = 2$)

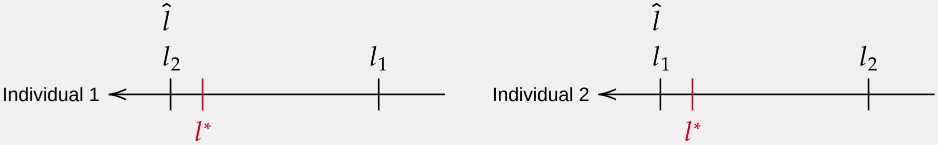
For all lotteries $\hat{l}, l_1, l_2 \in \Delta(A)$, if

- $\{\hat{l}, l_2\} \sim_1 \{\hat{l}\} \sim_1 \{l_2\}$ and $\{\hat{l}, l_1\} \succ_1 \{l_1\}$,
- $\{\hat{l}, l_1\} \sim_2 \{\hat{l}\} \sim_2 \{l_1\}$ and $\{\hat{l}, l_2\} \succ_2 \{l_2\}$,

there exists $l^* := \lambda_1 l_1 + \lambda_2 l_2 + (1 - \lambda_1 - \lambda_2)\,\hat{l}$ ($\lambda_1, \lambda_2 \in (0,1)$ with $\lambda_1 + \lambda_2 < 1$) such that

$$\{l^*, l_1, l_2\} \succ_0 \{l_1, l_2\}.$$

Interpretation: when $S_1 = \{s_1\}$, $S_2 = \{s_2\}$, and λ_1 and λ_2 are sufficiently small,



Axiom 3: Commitment Pareto

Commitment Pareto

For all lotteries $l, l' \in \Delta(A)$, if $\{l\} \succsim_i \{l'\}$ for all $i \in N$, then $\{l\} \succsim_0 \{l'\}$.

Axiom 4

Limitation for Flexibility

For each $X \in \mathcal{K}(\Delta(A))$, there exist $l_1, \ldots, l_m \in \Delta(A)$ with $m \leq |S_1 \times \cdots \times S_n|$ such that $X \sim_0 \{l_1, \cdots, l_m\}$.

- This axiom is **a tentative one to obtain a clear representation**.
  - It has no normative meaning.
- Imposing this axiom is equivalent to assuming

$$|S_0| \leq |S_1 \times \cdots \times S_n|.$$

Representation Theorem

DLR Representation: $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$.

Theorem

The DLR preference profile, $(\succsim_i)_{i \in N}$ and $\succsim_0$, satisfies the four axioms if and only if

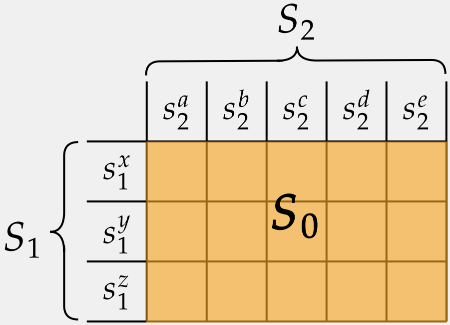
1. $S_0 = S_1 \times \cdots \times S_n$;
2. *for each* $s_0 = (s_i)_i \in S_0$, *there exists* $(\alpha_{i,s_0})_i \in (0,1)^n$ *such that*
$$u_0(\cdot\ , (s_i)_i) = \sum_{i \in N} \alpha_{i,s_0} u_i(\cdot\ , s_i);$$
3. *for each* $i \in N$ *and each* $s_i^* \in S_i$,
$$\sum_{s_0 = (s_j)_j \in S_0 : s_i = s_i^*} \alpha_{i,s_0} \pi_0(s_0) = \pi_i(s_i^*).$$

DLR Representation: $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$.

Theorem

The DLR preference profile, $(\succsim_i)_{i \in N}$ *and* $\succsim_0$*, satisfies the four axioms if and only if*

1. $S_0 = S_1 \times \cdots \times S_n$;

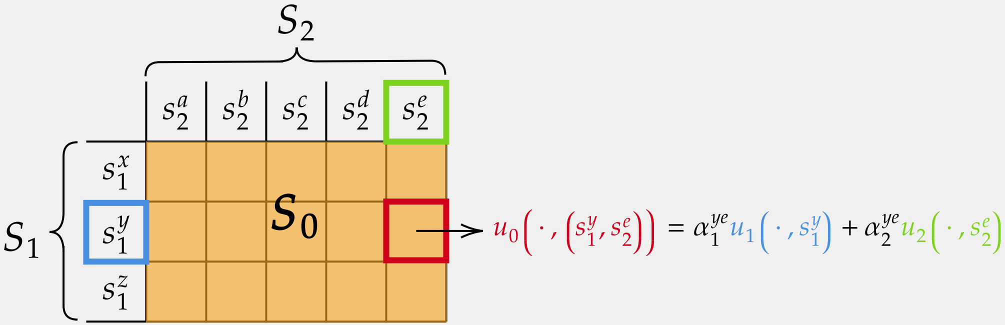


DLR Representation: $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$.

Theorem

The DLR preference profile, $(\succsim_i)_{i \in N}$ and $\succsim_0$, satisfies the four axioms if and only if

2. *for each $s_0 = (s_i)_i \in S_0$, $u_0(\cdot, (s_i)_i) = \sum_{i \in N} \alpha_{i,s_0} u_i(\cdot, s_i)$:*

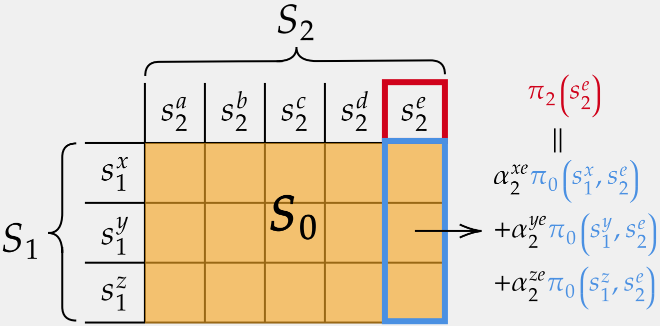


DLR Representation: $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$.

Theorem

The DLR preference profile, $(\succsim_i)_{i \in N}$ and $\succsim_0$, satisfies the four axioms if and only if

3. *for each $i \in N$ and each $s_i^* \in S_i$, $\sum_{s_0 = (s_j)_j \in S_0 : s_i = s_i^*} \alpha_{i,s_0} \pi_0(s_0) = \pi_i(s_i^*)$.*



Proof Intuition (1/3)

Pareto Indifference for Dominated Options

(1) $X \cup \{\hat{l}\} \sim_i X$ for some $i \in N$ and

(2) $\{\hat{l}, l\} \sim_j \{l\}$ for all $l \in X$ and all other individuals $j \in N \setminus \{i\}$,

$\Rightarrow X \cup \{\hat{l}\} \sim_0 X$.

≒ a Pareto principle for tastes over lotteries

$\Rightarrow$ For each $s_0 \in S_0$, there exists some $(s_i)_{i \in N}$ such that

$$u_0(\cdot\ , s_0) = \sum_{i \in N} \alpha_{i,s_0} u_i(\cdot\ , s_i)\,.$$

Proof Intuition (2/3)

Expansion toward Moderate Options (when $n = 2$)

1. $\{\hat{l}, l_2\} \sim_1 \{\hat{l}\} \sim_1 \{l_2\}$ and $\{\hat{l}, l_1\} \succ_1 \{l_1\}$,
2. $\{\hat{l}, l_1\} \sim_2 \{\hat{l}\} \sim_2 \{l_1\}$ and $\{\hat{l}, l_2\} \succ_2 \{l_2\}$,

$\Rightarrow\ ^{\exists} l^* := \lambda_1 l_1 + \lambda_2 l_2 + (1 - \lambda_1 - \lambda_2)\,\hat{l}$ such that $\{l^*, l_1, l_2\} \succ_0 \{l_1, l_2\}$.

- "Any $(u_i(\cdot\ , s_i))_{i \in N}$ has a disagreement $\Longrightarrow$ society needs a compromise lottery."

$\Rightarrow$ Society considers all of the combinations $S_1 \times \cdots \times S_n$.

$\Rightarrow S_0 \supset S_1 \times \cdots \times S_n$.

- **Limitation for Flexibility**: $|S_0| \leq |S_1 \times \cdots \times S_n|$.

$\Rightarrow S_0 = S_1 \times \cdots \times S_n$.

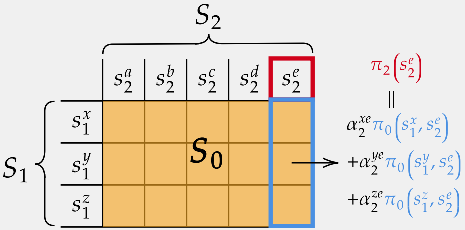
Proof Intuition (3/3)

Commitment Pareto: $\{l\} \succsim_i \{l'\}$ for all $i \in N \implies \{l\} \succsim_0 \{l'\}$.

$\Rightarrow$ In the evaluation, society has to maintain the ratio

$$\pi_i\left(s_i\right) u_i\left(\cdot\ , s_i\right) / \pi_i\left(s_i'\right) u_i\left(\cdot\ , s_i'\right).$$

$\Rightarrow$ $\sum_{s_0=(s_j)_j \in S_0 : s_i = s_i^*} \alpha_{i,s_0} \pi_0\left(s_0\right) = \pi_i\left(s_i^*\right)$ for each $s_i^* \in S_i$.



Proof of the Core Part

Two Core Axioms

- We only see the implications of the first two axioms:

Pareto Indifference for Dominated Options

(1) $X \cup \{\hat{l}\} \sim_i X$ for some $i \in N$ and

(2) $\{\hat{l}, l\} \sim_j \{l\}$ for all $l \in X$ and all other individuals $j \in N \setminus \{i\}$,

$\Rightarrow X \cup \{\hat{l}\} \sim_0 X$.

Expansion toward Moderate Options (when $n = 2$)

1. $\{\hat{l}, l_2\} \sim_1 \{\hat{l}\} \sim_1 \{l_2\}$ and $\{\hat{l}, l_1\} \succ_1 \{l_1\}$,
2. $\{\hat{l}, l_1\} \sim_2 \{\hat{l}\} \sim_2 \{l_1\}$ and $\{\hat{l}, l_2\} \succ_2 \{l_2\}$,

$\Rightarrow {}^{\exists} l^* := \lambda_1 l_1 + \lambda_2 l_2 + (1 - \lambda_1 - \lambda_2)\,\hat{l}$ such that $\{l^*, l_1, l_2\} \succ_0 \{l_1, l_2\}$.

Technical Assumption for the Proof

Richness Condition

For each $i \in N$ and each $s_i \in S_i$, there exist lotteries $l_{s_i}, l'_{s_i} \in \Delta(A)$ such that

- $u_i(l_{s_i}, s_i) > u_i(l'_{s_i}, s_i)$,
- $u_i(l_{s_i}, t_i) = u_i(l'_{s_i}, t_i)$ for all $t_i \neq s_i$, and
- $u_j(l_{s_i}, s_j) = u_j(l'_{s_i}, s_j)$ for all $j \neq i$ and all $s_j \in S_j$.

- In the paper, we adopt a weaker richness condition.
  - But here, we impose the above condition to simplify the proof.

Lemma (1/2): Ex-Post Utilitarianism

DLR Representation: $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$.

Lemma

If the DLR preference profile, $(\succsim_i)_{i \in I}$ and $\succsim_0$, satisfies Pareto Indifference for Dominated Options, then for each $s_0 \in S_0$, there exist $(s_i)_i \in S_1 \times \cdots \times S_n$ and $(\alpha_i)_i \in [0,1]^n$ with $\sum_{i \in N} \alpha_i = 1$ such that

$$u_0(\cdot\ , s_0) = \sum_{i \in N} \alpha_i u_i(\cdot\ , s_i).$$

Remarks:

- Under some $s_0 \in S_0$, society may assign zero weight to some individuals.
- For some profile $(s_i)_i \in S_1 \times \cdots \times S_n$, there may be no corresponding s_0.

Proof

DLR Representation: $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$.

- When $X = \{l\}$ in Pareto Indifference for Dominated Options,
  - $\{l, \hat{l}\} \sim_i \{l\}$ for all $i \in N \Longrightarrow \{l, \hat{l}\} \sim_0 \{l\}$.

  $\Leftrightarrow$ $u_i(l, s_i) \geq u_i(\hat{l}, s_i)$ for all $s_i \in S_i$ and all $i \in N \Longrightarrow u_0(l, s_0) \geq u_0(l', s_0)$ for all $s_0 \in S_0$.

$\Rightarrow$ For each $s_0 \in S_0$, by applying Harsanyi's Theorem,

$$u_0(\cdot\, , s_0) = \sum_{i \in N} \sum_{s_i \in S_i} \alpha_{s_i} u_i(\cdot\, , s_i)\,.$$

Proof (Continued)

- Suppose that for some $s_0 \in S_0$,

$$u_0(\cdot\ , s_0) = \underbrace{\alpha_{s_i}}_{>0} u_i(\cdot\ , s_i) + \underbrace{\alpha_{s_i'}}_{>0} u_i(\cdot\ , s_i') + \sum_{j \neq i} \sum_{s_j \in S_j} \alpha_{s_j} u_j(\cdot\ , s_j).$$

- Take $l, l', l'' \in \Delta(A)$ so that
  - $u_i(l, s_i) = u_i(l'', s_i) > u_i(l', s_i)$,
  - $u_i(l', s_i') = u_i(l'', s_i') > u_i(l, s_i')$,
  - $u_j(l'', s_j) = u_j(l, s_j) = u_j(l', s_j)$ for all $s_j \in \cup_{j \in N} S_j \backslash \{s_i, s_i'\}$.

1. Pareto Indifference for Dominated Options $\Longrightarrow \{l, l', l''\} \sim_0 \{l, l'\}$.
2. But, $u_0(l'', s_0) > u_0(l, s_0)$ and $u_0(l'', s_0)$. $\Longrightarrow \{l, l', l''\} \succ_0 \{l, l'\}$: a contradiction.

Lemma (2/2): Responsiveness to Every Profile of Individual States

DLR Representation: $U_i(X) = \sum_{s_i \in S_i} \pi_i(s_i) \max_{l \in X} u_i(l, s_i)$.

Lemma

Suppose that for each $s_0 \in S_0$, there exist $(s_i)_i \in S_1 \times \cdots \times S_n$ and $(\alpha_i)_i \in [0,1]^n$ with $\sum_{i \in N} \alpha_i = 1$ such that

$$u_0(\cdot\ , s_0) = \sum_{i \in N} \alpha_i u_i(\cdot\ , s_i)\,. \tag{1}$$

Then, if the DLR preference profile, $(\succsim_i)_{i \in I}$ and $\succsim_0$, satisfies Expansion toward Moderate Options, for each profile $(s_i)_i \in S_1 \times \cdots \times S_n$, there exists $s_0 \in S_0$ such that equation (1) holds where $\alpha_i > 0$ for all $i \in N$.

Remarks:

- Still, for some $(s_i)_i \in S_1 \times \cdots \times S_n$, there may exist multiple corresponding social states.

Proof

- Take any $s_1 \in S_1$, $s_2 \in S_2$ and $\hat{l}, l_1, l_2 \in \Delta(A)$ so that
  - $u_1(\hat{l}, s_1) = u_1(l_2, s_1) > u_1(l_1, s_1)$,
  - $u_2(\hat{l}, s_2) = u_2(l_1, s_2) > u_2(l_2, s_2)$,
  - $u_i(\hat{l}, s_i) = u_i(l_1, s_i) = u_i(l_i, s_i)$ for all $s_i \in (S_1 \cup S_2) \setminus \{s_1, s_2\}$.

1. Expansion toward Moderate Options $\Longrightarrow \{l^*, l_1, l_2\} \succ_0 \{l_1, l_2\}$.
2. $\nexists s_0 \in S_0$ such that $u_0(\cdot\,, s_0) = \alpha_1 u_1(\cdot\,, s_1) + \alpha_2 u_2(\cdot\,, s_2)$.

   $\Rightarrow$ $\nexists s_0 \in S_0$ such that $u_0(l^*, s_0) > u_0(l_1, s_0)$ and $u_0(l_2, s_0)$.

   $\Rightarrow$ $\{l^*, l_1, l_2\} \sim_0 \{l_1, l_2\}$: a contradiction.

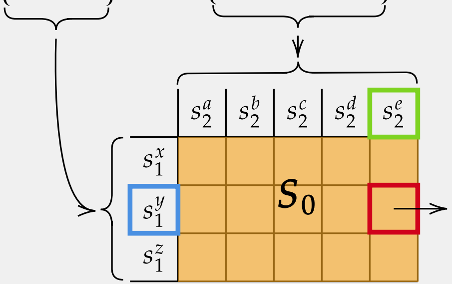
Summary

Question:

- How should society aggregate preferences over menus of options?
  - Especially, how should society construct a comprehensive state space?

Answer:

$$S_1 = \{s_1^x, s_1^y, s_1^z\}, \; S_2 = \{s_2^a, s_2^b, s_2^c, s_2^d, s_2^e\}$$



$$u_0\left(\cdot, (s_1^y, s_2^e)\right) = \alpha_1 u_1\left(\cdot, s_1^y\right) + \alpha_2 u_2\left(\cdot, s_2^e\right)$$

Reference I

Dekel, Eddie, Barton L. Lipman, and Aldo Rustichini, "Representing Preferences with a Unique Subjective State Space," *Econometrica*, 2001, *69* (4), 891–934.