Certainty Equivalent and Uncertainty Premium of Time-to-Build*

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Abstract

Time-to-build of an investment project induces a discrepancy between the timing of investment and that of revenue generation. Jeon (2024*b*) showed that uncertainty in the time-to-build *always* accelerates investment and enhances pre-investment firm value, regardless of its distribution. In this study, we examine *how much* the uncertainty advances the timing of investment and improves the firm value. Specifically, we show that there *always* exists a unique certainty equivalent of uncertain time-to-build and derive it in an analytic form. This enables us to derive the investment strategy with uncertain time-to-build in the form of the one that would have been adopted in the absence of such uncertainty. Even without full knowledge of the uncertainty, the firm can approximate the optimal investment strategy using only the mean and variance of time-to-build. We also clarify the positive impact of entropic risk measure of time-to-build on investment and derive the dual representation of the certainty equivalent of time-to-build based on relative entropy. Furthermore, we show that there *always* exists an uncertainty equivalent of fixed time-to-build. This implies that the firm can deduce the equivalent risk that its investment strategy, established without considering uncertainty in time-to-build, implicitly assumes. Lastly, we illustrate the practical application of our findings using some representative probability distributions and analyze the effects of the variance of time-to-build. In particular, we contrast the effects of uncertainty in demand with those of uncertainty in time-to-build, deriving the level of variance in time-to-build that offsets the negative impact of increased demand volatility on investment.

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1 Introduction

In 2015, Elon Musk made a bold promise that Tesla's vehicles would drive themselves in two years. In 2019, he made another promise, claiming that there would be a million robotaxis on the road in a year. After nearly a decade, neither have we seen fully self-driving technology from Tesla, nor do we see any of their robotaxis on the road.[1] At an event in October 2024, which had been delayed multiple times, they revealed a few prototypes of their robotaxis. Elon claimed that they would be available before 2027, which seems very unlikely to be fulfilled, considering his notorious record of unmet promises regarding timelines, as well as the complex regulatory requirements the technology must meet.

Elon and his company are not the only ones. Olkiluoto 3 in Finland is one of the largest nuclear reactors in Europe. Its construction began in 2005 with an estimated completion date in 2009 but it was finalized in 2022, resulting in a 13-year delay. Flamanville 3 in France is another example of a significant delay in the construction of a power plant. It started in 2007 with a goal of completion by 2012, but it still has not been finished yet (White (2024)). As can be seen from these examples, time-to-build is prevalent in real-world investment projects, and uncertainty is one of its inherent attributes.[2] Time-to-build has a significant impact on firm value because it introduces a discrepancy between the timing of investment and that of revenue generation. When its duration is uncertain, the firm's investment strategy must be established even more meticulously. Nevertheless, the effects of uncertainty in time-to-build on corporate investment have not been explored sufficiently.

To the best of our knowledge, Nishihara (2018) is the first study that shed light on the effects of uncertainty in time-to-build on investment. The paper analyzed a firm's research and development (R&D) investment decision, assuming its duration to follow a uniform distribution, and numerically showed that the uncertainty in the duration leads to earlier investment compared to the case with a fixed counterpart. Jeon (2024*b*) investigated the effects of uncertainty in time-to-build without any assumption on its distribution and analytically showed that uncertainty in time-to-build *always* accelerates investment and enhances pre-investment firm value.[3] Jeon (2024*a*) extended this framework by incorporating the firm's investment size decision in addition to the timing decision, and found that the positive impacts of uncertainty in time-to-build on investment remain intact. However, these studies did not demonstrate *how much* the uncertainty advances the timing of investment and improves firm value. In this study, we tackle this unanswered yet significant problem by clarifying the impacts of uncertainty in time-to-build on firm value in more detail.

First, we show that there always exists a unique *certainty equivalent* of uncertain time-to-build, regardless of its distribution. That is, there is a fixed time-to-build whose duration is shorter than the uncertain counterpart but yields the same firm value. Furthermore, we derive the certainty equivalent in

[1] Although Tesla has provided a driver-assistance system called Full Self-Driving (FSD), the name is misleading, as it remains at Level 2 automation, with Level 5 being fully autonomous driving based on the standards of the Society of Automotive Engineers (SAE) International.

[2] Examples are abundant. Recent ones include sluggish capacity expansion in the semiconductor industry, where demand surged during the COVID-19 pandemic, and significant production delays among new automakers following the rise of the electric vehicle market.

[3] Jeon (2024*b*) considered not only time-to-build but also regulation as internal and external factors that hinder immediate revenue generation after the investment, respectively. This study excludes the latter to simplify the model and its solution.

an analytic form. This enables us to derive the optimal investment strategy with uncertain time-to-build in the form of the investment strategy that would have been adopted in the absence of such uncertainty. Even without the full knowledge of the uncertainty in time-to-build (i.e., probability distribution), the certainty equivalent can be approximated using only a few moments, such as mean and variance, which greatly enhances the practicality. Moreover, we derive the tight upper and lower bounds of the certainty equivalent for given mean and variance of time-to-build. We also show that the degree of investment acceleration by the uncertainty in time-to-build decreases with the expected growth rate of demand and is independent of demand volatility.

Second, we characterize the entropic risk measure of time-to-build and show that it always accelerate investment and improves pre-investment firm value. It is defined as the maximum size of fixed time-to-build that can be added while ensuring that the firm value is above a certain level, and it is shown to be positively associated with the firm's optimal investment timing. We also derive the dual representation of the certainty equivalent of time-to-build based on relative entropy. Specifically, it is the sum of the expected duration of time-to-build under equivalent measure and the relative distance between the measures.

Third, we show that there always exists an *uncertainty equivalent* of fixed time-to-build. That is, there is an uncertain time-to-build whose expected duration is longer than the fixed counterpart but induces the identical firm value. Unlike the certainty equivalent, there can be many uncertainty equivalents for a given fixed time-to-build. The uncertainty equivalent enables the firm to deduce the equivalent risk that its investment strategy, established without considering uncertainty in time-to-build, implicitly assumes. We also show that for a given fixed time-to-build, there always exists an uncertain counterpart whose expected duration is longer yet yields higher firm value, which verifies the positive impacts of uncertainty in time-to-build.

Lastly, we apply the above arguments to representative probability distributions to demonstrate their practicality. We observe that the mean and variance of time-to-build are sufficient to approximate its certainty equivalent in many cases, unless the underlying distribution has a peak at either end of its support, such as an exponential distribution. Furthermore, we contrast the effects of uncertainty in demand with those of uncertainty in time-to-build. The former delays investment because it increases the value of the option to wait, whereas the latter accelerates investment because it increases the expected profits from the investment by the convexity of the discount factor with respect to the revenue generation timing. With these arguments, we derive the level of variance in time-to-build that offsets the negative impacts of increased demand volatility on investment.

The remainder of this paper is organized as follows. Section 2 reviews the literature on uncertainty-investment relationship and time-to-build. Section 3 introduces the setup of the model and Section 4 derives its solution. Specifically, Section 4.1 demonstrates the preliminary results based on a standard real options model and Section 4.2 contrasts the effects of uncertainty in time-to-build with those of uncertainty in demand. Section 4.3 derives the certainty equivalent of uncertain time-to-build and analyzes its sensitivity. Section 4.4 discusses the certainty equivalent from the perspective of entropic risk measure and Section 4.5 derives the uncertainty equivalent of fixed time-to-build. Section 5 applies the arguments discussed in Section 4 to representative probability distributions. Specifically, Sections 5.1 to 5.6 correspond to a uniform distribution, a triangular distribution, a log-normal distribution, an exponential distribution, a gamma distribution, and a scaled beta distribution, respectively. Section 5.7 focuses on the mean and variance of time-to-build and contrasts the effects of uncertainty in time-to-build and those of uncertainty in demand. Section 6 summarizes the main results and suggests possible future works. Appendix A presents all proofs and Appendix B summarizes the characteristics of the probability distributions discussed in Section 5.

2 Literature review

Majd and Pindyck (1987) was one of the first studies to examine the impact of time-to-build on corporate investment. They assumed a maximum rate at which a firm can invest and showed that such friction leads to a delay in investment. Bar-Ilan and Strange (1996*a*,*b*) supposed that a certain period of time must elapse after the investment to generate revenue, showing that uncertainty in demand can hasten investment in the presence of lags. They presumed a fixed time-to-build, and the one that accelerates investment is uncertainty in demand, not that in time-to-build. Furthermore, they assumed the firm's option to abandon the ongoing project, which truncates the downside risk of the project and yields a greater incentive to invest aggressively. Bar-Ilan and Strange (1998) extended their previous work to a two-stage investment project and found that the investment can be sequential in the presence of the firm's option to suspend the ongoing project. Pacheco-de-Almeida and Zemsky (2003) also studied a multi-stage investment in the presence of time-to-build and duopoly. They found that the investment behavior of firms can be either incremental or lumpy depending on the duration of time-to-build. These studies only considered a fixed time-to-build, leaving the effects of the uncertain counterpart unanswered.

Some studies adopted uncertain time-to-build in the discussion of corporate investment decision. Weeds (2002) investigated R&D competition in a duopoly market, assuming random discovery time for new technologies, and the uncertain lags are found to make negative impacts on the investment decision. Alvarez and Keppo (2002) examined a firm's irreversible investment with delivery lags in a generalized setup in which they are interdependent. Specifically, they assumed the lags increase with the level of demand shock and showed that the investment might be suboptimal depending on the level of demand shock, primarily because higher demands imply longer delivery lags. Jeon (2021*a*) investigated the effects of uncertain time-to-build on a levered firm's investment and financing decisions, showing that the default probability can be lower than the case without time-to-build. Jeon (2021*b*) studied a duopolistic market with asymmetric uncertain time-to-build and found the equilibrium in which the dominated firm with longer expected time-to-build becomes a leader. Jeon (2023) took account of learning effects in the discussion of capacity expansion with uncertain time-to-build.

Although these studies considered uncertain time-to-build in their discussion, the *sheer* effects of uncertainty in time-to-build were not addressed. To our knowledge, Nishihara (2018) is the first to discuss this issue. This study investigated a firm's R&D investment decision with uncertainty in market demands, competition, and R&D duration, and numerically showed that uncertainty in the duration, described by a uniform distribution, leads to earlier investment compared to the case in which the duration is fixed. Jeon (2024*b*) compared the optimal investment strategy and firm value with fixed time-to-build and those with uncertain time-to-build whose expected duration is identical with the fixed counterpart, without any assumption on the distribution of time-to-build. The comparison showed that uncertainty in time-to-build always accelerates investment and improves pre-investment firm value, regardless of its distribution. Jeon (2024a) found that the positive impact of uncertainty in time-to-build is robust even when the investment size decision is taken into account in addition to the timing decision.

Despite the difficulties of collecting data, a few studies empirically analyzed the effects and determinants of time-to-build. Koeva (2000) analyzed plant investment from various industries and found that time-to-build is about two years and is not sensitive to business cycles. Zhou (2000) found from empirical data that the presence of time-to-build can explain the positive correlation of investment. Salomon and Martin (2008) analyzed the determinants of time-to-build based on data from the semiconductor indsustry. They reported that the duration of time-to-build is associated with market competition, firm ownership, and firm/industry experience. Tsoukalas (2011) showed that in the presence of time-to-build, a firm's investment decision is significantly affected by the firm's cash flows. Kalouptsidi (2014) analyzed data from the bulk shipping industry and found that time-varying time-to-build decreases the level and volatility of investment. Oh and Yoon (2020) examined the U.S. residential investment during the 2002-2011 housing boom-bust cycle and found that the increase of time-to-build during the boom is due to construction bottlenecks whereas that during the bust is due to an increase of uncertainty. Charoenwong et al. (2024) utilized Japanese dataset to show that information acquisition and investment flexibility can reduce the negative impacts of time-to-build significantly.

3 Setup

Suppose that a risk-neutral firm is considering an investment project from which the following profit flow is generated:

$$\mathrm{d}X(t) = \mu X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t), \tag{1}$$

where μ and σ are positive constants and $(W(t))_{t\geq 0}$ is a standard Brownian motion on a filtered space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. The investment incurs lump-sum costs I and the variable costs of production are normalized to zero. The discount rate is $r(> \mu)$ to ensure finite value function, which is a standard assumption in real options literature.

The investment project does not yield revenue instantly after the investment because of the project's time-to-build. This can arise because of either the R&D for new technologies or the construction of manufacturing facilities of a large scale. Due to its inherent uncertainty, the size of time-to-build is a nonnegative random variable τ, which is assumed to be independent of $X(t)$ for simplicity.

4 Models and solutions

4.1 Preliminary results

The firm value with an option to invest in a project of which time-to-build is τ can be written as follows:

$$V_\tau(X) = \max_{T\geq 0} \mathbb{E}\Big[\int_{\hat{T}}^{\infty} e^{-rt}X(t)\mathrm{d}t - e^{-rT}I\Big|X(0) = X\Big]. \tag{2}$$

The investment timing can be characterized by the level of demand shock at which the firm invests in the project, and $T := \inf\{t > 0 | X(t) \geq X_\tau\}$ and $\hat{T} := T + \tau$ denote the timing of investment and revenue generation, respectively, where X_τ represents the corresponding investment threshold.

Due to the Markov property, the firm value at the investment timing for given demand shock X is

$$\mathbb{E}\Big[\int_\tau^\infty e^{-rt} X(t)\mathrm{d}t - I \Big| X(0) = X\Big] = \frac{X\delta(\tau)}{r-\mu} - I, \tag{3}$$

where $\delta(\tau) := \mathbb{E}[e^{-(r-\mu)\tau}]$ represents the discount factor with respect to the revenue generation timing, which plays a pivotal role in the discussion hereafter. Note that it is the Laplace transform of the time-to-build. Following the standard argument of real options, the firm value in (2) can be calculated as follows:[4]

$$V_\tau(X) = \begin{cases} \big[\frac{X_\tau \delta(\tau)}{r-\mu} - I\big]\big(\frac{X}{X_\tau}\big)^\gamma, & \text{if } X < X_\tau, \\ \frac{X\delta(\tau)}{r-\mu} - I, & \text{if } X \geq X_\tau, \end{cases} \tag{4}$$

where the optimal investment threshold is

$$X_\tau = \frac{\gamma(r-\mu)I}{(\gamma-1)\delta(\tau)}, \tag{5}$$

and

$$\gamma := \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\Big(\frac{1}{2} - \frac{\mu}{\sigma^2}\Big)^2 + \frac{2r}{\sigma^2}} \ \ (> 1). \tag{6}$$

4.2 The effects of uncertainty on investment

The effects of uncertainty in time-to-build on investment and firm value can be described as follows:

Lemma 1 *If τ_{n+1} is a mean-preserving spread of τ_n for $n \geq 0$ with a constant $\tau_0 = \bar{\tau}$, the following always holds:*

$$X_{\tau_{n+1}} < X_{\tau_n} \quad \textit{and} \quad V_{\tau_{n+1}}(X) > V_{\tau_n}(X) \qquad \textit{for all } n \geq 0. \tag{7}$$

Proof See Appendix A.1.

Lemma 1 implies that uncertainty in time-to-build *always* accelerates investment and enhances pre-investment firm value, which was first shown by Jeon (2024*b*). This is essentially due to the convexity of the discount factor with respect to the timing of revenue generation. That is, the good news (i.e., $\tau < \mathbb{E}[\tau]$) is discounted less while the bad news (i.e., $\tau > \mathbb{E}[\tau]$) is discounted significantly, yielding the asymmetric effects of uncertainty in time-to-build on the firm value. Note that this argument does not depend on the distribution of time-to-build τ.[5]

Hartman (1972, 1973) and Abel (1983) demonstrated that uncertainty *can* accelerate investment, focusing on uncertainties in the *state space*, such as market demands and output prices. In their papers, the convexity of the marginal profitability of capital, which results from the optimal adjustment of labor, induces the positive impacts of uncertainty in demands. Jeon (2024*b*,*a*) and this study shed light

[4] See Dixit and Pindyck (1994) for a comprehensive review regarding the real options theory.

[5] Jeon (2024*b*) verified the robustness of this result, showing that it still holds even when there are running costs during the phase of time-to-build and the firm has an option to abandon the ongoing project. Jeon (2024*a*) showed that the positive impacts of uncertainty in time-to-build remain intact even when the firm's investment size decision is considered in addition to the timing decision.

on the uncertainty in the *time dimension*, showing that uncertainty in revenue generation timing *always* accelerates investment, and it is also the convexity that drives the positive impacts of uncertainty in time-to-build.

In the standard real options literature in which the investment timing decision is mainly discussed, it is well-known that an increase in demand volatility (i.e., σ) delays investment. This negative impact of demand uncertainty on investment is in sharp contrast with the positive impact of uncertainty in time-to-build, and the economic intuition behind these opposing effects is as follows. When demand is uncertain, the firm obtains more information and resolves the uncertainty by waiting to invest. In other words, the value of the option to wait increases with demand uncertainty, and therefore, the firm delays investment as the market becomes more volatile. This can be seen from the fact that $\partial\gamma/\partial\sigma < 0$, and thereby $\gamma/(\gamma - 1)$ in (5), which represents the option value, increases with σ. Note that the expected profits from the investment in (3) are independent of σ. This implies that the option value is the sole channel through which uncertainty in demand negatively affects the investment decision.

Meanwhile, the firm can obtain more information regarding the timing of revenue generation and resolve the uncertainty only after the investment, but the amount and quality of the information do not depend on the investment timing. Thus, earlier investment due to the uncertainty in time-to-build is not associated with the value of the option to wait. This can be seen from the fact that γ in (6) is independent of τ. Note that the expected profits at the investment timing in (3) depends on τ. This implies that the expected profits from the investment, which depends on the convexity of the discount factor with respect to the revenue generation timing, are the sole channel through which uncertainty in time-to-build positively impacts the investment decision. This argument will be discussed in detail with numerical examples in Section 5.7.

Most empirical studies on uncertainty-investment relationship support the negative link between them (e.g., Leahy and Whited (1996), Guiso and Parigi (1999), Meinen and Roehe (2017)), but there are a few exceptions. For instance, Driver et al. (2008) used panel data from the British survey to test the effects of uncertainty on investment and found the positive impacts in the industries with high R&D and advertising intensities. Marmer and Slade (2018) analyzed the U.S. copper mining industry and reported the positive impact of uncertainty on investment when the project involves time-to-build. These studies suggest that time-to-build might be one of the the drivers behind the positive impacts of uncertainty on investment, although this hypothesis needs to be tested with empirical data.

4.3 Certainty equivalent of uncertain time-to-build

Now we examine *how much* uncertainty in time-to-build advances the timing of investment and improves the firm value.

Proposition 1 (Certainty equivalent) *For any uncertain time-to-build τ, there always exists a unique constant $\bar{\tau}_c (< \mathbb{E}[\tau])$ such that $\delta(\tau) = \delta(\bar{\tau}_c)$, or equivalently, $X_\tau = X_{\bar{\tau}_c}$ and $V_\tau(X) = V_{\bar{\tau}_c}(X)$. The certainty equivalent is derived as*

$$\bar{\tau}_c = -\frac{K_\tau(-(r-\mu))}{r-\mu}, \tag{8}$$

where $K_\tau(t)$ is the cumulant-generating function of τ:[6]

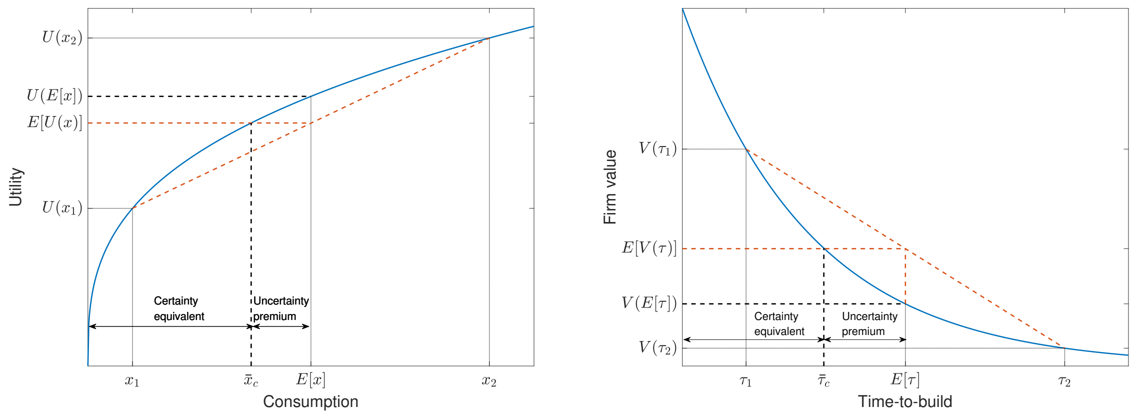
$$K_\tau(t) = \ln \mathbb{E}[e^{t\tau}] = \sum_{n=1}^{\infty} \frac{t^n \kappa_n}{n!}, \tag{9}$$

with κ_n denoting the n-th cumulant of τ.

PROOF See Appendix A.2.

Proposition 1 provides a practical framework for deriving the optimal investment strategy in the presence of uncertain time-to-build in a straightforward manner. Given prior knowledge of the uncertainty in time-to-build τ, the firm can derive the corresponding certainty equivalent $\bar{\tau}_c$ in (8) and apply it to the optimal investment strategy that would have been adopted in the absence of such uncertainty (i.e., $X_\tau = X_{\bar{\tau}_c}$). This tractable framework is applicable to *any* τ that has its probability density function.

Proposition 1 implies that the firm value with *longer* and *uncertain* time-to-build (i.e., $\mathbb{E}[\tau]$) is same as that with *shorter* and *fixed* time-to-build (i.e., $\bar{\tau}_c$) and that the unique correspondence (i.e., $X_\tau = X_{\bar{\tau}_c}$ and $V_\tau(X) = V_{\bar{\tau}_c}(X)$) *always* exists, regardless of the distribution of stochastic time-to-build. The degree to which uncertainty in time-to-build accelerates investment and thus improves firm value is measured by $\mathbb{E}[\tau] - \bar{\tau}_c$ (> 0), which is referred to as *uncertainty premium of time-to-build.*



(a) Uncertain consumption and utility (b) Uncertain time-to-build and firm value

Figure 1: Positive impacts of uncertainty in time-to-build on firm value

Figure 1 graphically illustrates the positive impacts of uncertainty in time-to-build and the existence of the certainty equivalent. To facilitate understanding, Figure 1a reviews the well-known negative impacts of uncertainty in consumption on utility. For a risk-averse investor, her utility function $U(x)$ is a function of consumption level x with $U' \geq 0$ and $U'' \leq 0$. Given possible outcomes of x_1 and x_2, the concavity of the utility function ensures $\mathbb{E}[U(x)] \leq U(\mathbb{E}[x])$ always holds, and there exists the certainty equivalent $\bar{x}_c$ such that $\mathbb{E}[U(x)] = U(\bar{x}_c)$ and $\bar{x}_c \leq \mathbb{E}[x]$. Figure 1b follows similar arguments. Firm value $V(\tau)$ is a function of time-to-build τ with $V' \leq 0$ and $V'' \geq 0$, and given possible outcomes of

[6]The cumulant-generating function is the natural logarithm of the moment-generating function $M_\tau(t) = \mathbb{E}[e^{t\tau}] = \sum_{n=0}^{\infty} \frac{t^n \mathbb{E}[\tau^n]}{n!}$. Since $r > \mu$, $M_\tau(-(r-\mu))(< 1)$ always exists and so does $K_\tau(-(r-\mu))(< 0)$, provided that the probability density function exists.

τ_1 and τ_2, the convexity ensures $\mathbb{E}[V(\tau)] \geq V(\mathbb{E}[\tau])$; there exists the certainty equivalent $\bar{\tau}_c$ such that $\mathbb{E}[V(\tau)] = V(\bar{\tau}_c)$ and $\bar{\tau}_c \geq \mathbb{E}[\tau]$.

The sensitivity of the certainty equivalent of uncertain time-to-build with respect to market demands can be addressed as follows:

Corollary 1 *The certainty equivalent of time-to-build increases with the expected growth rate of demand (i.e., μ). In other words, the uncertainty premium of time-to-build decreases with it. Both are independent of demand volatility (i.e., σ).*

PROOF See Appendix A.3.

This result implies that uncertainty in time-to-build accelerates investment more significantly when the market demand is expected to grow slowly. Technically speaking, this is because the convexity of the discount factor with respect to the timing of revenue generation—the main driver of the positive effects of uncertainty in time-to-build—decreases with the expected growth rate of demand (i.e., μ).[7] That is, the firm discounts the future cash flow more heavily when μ is low, and thus, earlier completion of the project is more appreciated and losses from the delay are discounted more substantially when μ is low. To sum up, the adjustment in the investment strategy due to the uncertainty of revenue generation timing needs to consider how much the demand is expected to grow over time but it should not reflect how volatile the demand is.

Proposition 1 allows us to summarize a direct relationship between time-to-build and firm value as follows:

Corollary 2 *Suppose the initial demand X is sufficiently low such that the investment is not triggered instantly. The firm value $V_\tau(X)$ being greater than $\bar{X}$ is equivalent to*

$$\bar{\tau}_c \leq \frac{\ln(A(X)/\bar{X})}{(r-\mu)\gamma}, \tag{10}$$

where

$$A(X) = \Big(\frac{\gamma-1}{I}\Big)^{\gamma-1}\Big(\frac{X}{\gamma(r-\mu)}\Big)^{\gamma}. \tag{11}$$

PROOF See Appendix A.4.

It is obvious that the right-hand side of (10) decreases with $\bar{X}$. This implies that the certainty equivalent of time-to-build needs to be smaller, or equivalently, the uncertainty premium of time-to-build needs to be greater to achieve a higher firm value.

Proposition 1 implies that the firm needs *perfect* prior information regarding the time-to-build (i.e., probability distribution) to derive the optimal investment strategy based on the certainty equivalent. However, in practice, it is rare for firms to have such perfect prior information regarding the uncertainty in time-to-build. Nevertheless, even without the full knowledge regarding such uncertainty, the firm can approximate the certainty equivalent using only a few moments of the time-to-build as follows:

[7] For $f(\tau) = \exp(-(r-\mu)\tau)$, the degree of convexity, measured by $\frac{1}{f}\frac{\partial^2 f}{\partial \tau^2} = (r-\mu)^2$, decreases with μ.

Corollary 3 *Given the mean and variance of time-to-build τ, denoted by m and v, respectively, the certainty equivalent of τ is approximated as follows:*

$$\tilde{\tau}_{c,2} := m - \frac{(r-\mu)v}{2}. \tag{12}$$

With the addition of skewness and excess kurtosis, denoted by s and e, respectively, it can be approximated more precisely as follows:

$$\tilde{\tau}_{c,3} := m - \frac{(r-\mu)v}{2}\Big(1 - \frac{(r-\mu)s\sqrt{v}}{3}\Big), \tag{13}$$

$$\tilde{\tau}_{c,4} := m - \frac{(r-\mu)v}{2}\Big(1 - \frac{(r-\mu)s\sqrt{v}}{3} + \frac{(r-\mu)^2 ev}{12}\Big), \tag{14}$$

where the approximation error is $\tilde{\tau}_{c,i} - \bar{\tau}_c$ for $i \in \{2, 3, 4\}$.

Proof See Appendix A.5.

Corollary 3 implies that if the firm has estimates of the mean and variance of time-to-build based on its prior investment experiences in similar fields, it can derive the optimal investment strategy that accounts for the uncertainty in time-to-build without any further information (i.e., the exact distribution). As will be shown in Section 5, the mean and variance are sufficient to approximate the certainty equivalent of uncertain time-to-build in many cases. In other words, the approximation error is negligible, unless the underlying distribution has a peak at either end of its support, such as an exponential distribution.

Based on Proposition 1 and Corollary 3, we can easily obtain the following result:

Corollary 4 *The certainty equivalent of time-to-build decreases with its dispersion. In other words, the uncertainty premium of time-to-build increases with its dispersion. Specifically, with the approximation up to the third moment in* (13)*, the uncertainty premium of time-to-build increases with the variance v if $s < 3/((r-\mu)\sqrt{v})$. With the approximation up to the fourth moment in* (14)*, it increases with the variance v if $s < 3/((r-\mu)\sqrt{v}) + (r-\mu)e\sqrt{v}/4$.*

Proof See Appendix A.6.

The positive impact of the variance of time-to-build on its uncertainty premium is straightforward; as noted in Lemma 1, the more dispersed time-to-build is, the more incentive to invest the firm has. The result in (13) shows that all else being equal, the skewness of time-to-build has a negative impact on its uncertainty premium. This is because the positively-skewed time-to-build implies that the distribution has a longer tail over the likelihoods of longer time-to-build, which reduces the firm's incentive to invest. The result in (14) implies that all else being equal, the excess kurtosis of time-to-build positively affects its uncertainty premium. This is because the greater excess kurtosis implies fatter tails, which increases the firm's incentive to invest due to the convexity effect described in Lemma 1.

Furthermore, for given mean and variance of time-to-build, we can derive the distribution-free upper and lower bounds of the certainty equivalent as follows:

Proposition 2 *For given mean m and variance v of time-to-build, the certainty equivalent of time-to-build is bounded as follows:*

$$-\frac{1}{r-\mu}\ln\left(\frac{e^{-(r-\mu)(m+v/m)}m^2+v}{m^2+v}\right) \leq \bar{\tau}_c \leq m. \quad (15)$$

In particular, both the upper and lower bounds are tight, and the lower bound strictly decreases with v.

Proof See Appendix A.7.

Corollary 4 suggests that the uncertainty premium of time-to-build might not increase with its variance. In fact, the following result can be obtained:

Proposition 3 *The certainty equivalent of time-to-build does not strictly decrease with variance of time-to-build. In other words, the uncertainty premium of time-to-build does not always strictly increase with its variance.*

Proof See Appendix A.8.

At a glance, the result of Proposition 3 might seem to contradict Lemma 1, but it does not. A mean-preserving spread of given time-to-build always accelerates investment, as addressed in Lemma 1, and it has a greater variance than the given time-to-build.[8] However, this does not imply that time-to-build with a greater variance always accelerates investment. This is because a random variable with the same mean yet a greater variance is not necessarily a mean-preserving spread of the counterpart.[9]

4.4 Entropic risk measure of time-to-build

In general, entropic risk measure is defined from the perspective of a risk-averse investor's expected utility. Specifically, it draws on exponential utility function $u_\theta(x) = 1 - e^{-\theta x}$ where $\theta(>0)$ is a risk aversion parameter, which strictly *increases* with x and is *concave* with respect to it, and it is defined as follows:

$$\rho_\theta(x) := \inf\{z \in \mathbb{R} \,|\, \mathbb{E}[u_\theta(x+z)] \geq 0\}. \quad (16)$$

It is the minimum amount of additional capital to ensure that the expected utility is above a certain level. Therefore, the higher (16) is, the riskier x is.

Following similar arguments, we can define entropic risk measure of time-to-build. Since the firm value *decreases* with the size of time-to-build τ and is *convex* with respect to it, we can consider $v_\theta(\tau) = e^{-\theta\tau} - 1(= -u_\theta(\tau))$. With these, entropic risk measure of time-to-build τ can be defined as follows:

$$\sup\{z \in \mathbb{R} \,|\, \mathbb{E}[v_\theta(\tau+z)] \geq 0\} = \inf\{z \in \mathbb{R} \,|\, \mathbb{E}[u_\theta(\tau+z)] \geq 0\}, \quad (17)$$

which amounts to the right-hand side of (16) with τ instead of x. It represents the maximum size of fixed time-to-build that can be added while ensuring that the firm value is above a certain level. As

[8]As in Lemma 1, suppose $\tau_{n+1} = \tau_n + \epsilon_{n+1}$ where $\mathbb{E}[\epsilon_{n+1}|\tau_n] = 0$. By the law of iterated expectations, $\mathbb{E}[\epsilon_{n+1}] = \mathbb{E}[\mathbb{E}[\epsilon_{n+1}|\tau_n]] = 0$ holds, and it is straightforward to show that $\mathrm{Var}(\tau_{n+1}) = \mathrm{Var}(\tau_n) + \mathrm{Var}(\epsilon_{n+1}) > \mathrm{Var}(\tau_n)$ since $\mathrm{Cov}(\tau_n, \epsilon_{n+1}) = \mathbb{E}[\tau_n\epsilon_{n+1}] - \mathbb{E}[\tau_n]\mathbb{E}[\epsilon_{n+1}] = \mathbb{E}[\tau_n\mathbb{E}[\epsilon_{n+1}|\tau_n]] = 0$.

[9]Namely, τ_{n+1} is not necessarily a mean-preserving spread of τ_n even if $\mathbb{E}[\tau_{n+1}] = \mathbb{E}[\tau_n]$ and $\mathrm{Var}(\tau_{n+1}) > \mathrm{Var}(\tau_n)$ hold. Refer to Appendix A.8 for a specific example.

noted in Lemma 1, uncertainty in time-to-build always increases the firm value. In other words, the firm has more slack in terms of time-to-build when it is riskier. Therefore, the higher (17) is, the riskier time-to-build is.[10]

With these arguments, we can describe the positive impact of uncertainty in time-to-build in terms of its entropic risk measure:

Proposition 4 *The certainty equivalent strictly decreases with entropic risk measure of time-to-build. In other words, the uncertainty premium of time-to-build strictly increases with its entropic risk measure.*

Proof See Appendix A.9.

Furthermore, we can express the certainty equivalent of time-to-build in terms of relative entropy:

Proposition 5 *The certainty equivalent of time-to-build τ can be represented as follows:*

$$\bar{\tau}_c = \inf_{\mathbb{Q}\in\mathcal{M}(\mathbb{P})} \left\{ \mathbb{E}^{\mathbb{Q}}[\tau] + \frac{1}{r-\mu} H(\mathbb{Q}|\mathbb{P}) \right\}, \tag{18}$$

where $\mathcal{M}(\mathbb{P})$ is the set of probability measures on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to $\mathbb{P}$, and

$$H(\mathbb{Q}|\mathbb{P}) := \mathbb{E}^{\mathbb{Q}}\Big[\ln \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \Big] \tag{19}$$

denotes the relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$, also known as Kullback-Leibler divergence, with $\mathbb{E}^{\mathbb{Q}}$ denoting the expectation under measure $\mathbb{Q}$.

Proof See Appendix A.10.

Proposition 5 implies that the certainty equivalent of time-to-build can be expressed as the sum of the expectation of time-to-build under equivalent measure and the distance between the two measures, minimized over the set of equivalent measures. Note that $\mathbb{E}^{\mathbb{Q}}[\tau] \leq \bar{\tau}_c < \mathbb{E}[\tau]$ always holds since $H(\mathbb{Q}|\mathbb{P}) \geq 0$. That is, the equivalent measure $\mathbb{Q}$ is chosen such that the expectation of time-to-build under the measure becomes smaller than under physical measure $\mathbb{P}$, but it is penalized by the relative distance between the two measures.

With the relevance of certainty equivalent of time-to-build and relative entropy described in Proposition 5, one might conjecture that the uncertainty premium of time-to-build, which measures how much investment is accelerated by the uncertainty of time-to-build, would increase with entropy of time-to-build. Recall that differential entropy of a continuous random variable x with a probability density function $f(x)$ is defined as $H(x) := -\int f(x) \ln f(x) \mathrm{d}x$, and it measures the average level of uncertainty, or more pricisely, information associated with the possible outcome of the random variable.[11] However, the following result is obtained:

[10] Although we measure the uncertainty of time-to-build based on the concept of entropic risk measure, our discussion is not directly associated with the decision-making of a risk-averse agent. As noted in Section 3, we assume a risk-neutral firm.

[11] For a discrete random variable x with a probability mass function $p(x)$, Shannon's entropy is defined as $-\sum p(x) \ln p(x) (\geq 0)$. Unlike Shannon's entropy, differential entropy can be negative since $f(x) > 1$ is possible.

Proposition 6 *The certainty equivalent of time-to-build does not strictly decrease with entropy of time-to-build. In other words, the uncertainty premium of time-to-build does not always strictly increase with its entropy.*

Proof See Appendix A.11.

In fact, as will be shown in Section 5, the opposite can hold depending on the distribution of time-to-build. That is, a decrease of entropy of time-to-build can lead to an increase of uncertainty premium of time-to-build. For instance, when the distribution is bimodal with most density being concentrated around both ends of its support, its variance becomes substantial, which in most cases increases the uncertainty premium of time-to-build, but its entropy is relatively insignificant.

4.5 Uncertainty equivalent of fixed time-to-build

Proposition 1 presents the firm's optimal investment decision, given precise knowledge of the uncertainty in time-to-build (i.e., probability distribution), in the form of the investment strategy that would have been implemented in the absence of such uncertainty. In practice, the opposite is more plausible; the firm has established the optimal investment strategy without consideration of uncertainty in time-to-build but does not know the extent of uncertainty that such an investment strategy implicitly assumes.

From this perspective, we can derive the following result:

Proposition 7 (Uncertainty equivalent) *For any fixed time-to-build $\bar{\tau}$ (> 0), there always exists a nonnegative random variable τ_u with $\mathbb{E}[\tau_u] > \bar{\tau}$ such that $\delta(\bar{\tau}) = \delta(\tau_u)$, or equivalently, $X_{\bar{\tau}} = X_{\tau_u}$ and $V_{\bar{\tau}}(X) = V_{\tau_u}(X)$. Specifically, the uncertainty equivalent is derived from*

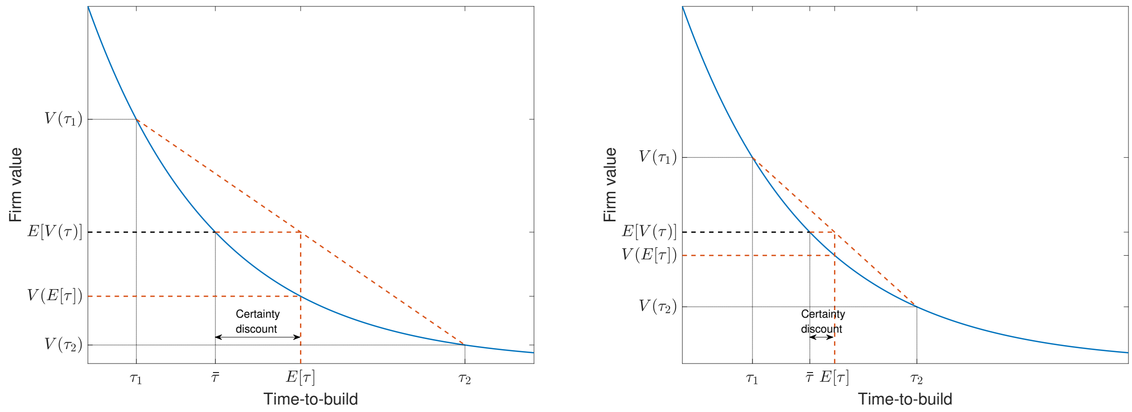
$$K_{\tau_u}(-(r-\mu)) = -(r-\mu)\bar{\tau}. \tag{20}$$

Proof See Appendix A.12.

Note that Proposition 7 proves the existence of the uncertainty equivalent but its uniqueness is not guaranteed. Namely, there can be many uncertainty equivalents that satisfy (20) for given fixed time-to-build. The degree to which the investment is disincentivized by the certainty of time-to-build compared to the case with uncertain time-to-build τ_u is measured by $\bar{\tau} - \mathbb{E}[\tau_u] (< 0)$, and its magnitude, $\mathbb{E}[\tau_u] - \bar{\tau}$ (> 0), is referred to as *certainty discount of time-to-build.*

Figure 2 graphically illustrates the multiplicity of the uncertainty equivalent for a fixed time-to-build. For a given constant $\bar{\tau}$, there can exist a random variable with greater mean and variance that satisfies (20) and the other one with smaller mean and variance yet satisfies (20); they need not follow the same distribution as well.

Proposition 7 suggests a guideline for firms to evaluate their investment strategies that have been established without consideration of uncertainty in time-to-build, specifying the equivalent risks they would essentially take under such an investment policy.



(a) Uncertainty equivalent with greater mean and variance

(b) Uncertainty equivalent with smaller mean and variance

Figure 2: Examples of multiple uncertainty equivalents for a fixed time-to-build

Based on Proposition 7, we can derive the following result:

Corollary 5 *The expected value of the uncertainty equivalent τ_u for a fixed time-to-build $\bar{\tau}$ increases with the dispersion of τ_u. In other words, the certainty discount of the fixed time-to-build increases with the dispersion of τ_u.*

Proof See Appendix A.13.

Following similar arguments from Proposition 7, we can also obtain the following result:

Corollary 6 *For any fixed time-to-build $\bar{\tau}$ (> 0), there always exists a nonnegative random variable τ_w with $\mathbb{E}[\tau_w] > \bar{\tau}$ such that $\delta(\bar{\tau}) < \delta(\tau_w)$, or equivalently, $X_{\bar{\tau}} > X_{\tau_w}$ and $V_{\bar{\tau}}(X) < V_{\tau_w}(X)$.*

Proof See Appendix A.14.

This implies that for any fixed time-to-build, there always exists uncertain time-to-build whose expected duration is *longer* than the fixed counterpart but induces *higher* firm value.

5 Probability distributions of time-to-build

In this section, we illustrate the practical application of the results from Section 4 using representative probability distributions. Throughout this section, we adopt the following parameters for describing the investment project. They are in a moderate range that can be easily found from real options literature.

Notation	Value	Description
r	0.08	Risk-free rate
μ	0.02	Expected growth rate of demand shock
σ	0.2	Volatility of demand shock
I	3	Lump-sum investment costs
X	0.1	Initial demand shock

Table 1: Benchmark parameters for numerical calculation

5.1 Uniform distribution

Suppose the firm knows that the time-to-build of its investment project has equal likelihood between the minimum a and maximum b. That is, assume that τ follows a uniform distribution with parameters (a, b) where $0 \leq a < b$. Its probability density function is

$$f(\tau) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq \tau \leq b, \\ 0 & \text{otherwise,} \end{cases} \tag{21}$$

which is described in Figure 3a, and its moment-generating function is

$$M_\tau(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}, \tag{22}$$

which amounts to the cumulant-generating function $K_\tau(t) = \ln M_\tau(t)$.

Recall that Propositions 3 and 6 showed an increase of variance and entropy of time-to-build does not always accelerate investment. However, it always has the positive impacts on investment when time-to-build follows a uniform distribution:

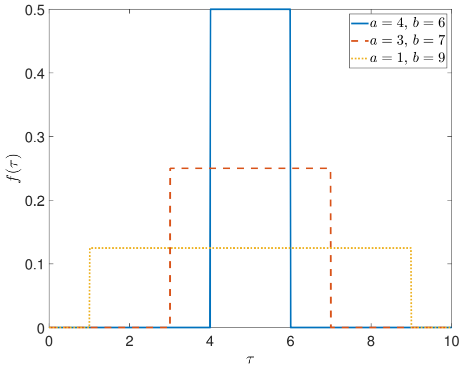
Proposition 8 *When time-to-build follows a uniform distribution, the certainty equivalent strictly decreases with its variance and differential entropy. In other words, the uncertainty premium strictly increases with its variance and differential entropy.*

Proof See Appendix A.15.

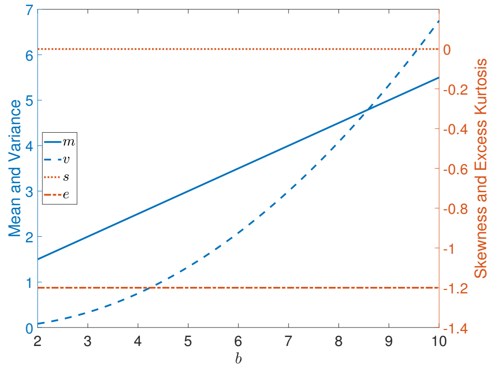
The certainty equivalent of uniformly distributed time-to-build can be calculated by (8) with (22). Figure 3c presents the certainty equivalent (i.e., $\bar{\tau}_c$) for a given minimum (i.e., $a = 1$). It is obvious that $\mathbb{E}[\tau]$ increases with the maximum b, and so does $\bar{\tau}_c$. By comparing Figures 3b, 3c and 3e, we can see that the uncertainty premium of time-to-build (i.e., $\mathbb{E}[\tau] - \bar{\tau}_c$) strictly increases with its variance and differential entropy, which verifies Proposition 8.

The uncertainty equivalent of fixed time-to-build, assumed to follow a uniform distribution, can be found by determining (a, b) that satisfies (20) with (22). Since a uniform distribution is defined by two parameters, if the firm assumes that time-to-build follows this distribution and is certain of its minimum (i.e., a), it naturally determines the maximum of the uncertainty equivalent (i.e., b), the worst-case scenario, corresponding to the fixed time-to-build. This is illustrated in Figure 3d for $\bar{\tau} = 5$. As noted in Section 4.5, there can exist many uncertainty equivalents for a fixed time-to-build, which is described by the solid line in Figure 3d. Recall that the uncertainty equivalent of time-to-build has a longer (expected) duration than the fixed one but yields the same firm value. Furthermore, the shaded area corresponds to the case in which uncertain time-to-build is longer (in expectation) than the fixed counterpart but induces higher firm value (i.e., $\bar{\tau} < \mathbb{E}[\tau]$ and $V_{\bar{\tau}} < V_\tau$), which confirms Corollary 6.

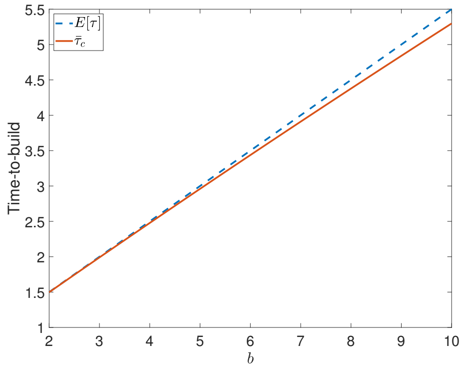
Figures 3f and 3g present the optimal investment threshold and firm value along with their approximation based on the moments described in Figure 3b, and they show that the approximation error is negligible. That is, the firm can essentially establish the optimal investment strategy taking account of uncertain time-to-build solely based on its mean and variance. Note that the approximation described in Figures 3f and 3g is solely based on the moments of the corresponding distribution without assuming any specific distribution; this is the same for other figures regarding the approximation hereafter.



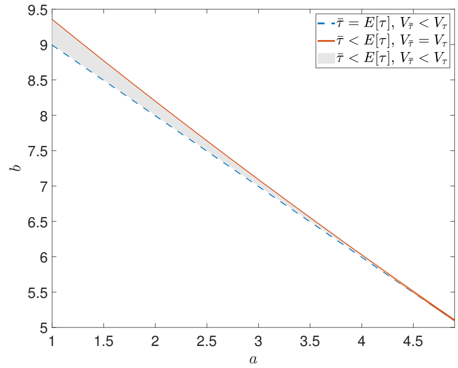
(a) Probability density function



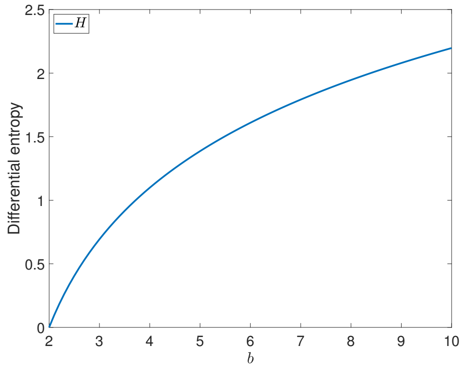
(b) Four common moments for $a = 1$



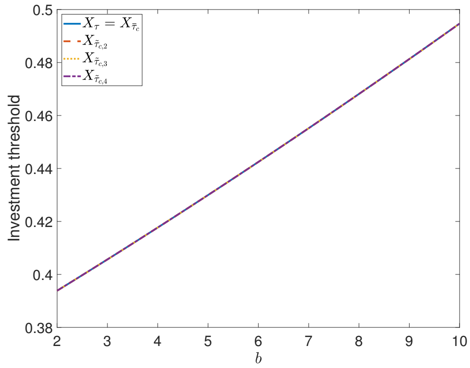
(c) Certainty equivalent of time-to-build for $a = 1$



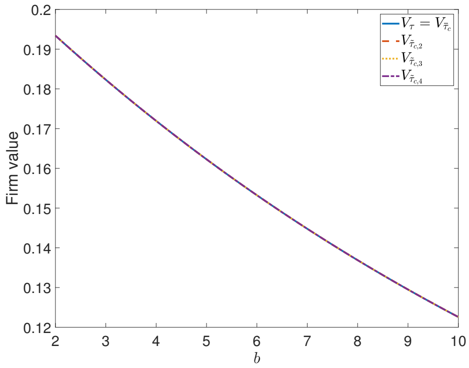
(d) Uncertainty equivalent of time-to-build for $\bar{\tau} = 5$



(e) Differential entropy for $a = 1$



(f) Investment threshold and its approximation for $a = 1$



(g) Firm value and its approximation for $a = 1$

Figure 3: When time-to-build follows a uniform distribution with minimum a and maximum b

5.2 Triangular distribution

Suppose that the firm knows the minimum, maximum, and mode of the time-to-build of its project, denoted by a, b, and c, respectively, and that its likelihood is unimodal and piece-wise linear. That is, assume that τ follows a triangular distribution with parameters (a, b, c) with $0 \leq a \leq c \leq b$ and $a < b$. Its probability density function is

$$f(\tau) = \begin{cases} \frac{2(\tau-a)}{(b-a)(c-a)} & \text{if } a \leq \tau < c, \\ \frac{2}{b-a} & \text{if } \tau = c, \\ \frac{2(b-\tau)}{(b-a)(b-c)} & \text{if } c < \tau \leq b, \\ 0 & \text{otherwise,} \end{cases} \tag{23}$$

which is depicted in Figure 4a, and its moment-generating function is

$$M_\tau(t) = \frac{2\{(b-c)e^{at} - (b-a)e^{ct} + (c-a)e^{bt}\}}{(b-a)(c-a)(b-c)t^2}. \tag{24}$$

As in Proposition 8, an increase of variance and differential entropy of time-to-build always accelerates investment when it follows a symmetric triangular distribution:

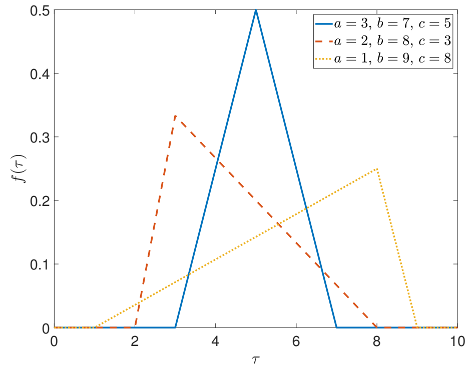
Proposition 9 *When time-to-build follows a symmetric triangular distribution, the certainty equivalent strictly decreases with its variance and differential entropy. In other words, the uncertainty premium strictly increases with its variance and differential entropy.*

Proof See Appendix A.16.

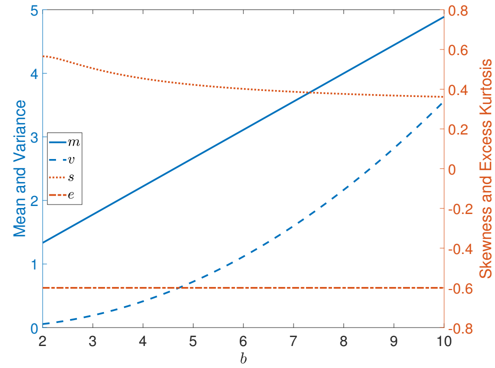
The certainty equivalent of time-to-build following a triangular distribution can be obtained by (8) with (24). Figure 4c presents the certainty equivalent assuming the minimum $a = 1$ and the mode $c = (a + b)/3$. Comparing Figures 4b, 4c and 4e, we can see that the uncertainty premium increases with the variance and differential entropy of time-to-build (Proposition 9).

The uncertainty equivalent of fixed time-to-build, assumed to follow a triangular distribution, can be found by determining (a, b, c) that satisfies (20) with (24). If the firm assumes that time-to-build follows this distribution and is sure of its minimum and mode (i.e., a and c), it can instantly deduce the worst-case scenario of the uncertainty equivalent (i.e., b) corresponding to the fixed time-to-build, which is described in Figure 4d for $\bar{\tau} = 5$. We can see that the worst-case scenario (i.e, b) decreases with the most likely scenario (i.e., c) for given best-case scenario (i.e., $a = 1$) and fixed time-to-build (i.e., $\bar{\tau} = 5$). The shaded area represents the uncertain time-to-build whose expected duration is longer than the fixed counterpart yet induces higher firm value (i.e., $\bar{\tau} < \mathbb{E}[\tau]$ and $V_{\bar{\tau}} < V_\tau$), which supports Corollary 6.

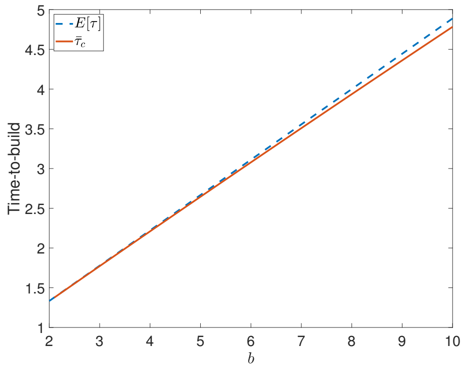
Figures 4f and 4g present the optimal investment threshold and firm value along with their approximation based on the moments described in Figure 4b. As in Section 5.1, the approximation error is negligible, which implies that the mean and variance of time-to-build are sufficient to derive the optimal investment strategy.



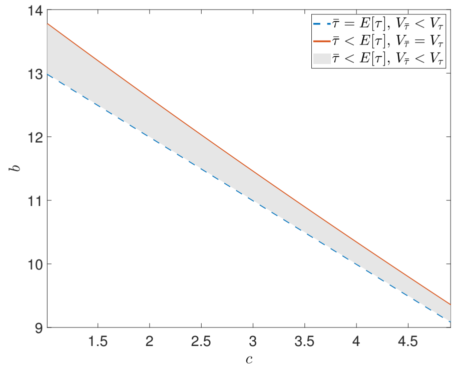
(a) Probability density function



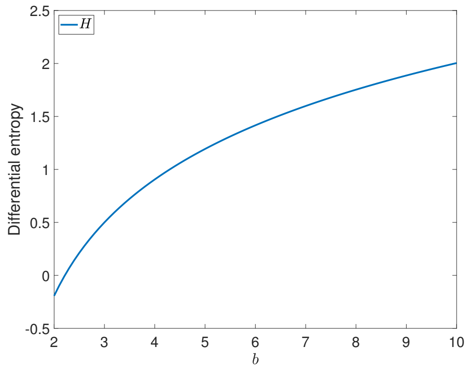
(b) Four common moments for $a = 1$ and $c = (a + b)/3$



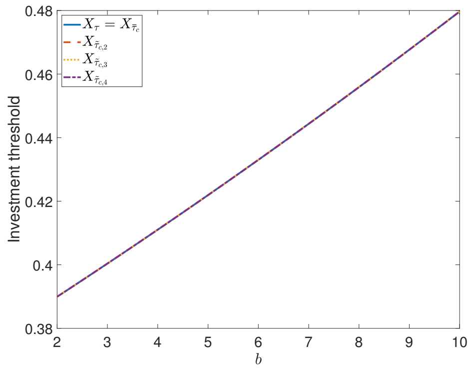
(c) Certainty equivalent of time-to-build for $a = 1$ and $c = (a + b)/3$



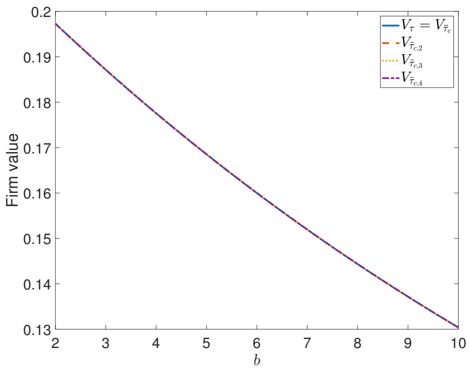
(d) Uncertainty equivalent of time-to-build for $\bar{\tau} = 5$ and $a = 1$



(e) Differential entropy for $a = 1$ and $c = (a + b)/3$



(f) Investment threshold and its approximation for $a = 1$ and $c = (a + b)/3$



(g) Firm value and its approximation for $a = 1$ and $c = (a + b)/3$

Figure 4: When time-to-build follows a triangular distribution with minimum a, maximum b, and mode c

5.3 Log-normal distribution

Suppose the firm knows that time-to-build τ follows a log-normal distribution with parameters μ_τ and $\sigma_\tau(>0)$. That is, $\ln\tau$ follows a normal distribution with mean μ_τ and variance σ_τ^2. It is a unimodal distribution on $(0,\infty)$, and $\tau\to\infty$ can describe the failure of R&D project. Its probability density function is

$$f(\tau)=\begin{cases}\frac{1}{\tau\sigma_\tau\sqrt{2\pi}}e^{-\frac{(\ln\tau-\mu_\tau)^2}{2\sigma_\tau^2}} & \text{if } \tau>0,\\ 0, & \text{otherwise,}\end{cases} \tag{25}$$

which is described in Figure 5a. Its moment-generating function (i.e., $M_\tau(t)=\mathbb{E}[e^{t\tau}]$) does not exist for $t\geq 0$ since the defining integral diverges. Although $\mathbb{E}[e^{t\tau}]$ converges for $t<0$ due to $\tau\in(0,\infty)$, its closed-form expression has not been found yet.[12] Asmussen et al. (2016) suggested the following approximation of the moment-generating function:

$$M_\tau(t)\approx\frac{\exp\left(-\frac{\{W(-t\sigma_\tau^2e^{\mu_\tau})\}^2+2W(-t\sigma_\tau^2e^{\mu_\tau})}{2\sigma_\tau^2}\right)}{\sqrt{1+W(-t\sigma_\tau^2e^{\mu_\tau})}}, \tag{26}$$

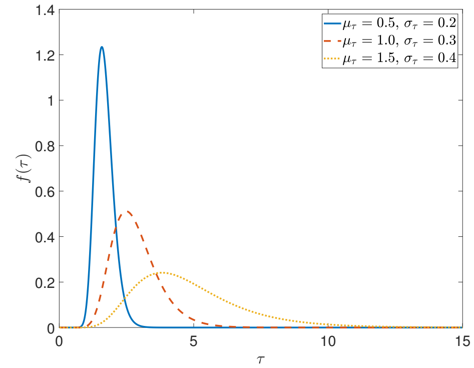
where $W(x)$ is the Lambert W function defined as the solution of $W(x)e^{W(x)}=x$, and we adopt this approximation to derive the certainty equivalent and uncertainty premium of time-to-build following log-normal distribution.

The certainty equivalent of time-to-build following a log-normal distribution can be found by (8) with (26), which is described in Figure 5c for $\mu_\tau=1$. Its comparison with Figures 5b and 5e numerically shows that the uncertainty premium of time-to-build increases with the variance and differential entropy.

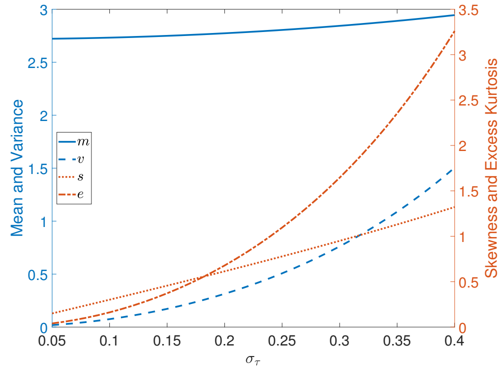
The uncertainty equivalent of time-to-build, assumed to follow a log-normal distribution, can be found by obtaining (μ_τ,σ_τ) that satisfies (20) with (26). If the firm suppose that the uncertainty equivalent follows this distribution and is certain of the mean of time-to-build, it can specify the candidates of the uncertainty equivalent. Figure 5d presents the uncertainty equivalent of fixed time-to-build $\bar{\tau}=5$, and we can see that μ_τ of the uncertainty equivalent decreases with σ_τ. The shaded area in Figure 5d represents the uncertain time-to-build whose expected duration is longer than the fixed counterpart yet yields higher firm value (i.e., $\bar{\tau}<\mathbb{E}[\tau]$ and $V_{\bar{\tau}}<V_\tau$), consistent with Corollary 6.

Figures 5f and 5g present the optimal investment threshold and firm value along with their approximation based on the moments illustrated in Figure 5b, and we can see that the approximation errors are negligible.

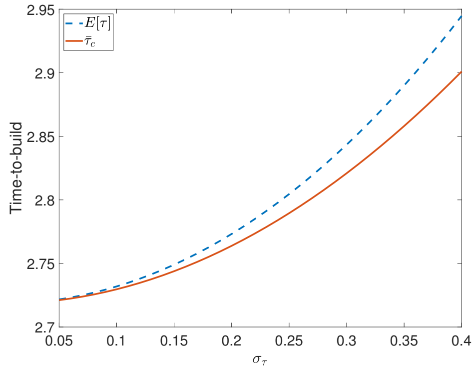
[12] All moments of the log-normal distribution exist (i.e., $\mathbb{E}[\tau^n]=e^{n\mu_\tau+n^2\sigma_\tau^2/2}$), but the log-normal distribution is not determined by its moments (e.g., Heyde (1963)). This implies that it cannot have a defined moment-generating function in a neighborhood of zero.



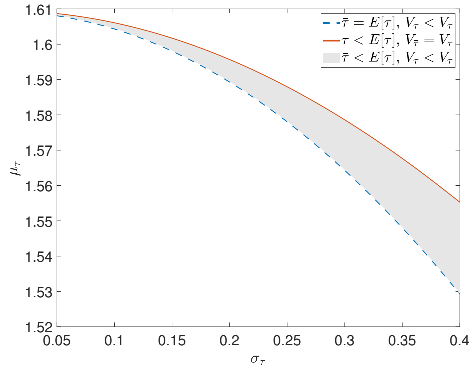
(a) Probability density function



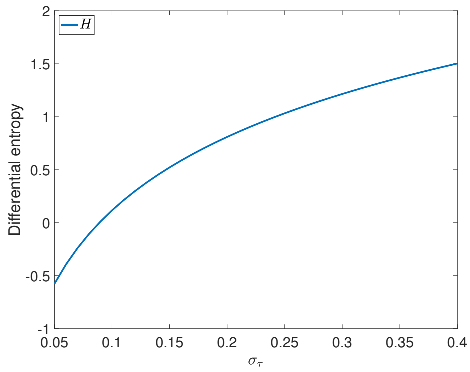
(b) Four common moments for $\mu_\tau = 1$



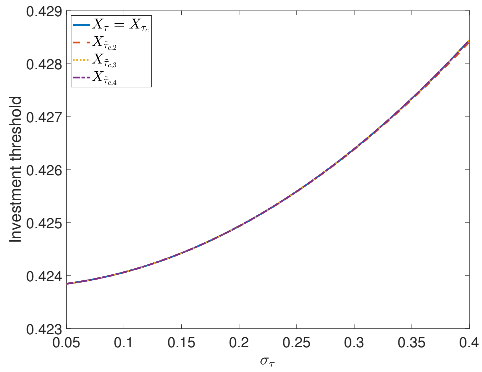
(c) Certainty equivalent of time-to-build for $\mu_\tau = 1$



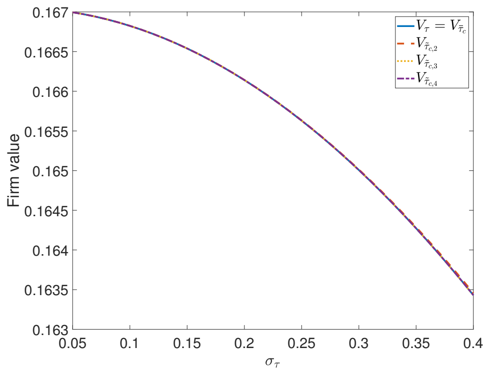
(d) Uncertainty equivalent of time-to-build for $\bar{\tau} = 5$



(e) Differential entropy for $\mu_\tau = 1$



(f) Investment threshold and its approximation for $\mu_\tau = 1$



(g) Firm value and its approximation for $\mu_\tau = 1$

Figure 5: When time-to-build follows a log-normal distribution with parameters μ_τ and σ_τ

5.4 Exponential distribution

Suppose the firm knows that the time-to-build follows an exponential distribution with a parameter $\lambda > 0$, which is well-known for its tractability including the memoryless property.[13] Its support is $[0, \infty)$, and the likelihood is significantly more concentrated towards 0 than towards ∞. Its probability density function is

$$f(\tau) = \begin{cases} \lambda e^{-\lambda\tau} & \text{if } \tau \geq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{27}$$

which is described in Figure 6a, and its moment-generating function is

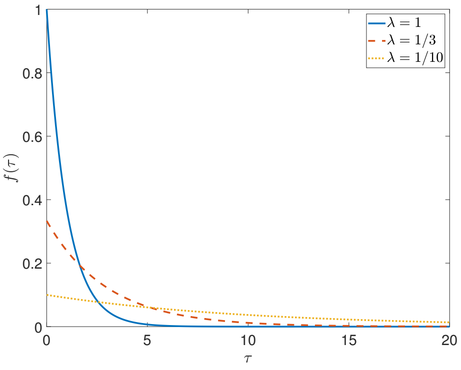
$$M_\tau(t) = \frac{\lambda}{\lambda - t} \qquad \text{for } t < \lambda. \tag{28}$$

The certainty equivalent of time-to-build following an exponential distribution can be found by (8) with (28). Specifically, it is given by $\bar{\tau}_c = \ln(1 + (r - \mu)/\lambda)/(r - \mu)$, and it is described in Figure 6c. The comparison of Figures 6b, 6c and 6e numerically shows that the uncertainty premium of time-to-build increases with its variance and differential entropy.

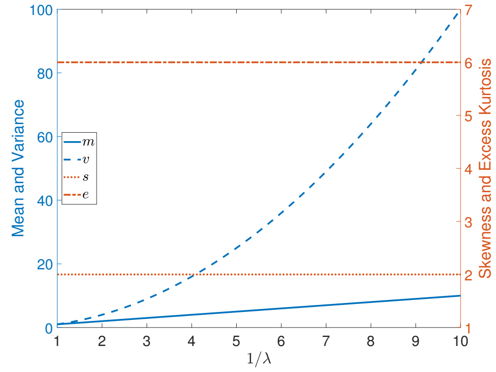
The uncertainty equivalent of fixed time-to-build, assumed to follow this distribution, can be found by determining λ that satisfies (20) with (28). In fact, since there is a single parameter, the uncertainty equivalent for a given fixed time-to-build following this distribution is *unique*, and so are its mean and variance (i.e., $1/\lambda$ and $1/\lambda^2$) and other n-th moments. Figure 6d presents the uncertainty equivalent for a various level of fixed time-to-build $\bar{\tau}$. As in Sections 5.1 to 5.3, the shaded area in Figure 6d corresponds to the uncertain time-to-build of which expected duration is longer than the fixed one yet induces higher firm value (i.e., $\bar{\tau} < \mathbb{E}[\tau]$ and $V_{\bar{\tau}} < V_\tau$), supporting Corollary 6.

Figures 6f and 6g present the optimal investment threshold and firm value considering uncertain time-to-build along with their approximation based on the moments in Figure 6b. Unlike Sections 5.1 to 5.3, we can see that there is a significant approximation error, which increases with the size of time-to-build. It is obvious that the size of the error decreases with the number of moments used for the approximation.

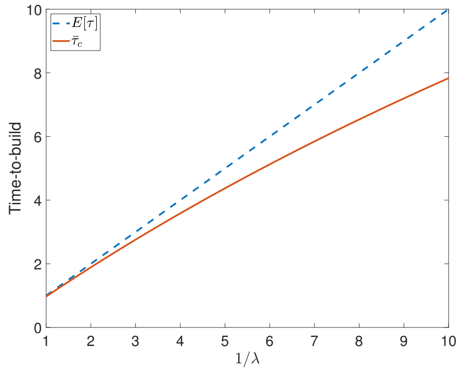
[13] The memoryless property becomes valuable when there is another decision-making after the investment is made yet before the project is finished, such as the default decision of a debt-financed firm or a follow-up investment. This is because it makes the subsequent decision-making time-independent so that we can solve an ordinary differential equation instead of a partial differential equation. For simplicity, this study only considers a single decision-making.



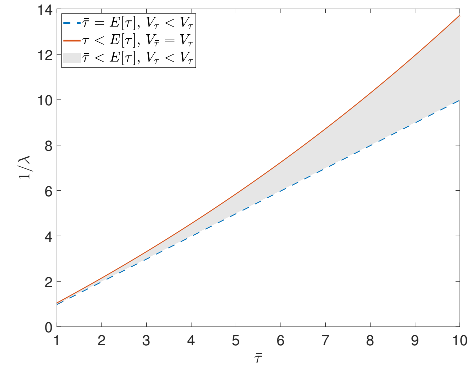
(a) Probability density function



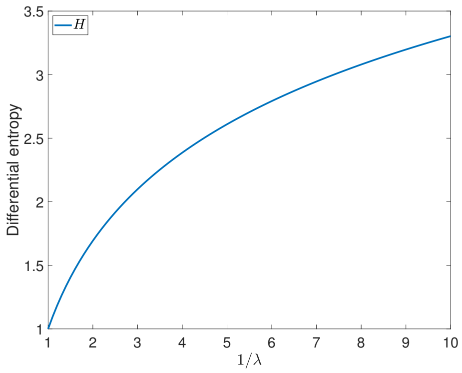
(b) Four common moments



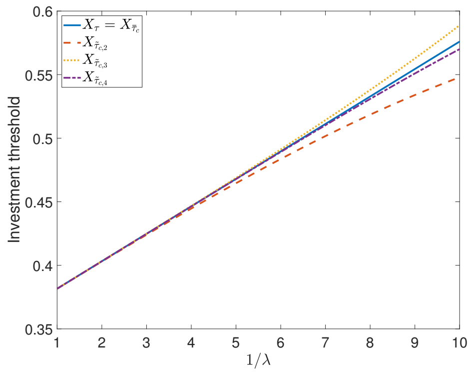
(c) Certainty equivalent of time-to-build



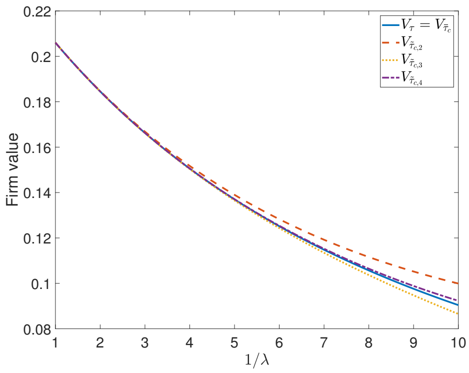
(d) Uncertainty equivalent of time-to-build for a various level of $\bar{\tau}$



(e) Differential entropy



(f) Investment threshold and its approximation



(g) Firm value and its approximation

Figure 6: When time-to-build follows an exponential distribution with a parameter λ

5.5 Gamma distribution

Suppose the firm knows that the time-to-build τ follows a gamma distribution with parameters (k, θ) where $k > 0$ is a shape parameter and $\theta > 0$ is a scale parameter. The two-parameter distribution is well-known for its versatility, encompassing many distributions as its special cases.[14] Its support is $(0, \infty)$, which allows us to model the potential for R&D failure (i.e., $\tau \to \infty$). However, with the inclusion of an additional parameter, it can describe a much broader range of likelihoods of time-to-build. Specifically, unlike the exponential distribution, the likelihood does not nave to be concentrated near 0. Its probability density function is

$$f(\tau) = \begin{cases} \frac{\tau^{k-1} e^{-\tau/\theta}}{\Gamma(k)\theta^k} & \text{if } \tau \geq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{29}$$

where $\Gamma(\cdot)$ is gamma function, which is described in Figure 7a, and its moment-generating function is

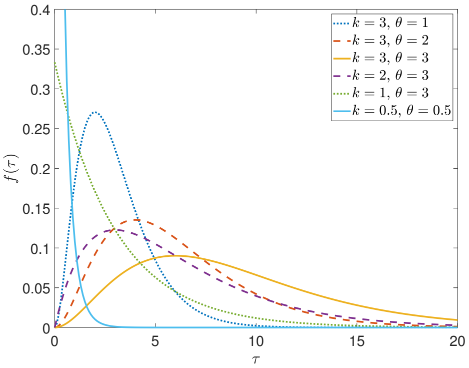
$$M_\tau(t) = (1 - \theta t)^{-k} \qquad \text{for } t < \frac{1}{\theta}. \tag{30}$$

The certainty equivalent of time-to-build following a gamma distribution can be found from (8) with (30), which is described in Figure 7c. Its comparison with Figures 7b and 7e numerically shows that the uncertainty premium of time-to-build increases with its variance and differential entropy.

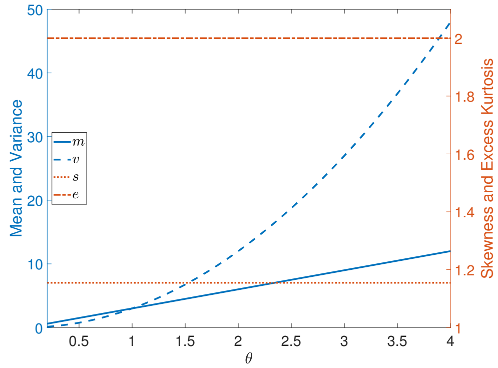
The uncertainty equivalent of fixed time-to-build, assumed to follow this distribution, can be found by determining (k, θ) that satisfies (20) with (30). If the firm supposes that the uncertainty equivalent follows this distribution without knowing the parameters but is certain of the mean of time-to-build (i.e., $k\theta$), it can specify the candidates of the uncertainty equivalent. Figure 7d presents the combination of k and θ that yields the uncertainty equivalent of fixed time-to-build $\bar{\tau} = 5$, and the shaded area corresponds to the case discussed in Corollary 6.

Figures 7f and 7g illustrate the optimal investment threshold and firm value taking uncertain time-to-build into account along with their approximation based on the moments in Figure 7b. They show that the approximation solely based on the mean and variance of time-to-build yields a nonnegligible error, but it becomes insignificant when the skewness is taken into account.

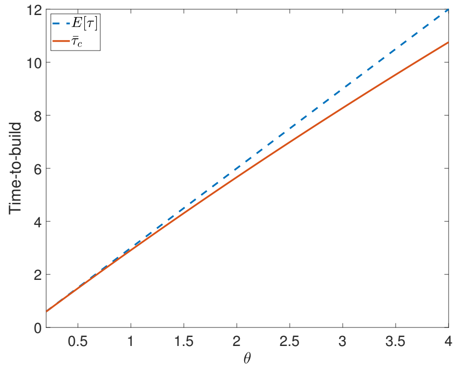
[14] With $k = 1$, it becomes the exponential distribution discussed in Section 5.4. When k is an integer, it is known as the Erlang distribution. When $k = \nu/2$ and $\theta = 2$, it corresponds to the chi-squared distribution with a parameter ν.



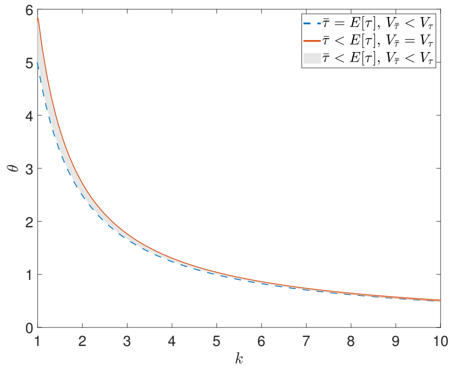
(a) Probability density function



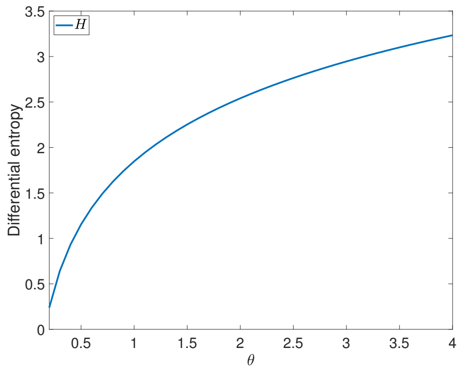
(b) Four common moments for $k = 3$



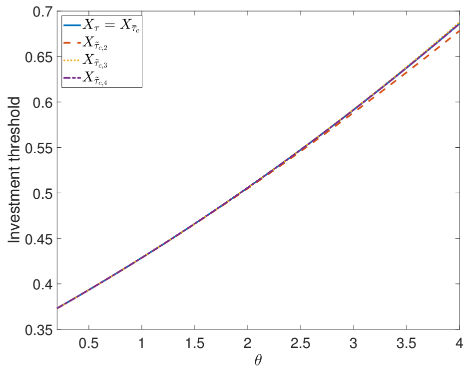
(c) Certainty equivalent of time-to-build for $k = 3$



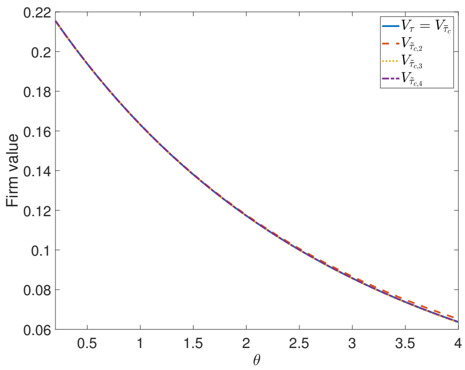
(d) Uncertainty equivalent of time-to-build for $\bar{\tau} = 5$



(e) Differential entropy for $k = 3$



(f) Investment threshold and its approximation for $k = 3$



(g) Firm value and its approximation for $k = 3$

Figure 7: When time-to-build follows a gamma distribution with shape parameter k and scale parameter θ

5.6 Scaled beta distribution

Despite its flexibility, the gamma distribution might not be suitable for describing the time-to-build of some investment projects, primarily because of its semi-infinite support. For this reason, we consider a scaled beta distribution.

Suppose ν follows a beta distribution with parameters (α, β) where $\alpha, \beta > 0$ are shape parameters. Since $\nu \in [0, 1]$, we can scale it to $\tau := (c-a)\nu + a$ so that $\tau \in [a, c]$ where $0 \leq a < c$. The probability density function of ν is

$$f(\nu) = \begin{cases} \frac{\nu^{\alpha-1}(1-\nu)^{\beta-1}}{B(\alpha,\beta)} & \text{if } 0 \leq \tau \leq 1, \\ 0 & \text{otherwise,} \end{cases} \tag{31}$$

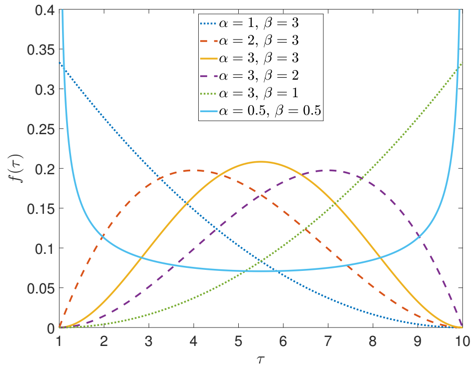
where $B(\cdot, \cdot)$ is beta function, and that of τ is $f(\tau) = f(\nu)/(c-a)$ on its support $[a, c]$, which is described in Figure 8a. The moment-generating function of τ following the scaled beta distribution with parameters (α, β, a, c) is

$$\begin{aligned} M_\tau(t) &= \int_a^c e^{t\tau} f(\tau) \mathrm{d}\tau \\ &= \frac{1}{B(\alpha,\beta)} \int_0^1 e^{t\{(c-a)\nu+a\}} \nu^{\alpha-1}(1-\nu)^{\beta-1} \mathrm{d}\nu \\ &= \frac{1}{B(\alpha,\beta)} \sum_{n=0}^\infty \frac{(ta)^n}{n!} \sum_{k=0}^\infty \frac{\{t(c-a)\}^k}{k!} \int_0^1 \nu^{\alpha+k-1}(1-\nu)^{\beta-1} \mathrm{d}\nu \\ &= \sum_{n=0}^\infty \frac{(ta)^n}{n!} \sum_{k=0}^\infty \frac{\{t(c-a)\}^k}{k!} \frac{B(\alpha+k,\beta)}{B(\alpha,\beta)} \\ &= \sum_{n=0}^\infty \frac{(ta)^n}{n!} \Big[1 + \sum_{k=1}^\infty \Big(\prod_{s=0}^{k-1} \frac{\alpha+s}{\alpha+\beta+s}\Big) \frac{\{t(c-a)\}^k}{k!}\Big]. \end{aligned} \tag{32}$$

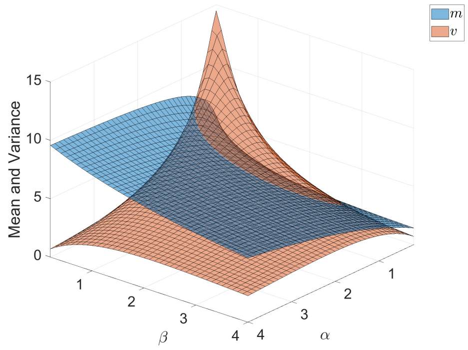
The certainty equivalent of time-to-build following the scaled beta distribution can be found from (8) with (32). Figure 8d presents the certainty equivalent of time-to-build following this distribution on $[1, 10]$. Although the skewness and excess kurtosis greatly vary depending on α and β (Figure 8c), the comparision of Figures 8b and 8e reveals that the uncertainty premium of time-to-build is mainly driven by its variance. Figure 8e also shows that the uncertainty premium of time-to-build is highest when both α and β are below 1 so that it becomes bimodal with peaks at both ends of the support $[a, c]$. The comparison of Figures 8e and 8g shows that the uncertainty premium of time-to-build does not strictly increases with its differential entropy, supporting Proposition 6. Note that the differential entropy decreases significantly when both α and β are below 1, where the variance becomes substantial.

The uncertainty equivalent of fixed time-to-build, assumed to follow this distribution, can be found by determining (α, β, a, c) that satisfies (20) with (32). Figure 8f presents the uncertainty equivalent of time-to-build following this distribution on $[1, 10]$ for fixed time-to-build $\bar{\tau} = 5$. The shaded area in Figure 8f corresponds to the case discussed in Corollary 6.

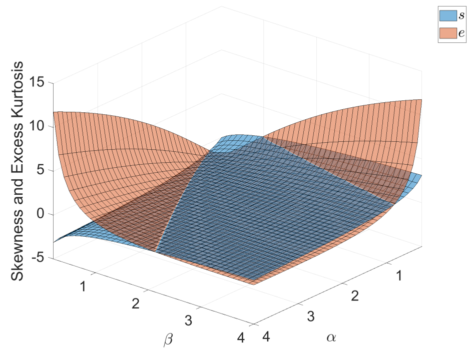
Figures 8h and 8i present the optimal investment threshold and firm value with uncertain time-to-build following this distribution along with their approximation, showing that the approximation error is insignificant.



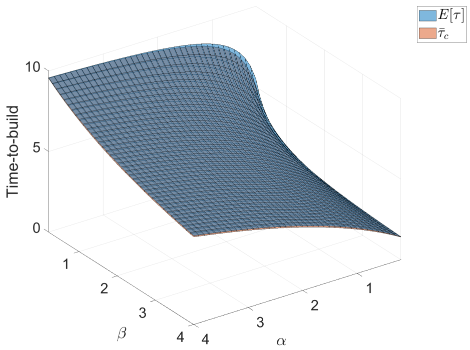
(a) Probability density function for $a = 1$ and $c = 10$



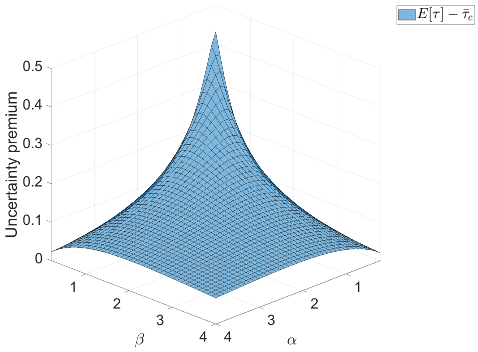
(b) Mean and variance for $a = 1$ and $c = 10$



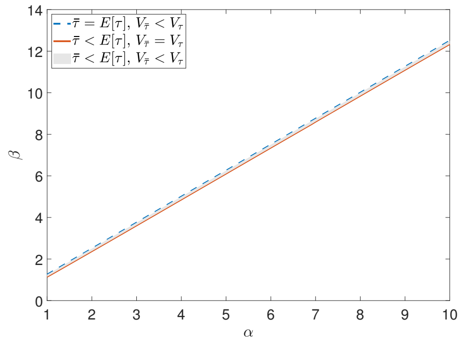
(c) Skewness and excess kurtosis for $a = 1$ and $c = 10$



(d) Certainty equivalent of time-to-build for $a = 1$ and $c = 10$

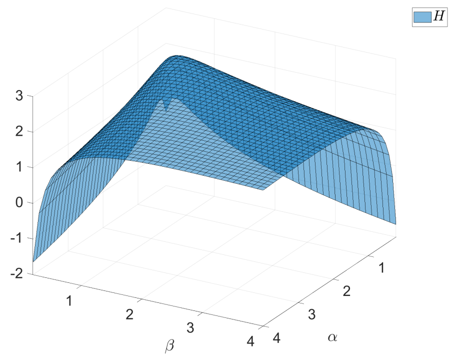


(e) Uncertainty premium of time-to-build for $a = 1$ and $c = 10$

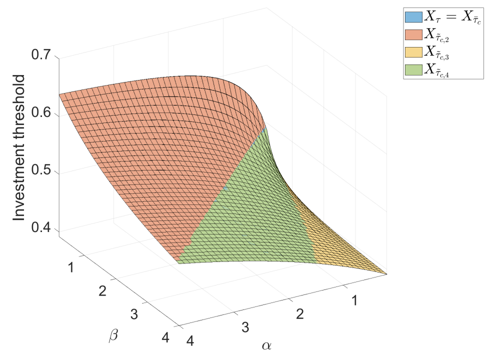


(f) Uncertainty equivalent of time-to-build for $\bar{\tau} = 5$, $a = 1$, and $c = 10$

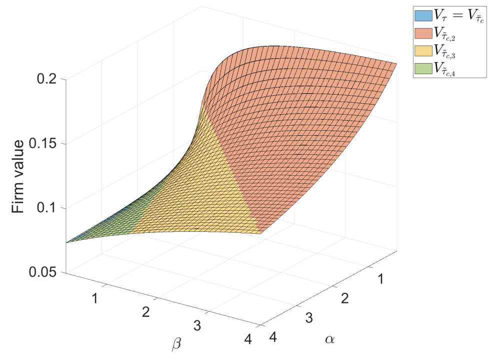
Figure 8: When time-to-build follows a scaled beta distribution with minimum a, maximum c, and shape parameters α and β



(g) Differential entropy for $a = 1$ and $c = 10$



(h) Investment threshold and its approximation for $a = 1$ and $c = 10$



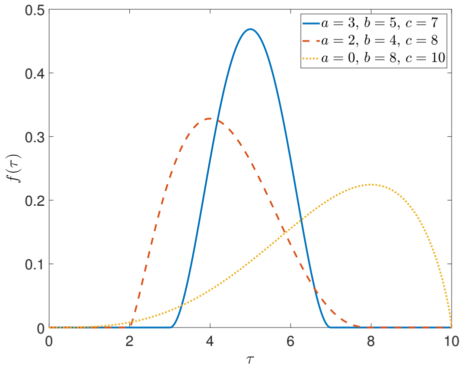
(i) Firm value and its approximation for $a = 1$ and $c = 10$

Figure 8: When time-to-build follows a scaled beta distribution with minimum a, maximum c, and shape parameters α and β

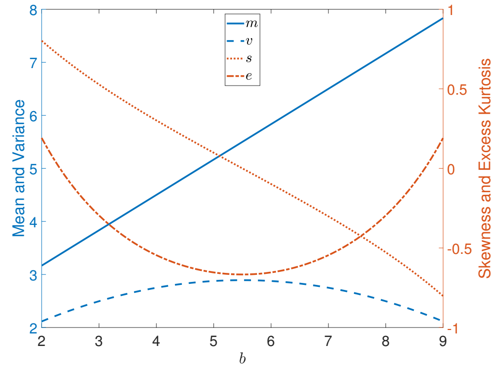
PERT distribution, which was developed for program evaluation and review technique, is a special case of the scaled beta distribution. Specifically, if τ follows the PERT distribution with parameters (a, b, c) with $0 \leq a < b < c < \infty$ where a and c are the minimum and maximum of time-to-build, respectively, and b is its mode, its probability density function coincides with that of the scaled beta distribution with $\alpha = 1 + 4(b-a)/(c-a)$ and $\beta = 1 + 4(c-b)/(c-a)$, which is described in Figure 9a. Unlike the triangular distribution discussed in Section 5.2, it is a *smooth* unimodal distribution, and it can be suitable for describing time-to-build having gradual changes in likelihoods with a single mode between a potentially nonzero minimum (i.e., a) and a finite worst-case scenario (i.e., c).

The certainty equivalent of time-to-build following the PERT distribution can be found following the same manner as before, which is illustrated in Figure 9c. As can be seen from Figure 9b, the variance does not vary significantly in accordance with the mode b within the fixed support $[a, c]$. This amounts to the insignificant change in the uncertainty premium (i.e., $\mathbb{E}[\tau] - \bar{\tau}_c$) in Figure 9c.

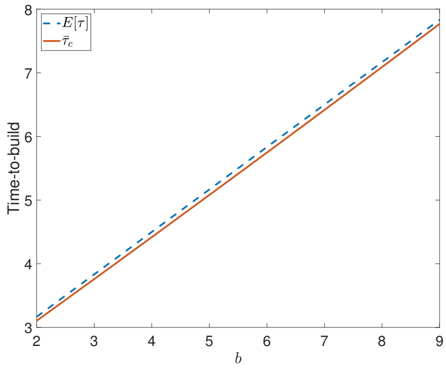
The uncertainty equivalent of fixed time-to-build, assumed to follow the PERT distribution, can be found by determining (a, b, c) that satisfies (20) and (32) with $\alpha = 1 + 4(b-a)/(c-a)$ and $\beta = 1 + 4(c-b)/(c-a)$. If the firm supposes that the uncertainty equivalent follows this distribution and is sure of the best-case scenario (i.e., a) and its most likely one (i.e., b), it can specify the worst-case scenario (i.e., c) of the uncertainty equivalent, which is presented in Figure 9d. Figures 9f and 9g describe the optimal investment strategy and firm value taking uncertain time-to-build into account, along with their approximation, and they reveal that the approximation is negligible.



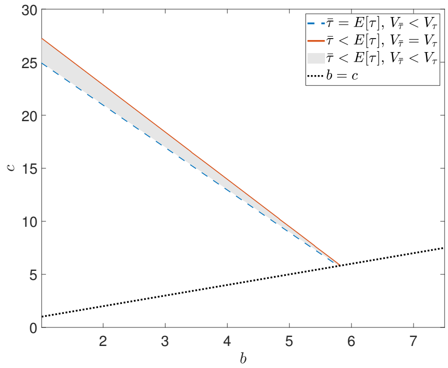
(a) Probability density function



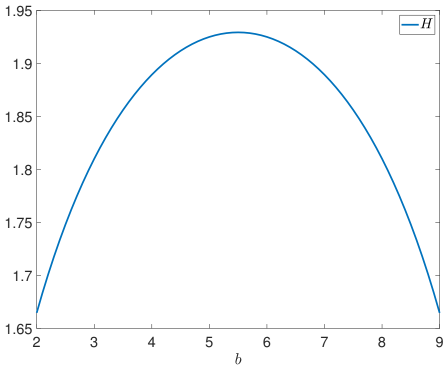
(b) Four common moments for $a = 1$ and $c = 10$



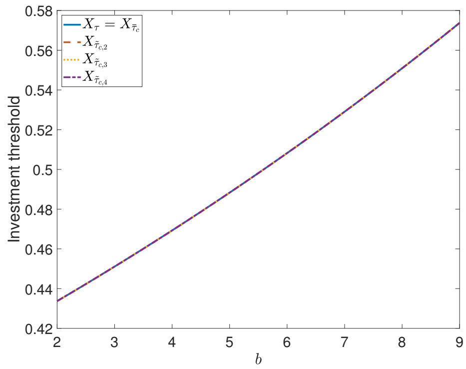
(c) Certainty equivalent of time-to-build for $a = 1$ and $c = 10$



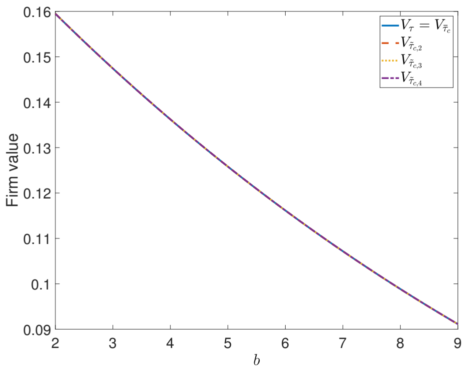
(d) Uncertainty equivalent of time-to-build for $\bar{\tau} = 5$ and $a = 1$



(e) Differential entropy for $a = 1$ and $c = 10$



(f) Investment threshold and its approximation for $a = 1$ and $c = 10$



(g) Firm value and its approximation for $a = 1$ and $c = 10$

Figure 9: When time-to-build follows a PERT distribution with minimum a, mode b, and maximum c

5.7 Mean and variance of time-to-build

Now we focus on the two most important moments of time-to-build: its mean and variance. Proposition 3 showed that without an assumption regarding the distribution of time-to-build, an increase in the variance of time-to-build does not always accelerate investment. However, we have seen from Sections 5.1 to 5.6 that such an increase leads to earlier investment for the well-known probability distributions.

Figure 10 illustrates the tight upper and lower bounds of the certainty equivalent of time-to-build for given mean and variance, which are demonstrated in Proposition 2, along with the corresponding certainty equivalent for representative probability distributions and its approximation based on the mean and variance (i.e., $\tilde{\tau}_{c,2}$ in (12)). Specifically, we fix the mean of time-to-build m and vary its variance v, demonstrating how the bounds change along with the certainty equivalent. Due to the degree of freedom, we choose distributions that can be characterized by two parameters: a uniform distribution, a symmetric triangular distribution, a log-normal distribution, a gamma distribution, and a symmetric PERT distribution.

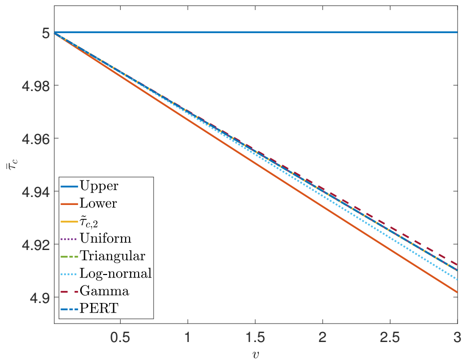
Figure 10a shows that the lower bound decreases with the variance v while the upper bound remains constant. It also clarifies that the certainty equivalent of time-to-build decreases with variance for these distributions, although this might not hold in an extreme case as the counterexample from the proof of Proposition 3. Note that the accuracy of the approximation of the certainty equivalent based on the mean and variance is significantly high. Its approximation error is essentially zero except for the log-normal distribution and the gamma distribution of which support is semi-infinite. Figures 10b to 10f describe the corresponding parameters for each distribution that satisfy the mean m and variance v.

Figure 11 presents the examples of the uncertainty equivalent discussed in Proposition 7. Specifically, Figure 11a depicts the level of variance v, combined with the mean m, that induces the same optimal investment threshold and firm value as the ones with a fixed time-to-build $\bar{\tau}$. This figure, along with Figures 11b and 11c, clarifies that even though the expected duration of time-to-build lengthens, the optimal investment timing and firm value can remain the same as long as its uncertainty increases significantly. It also demonstrates that the variance of the uncertainty equivalent depends on the distribution governing time-to-build. The variance of a gamma distribution is found to be significantly higher than other distributions, primarily because of its semi-infinite support.

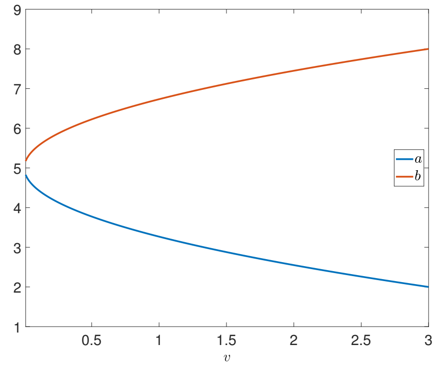
Figures 11d to 11h present the corresponding parameters for each distribution. Note that some of the distributions are unable to yield the uncertainty equivalent for mean m that is substantially greater than $\bar{\tau}$ due to their restrictions on the parameters, such as a nonnegative minimum. A log-normal distribution and a gamma distribution are relatively flexible for yielding the uncertainty equivalent owing to their versatility. Note that the uncertainty equivalents do not need to follow the distributions illustrated in Figure 11; they are only a fraction of many alternatives that induce the same investment decision and firm value for a given $\bar{\tau}$.

As discussed in Section 4.2, an increase of demand uncertainty (i.e., σ) delays the investment yet increases the firm value. This is because the firm's option to wait becomes more valuable when the market demands are uncertain. By contrast, Lemma 1 demonstrates that an increase of uncertainty in time-to-build advances the investment timing and improves the firm value. That is, the uncertainty of demand and that of time-to-build induce the opposite effects on the investment timing yet both yields the positive impacts on the firm value.

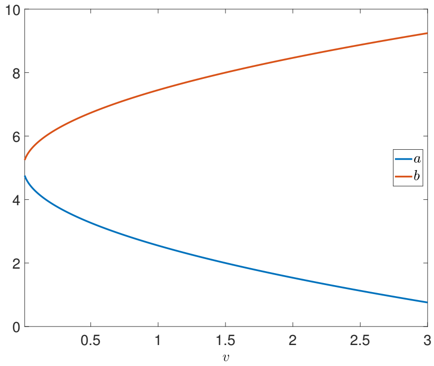
With these arguments, Figure 12 sheds light on the contrasting effects of the two different types of uncertainty. Specifically, Figure 12a describes the level of variance v that offsets the negative impacts of an increase in σ on the investment timing. It shows that the variance of time-to-build required to offset the impacts of increased demand volatility differs depending on the distribution of time-to-build. A gamma distribution is found to require significantly higher variance than other distributions, mainly because of its semi-infinite support. Figures 12b and 12c clarify that the effects of uncertainty on investment timing from the two different channels are canceled out, while the firm value significantly improves due to the uncertainty from the both channels. Figures 12d to 12h present the corresponding parameters for each distribution. Note that some of the distributions are unable to offset the negative impacts of a significant increase in demand volatility due to their restrictions on the parameters.



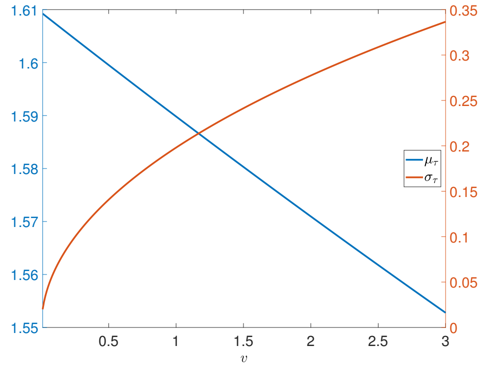
(a) Upper and lower bounds of certainty equivalent



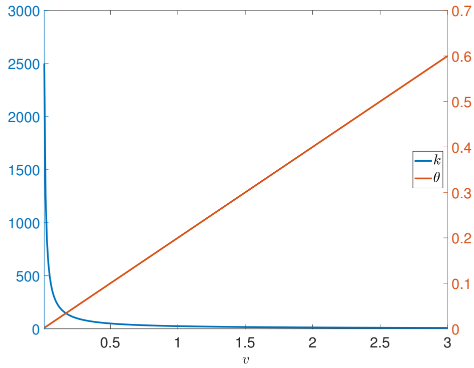
(b) Corresponding parameters for uniform distribution (a and b)



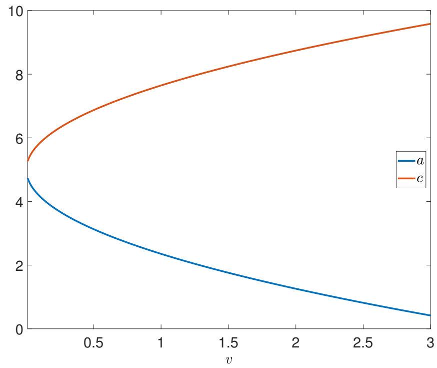
(c) Corresponding parameters for symmetric triangular distribution (a and b)



(d) Corresponding parameters for log-normal distribution (μ_τ and σ_τ)

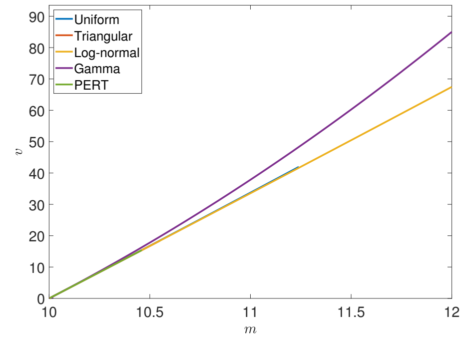


(e) Corresponding parameters for gamma distribution (k and θ)

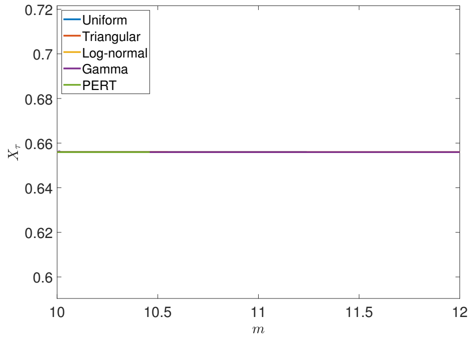


(f) Corresponding parameters for symmetric PERT distribution (a and c)

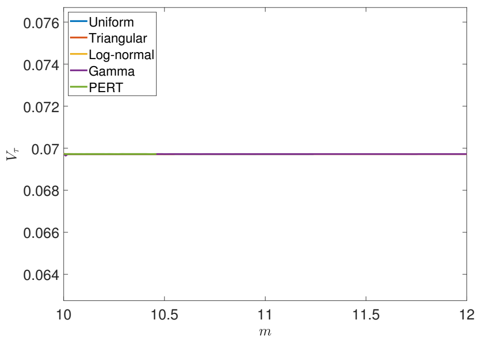
Figure 10: Tight upper and lower bounds of certainty equivalent of time-to-build and the certainty equivalents for representative probability distributions with mean $m = 5$ and different levels of variance v



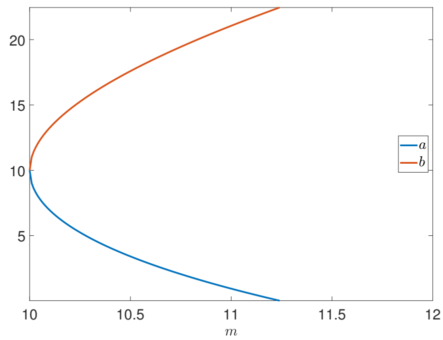
(a) Variance v of uncertainty equivalent with mean m for given $\bar{\tau}$



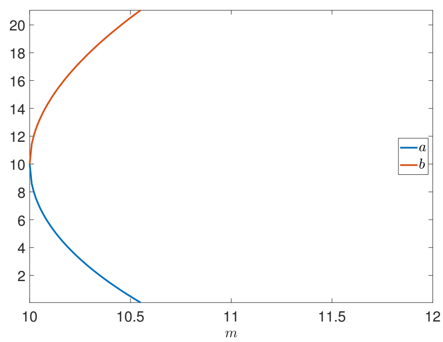
(b) Investment threshold with uncertainty equivalent following each distribution



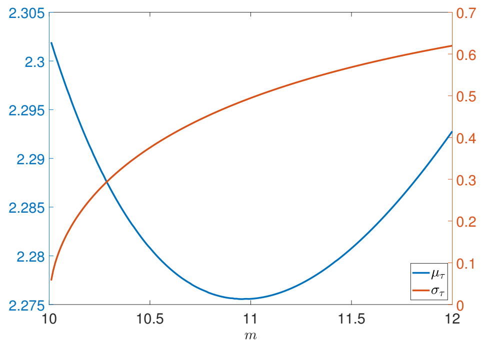
(c) Firm value with uncertainty equivalent following each distribution



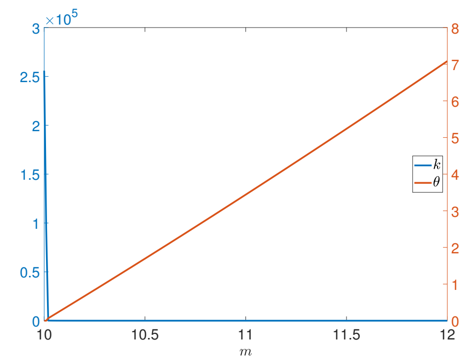
(d) Corresponding parameters for uniform distribution (a and b)



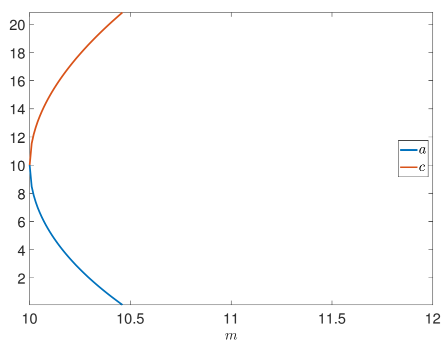
(e) Corresponding parameters for symmetric triangular distribution (a and b)



(f) Corresponding parameters for log-normal distribution (μ_τ and σ_τ)

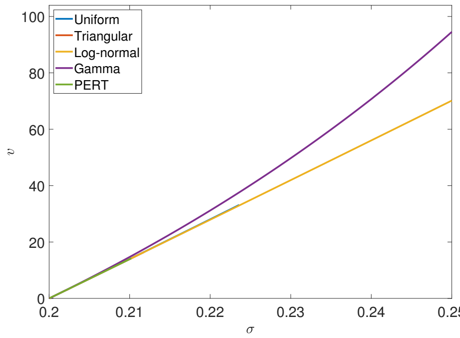


(g) Corresponding parameters for gamma distribution (k and θ)

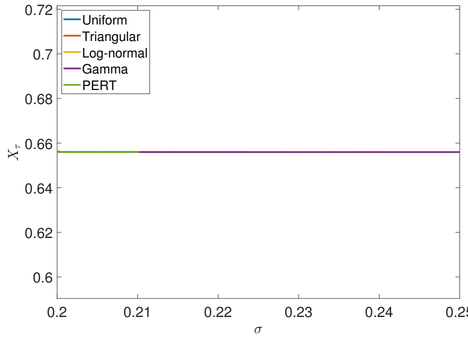


(h) Corresponding parameters for symmetric PERT distribution (a and c)

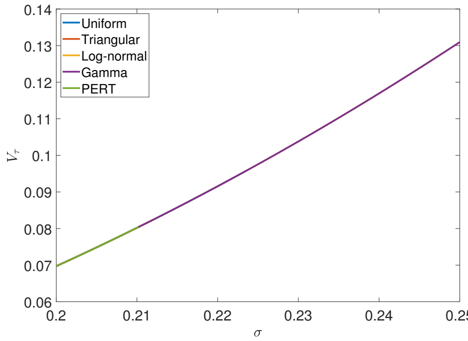
Figure 11: Examples of uncertainty equivalent with mean m for $\bar{\tau} = 10$



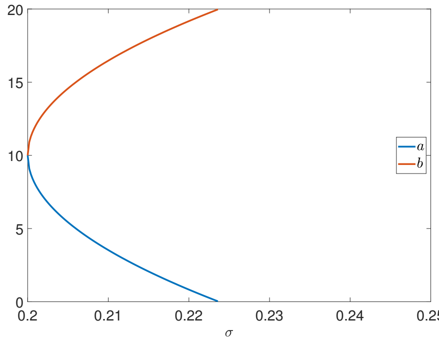
(a) Variance v that induces the same investment threshold for σ



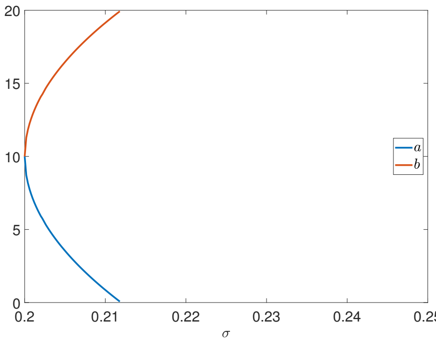
(b) Investment threshold for each distribution



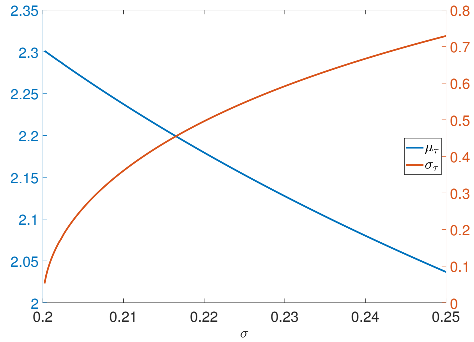
(c) Firm value for each distribution



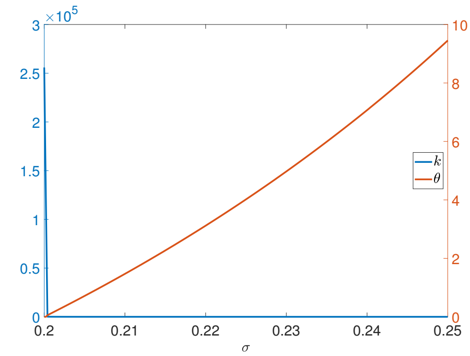
(d) Corresponding parameters for uniform distribution (a and b)



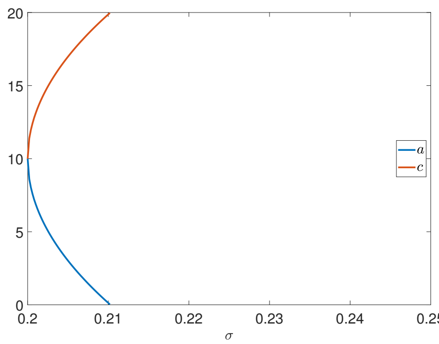
(e) Corresponding parameters for symmetric triangular distribution (a and b)



(f) Corresponding parameters for log-normal distribution (μ_τ and σ_τ)



(g) Corresponding parameters for gamma distribution (k and θ)



(h) Corresponding parameters for symmetric PERT distribution (a and c)

Figure 12: The level of variance v that induces the same optimal investment threshold for mean $m = 10$ and different levels of demand volatility σ with the base line $\sigma = 0.2$

6 Conclusion

This study investigated the impacts of uncertainty in time-to-build on corporate investment, clarifying *how much* the uncertainty accelerates investment and improves the firm value. We showed that there always exists a unique certainty equivalent of uncertain time-to-build, regardless of its distribution, and derived it in an analytic form. With this, firms can establish the optimal investment strategy with uncertain time-to-build in the form of the investment strategy that would have been adopted in the absence of such uncertainty. Even without knowing the exact distribution, the certainty equivalent can be approximated based on only a few moments, such as mean and variance, which enhances the practicality significantly. Meanwhile, we characterized the entropic risk measure of time-to-build and showed its positive impacts on investment. We also found the dual representation of the certainty equivalent of time-to-build based on relative entropy. Furthermore, we showed that for a given fixed time-to-build, there always exists an uncertainty equivalent. This enables firms to deduce the equivalent risk that its investment strategy, established without considering uncertainty in time-to-build, impliclty assumes. Lastly, we applied these arguments to representative probability distributions to demonstrate the practicality and analyzed the effects of the variance of time-to-build on investment. In particular, we derived the variance of time-to-build that offsets the negative impacts of demand uncertainty on investment.

Many problems still remain to be explored. For instance, we only considered a monopolistic firm for simplicity. Preemptive incentive due to market competition would significantly alter firms' optimal investment strategies, as well as the impacts of time-to-build on them. However, introducing competition would substantially reduce the tractability of the model. As noted in Section 5.4, another layer of decision-making that occurs after the investment yet before its completion directly depends on the remaining time-to-build. Thus, an analytic solution is unlikely, unless the underlying distribution is assumed to have the memoryless property. We also assumed an all-equity firm, but the impacts of uncertainty in time-to-build on financing and default decisions for a levered firm need to be addressed. Jeon (2021*a*) investigated the effects of uncertain time-to-build on a firm's investment and default decisions, showing that it can lead to a lower default probability compared to the case without time-to-build, mainly due to more conservative investment decisions. However, the study did not clarify the pure effects of uncertainty in time-to-build by comparing it with the case of fixed time-to-build. Future works need to address this question, though it will encounter the same technical difficulties mentioned above. More importantly, we assumed the independence between the demand shock and time-to-build for tractability. Follow-up research needs to test whether the same result holds without the independence assumption. Lastly, in spite of the difficulties of collecting data, empirical analysis needs to be carried out to test the theoretical results discussed in this paper. It is hoped that this study will serve as a platform for investigating these issues in the future.

A Proofs

A.1 Proof of Lemma 1

By the definition, $\tau_{n+1} = \tau_n + \epsilon_{n+1}$ where $\mathbb{E}[\epsilon_{n+1}|\tau_n] = 0$. Suppose that τ_n has a cumulative distribution function F_n for $n \geq 1$. Since $f(\tau) := e^{-(r-\mu)\tau}$ is a strictly convex function, Jensen's inequality ensures the following always holds for all $n \geq 0$:

$$\begin{aligned}\delta(\tau_{n+1}) &= \mathbb{E}[f(\tau_{n+1})] = \int f(\tau_{n+1})\mathrm{d}F_{n+1}(\tau_{n+1}) = \int \mathbb{E}[f(\tau_n + \epsilon_{n+1}|\tau_n)]\mathrm{d}F_n(\tau_n) \\ &> \int f(\mathbb{E}[\tau_n + \epsilon_{n+1}|\tau_n])\mathrm{d}F_n(\tau_n) = \int f(\tau_n)\mathrm{d}F_n(\tau_n) = \mathbb{E}[f(\tau_n)] = \delta(\tau_n). \end{aligned} \tag{33}$$

With $\delta(\tau_{n+1}) > \delta(\tau_n)$, it is straightforward that $X_{\tau_{n+1}} < X_{\tau_n}$ and $V_{\tau_{n+1}}(X) > V_{\tau_n}(X)$.

A.2 Proof of Proposition 1

By Lemma 1, $f(\mathbb{E}[\tau]) < \mathbb{E}[f(\tau)] = \delta(\tau)$ always holds, and $f(\tau)$ strictly decreases with τ. Thus, there exists a constant $\bar{\tau}_c(< \mathbb{E}[\tau])$ such that $\delta(\bar{\tau}_c) = \delta(\tau)$. The monotonicity of $f(\tau)$ ensures its uniqueness.

Meanwhile, the definition of $\delta(\tau)$ and that of the moment-generating function in (9) imply $\delta(\tau) = M_\tau(-(r-\mu))$, from which we obtain (8).

A.3 Proof of Corollary 1

Let us define $u(z) := K_\tau(z)/z$ for $z < 0$. It is straightforward that $u'(z) = v(z)/z^2$ where $v(z) := K'_\tau(z)z - K_\tau(z)$. Due to the convexity of the cumulant-generating function, we have $v'(z) = K''_\tau(z)z \leq 0$ for $z < 0$, which amounts to $v(z) \geq v(0) = 0$ for $z < 0$. Therefore, we obtain $u'(z) \geq 0$. Note that $\bar{\tau}_c = u(-(r-\mu))$, and thus, $\bar{\tau}$ increases with μ, and the independence with respect to σ is evident.

A.4 Proof of Corollary 2

Plugging (5) into (4), it is straightforward that $V_\tau(X) = A(X)(\delta(\tau))^\gamma$ where $A(X)$ is given by (11). Thus, $V_\tau(X) \geq \bar{X}$ is equivalent to $\delta(\tau) \geq (\bar{X}/A(X))^{1/\gamma}$. By definition, $\delta(\tau) = \delta(\bar{\tau}_c) = \exp(-(r-\mu)\bar{\tau}_c)$ and $\delta(\tau) = M_\tau(-(r-\mu)) = \exp(K_\tau(-(r-\mu)))$, which amounts to (10).

A.5 Proof of Corollary 3

Combining the cumulants $\kappa_1 = \mathbb{E}[\tau]$, $\kappa_2 = \mathbb{E}[(\tau - \mathbb{E}[\tau])^2]$, $\kappa_3 = \mathbb{E}[(\tau - \mathbb{E}[\tau])^3]$, and $\kappa_4 = \mathbb{E}[(\tau - \mathbb{E}[\tau])^4] - 3(\mathbb{E}[(\tau - \mathbb{E}[\tau])^2])^2$ with (8) and (9), we can easily obtain (12) through (14).

A.6 Proof of Corollary 4

Suppose $\tau_{n+1} = \tau_n + \epsilon_{n+1}$ with $\mathbb{E}[\epsilon_{n+1}|\tau_n] = 0$. Proposition 1 implies that for τ_n, there exists a unique constant $\bar{\tau}_{c,n} = \mathbb{E}[\tau_n] - c_n$ where $c_n > 0$ such that $\delta(\bar{\tau}_{n,c}) = \delta(\tau_n)$. By Lemma 1, $\delta(\tau_n) < \delta(\tau_{n+1})$, or equivalently, $f(\mathbb{E}[\tau_n] - c_n) < f(\mathbb{E}[\tau_{n+1}] - c_{n+1})$. Because $f(\tau)$ strictly decreases with τ, we have $c_{n+1} > c_n$, which implies that the uncertainty premium of time-to-build increases with its dispersion.

The approximation of the certainty equivalent in (13) and (14) decrease with v when the terms in the parentheses are positive.

A.7 Proof of Proposition 2

Suppose τ is a nonnegative random variable with mean m and variance v. Jensen's inequality ensures the following holds:

$$e^{-(r-\mu)m} \leq \mathbb{E}[e^{-(r-\mu)\tau}]. \tag{34}$$

The tightness of (34) can be shown as follows. Suppose τ follows a two-point distribution with possible outcomes of $m+n$ and $m-l(n)$ with probabilities vn^{-2} and $1-vn^{-2}$, respectively, where n is sufficiently large and $l(n)$ is chosen such that $\mathbb{E}[\tau] = m$ (i.e., $l(n) = (m+n)v/(n^2-v)$). Since $\lim_{n\to\infty} l(n) = 0$, the following holds:

$$\mathbb{E}[(\tau-m)^2] = n^2 \cdot \frac{v}{n^2} + (l(n))^2\Big(1-\frac{v}{n^2}\Big) \xrightarrow{n\to\infty} v, \tag{35}$$

$$\mathbb{E}[e^{-(r-\mu)\tau}] = e^{-(r-\mu)(m+n)}\frac{v}{n^2} + e^{-(r-\mu)(m-l(n))}\Big(1-\frac{v}{n^2}\Big) \xrightarrow{n\to\infty} e^{-(r-\mu)m}. \tag{36}$$

Meanwhile, let us define a quadratic function $g(\tau) := a\tau^2 + b\tau + 1$ where

$$a = \frac{1 - e^{-(r-\mu)\tau_0}(1+(r-\mu)\tau_0)}{\tau_0^2}, \tag{37}$$

$$b = \frac{-2 + e^{-(r-\mu)\tau_0}(2+(r-\mu)\tau_0)}{\tau_0}, \tag{38}$$

$$\tau_0 = m + \frac{v}{n}(>0). \tag{39}$$

Then, for $f(\tau) := e^{-(r-\mu)\tau}$, it is straightforward to show the following:

$$g(0) - f(0) = 0, \tag{40}$$

$$g(\tau_0) - f(\tau_0) = 0, \tag{41}$$

$$g'(\tau_0) - f'(\tau_0) = 0. \tag{42}$$

We can also show that $g''(\tau) - f''(\tau) = 2a - (r-\mu)^2 e^{-(r-\mu)\tau}$ is an increasing function of τ and that

$$g''(\tau_0) - f''(\tau_0) = \frac{2 - e^{-(r-\mu)\tau_0}\{2 + 2(r-\mu)\tau_0 + (r-\mu)^2\tau_0^2\}}{\tau_0^2} > 0. \tag{43}$$

The inequality in (43) holds because for $\tau > 0$,

$$h(\tau) := 2 - e^{-(r-\mu)\tau}\{2 + 2(r-\mu)\tau + (r-\mu)^2\tau^2\} > h(0) = 0 \tag{44}$$

since $h'(\tau) > 0$ for $\tau > 0$. From (40), (41), (42), and the monotonic increase of $g''(\tau) - f''(\tau)$, we can show that $g(\tau) - f(\tau)$ for $\tau \geq 0$ takes the minimum value of 0 at $\tau = 0$ and $\tau = \tau_0$, implying that $g(\tau) \geq f(\tau)$ for $\tau \geq 0$. Thus, we have

$$\mathbb{E}[e^{-(r-\mu)\tau}] = \mathbb{E}[f(\tau)] \leq \mathbb{E}[g(\tau)] = a(m^2+v) + bm + 1 = \frac{e^{-(r-\mu)(m+v/m)}m^2 + v}{m^2+v}. \tag{45}$$

The tightness of (45) can be shown as follows. Suppose τ follows a two-point distribution with possible outcomes of 0 and $m + v/m$ with probabilities $v/(m^2+v)$ and $m^2/(m^2+v)$, respectively. This satisfies $\mathbb{E}[\tau] = m$ and $\mathbb{E}[\tau^2] = v + m^2$, and $\mathbb{E}[e^{-(r-\mu)\tau}]$ coincides the right-hand side of (45).

By combining (34) and (45) and rewriting them in terms of the certainty equivalent in (8), we can obtain (15). The left-hand side of (15) is $-\ln(p(v))/(r-\mu)$ where $p(v) = (e^{-(r-\mu)(m+v/m)}m^2 + v)/(m^2+v)$, and it is straightforward to show $\partial p/\partial v = \{m^2(1-e^{-c}(1+c))\}/(m^2+v)^2$ where $c = (r-\mu)(m+v/m) > 0$. For $q(c) := e^{-c}(1+c)$, $\partial q/\partial c < 0$ and $q(0) = 1$, and thus, $q(c) < 1$ for $c > 0$. This implies $\partial p/\partial v > 0$, and thus, the left-hand side of (15) strictly decreases with v.

A.8 Proof of Proposition 3

Suppose that τ follows a two-point distribution with possible outcomes of n and $m - l(n)$ with probabilities $n^{-1.5}$ and $1 - n^{-1.5}$, respectively, where n is sufficiently large and $l(n)$ is chosen such that $\mathbb{E}[\tau] = m$ (i.e., $l(n) = (n-m)/(n^{1.5}-1)$). Since $\lim_{n\to\infty} l(n) = 0$, the following holds:

$$\mathbb{E}[\tau^2] = \frac{n^2}{n^{1.5}} + (m - l(n))^2\Big(1 - \frac{1}{n^{1.5}}\Big) \xrightarrow{n\to\infty} \infty, \tag{46}$$

$$\mathbb{E}[e^{-(r-\mu)\tau}] = \frac{e^{-(r-\mu)n}}{n^{1.5}} + e^{-(r-\mu)(m-l(n))}\Big(1 - \frac{1}{n^{1.5}}\Big) \xrightarrow{n\to\infty} e^{-(r-\mu)m}. \tag{47}$$

That is, as n increases, variance of τ increases, but its certainty equivalent also increases, converging to m. In other words, it is possible that uncertainty premium of time-to-build can decrease and converge to 0 as its variance increases.

A.9 Proof of Proposition 4

Given the definition in (16), it is straightforward to derive the following:

$$\rho_\theta(x) = \frac{K_x(-\theta)}{\theta}. \tag{48}$$

With (17), and (48), we can express the certainty equivalent of time-to-build in (8) as follows:

$$\bar{\tau}_c = -\rho_{r-\mu}(\tau). \tag{49}$$

Note that $\rho_{r-\mu}(\tau) \leq 0$ since $\tau \geq 0$ and $r > \mu$.

A.10 Proof of Proposition 5

A risk measure $\rho(x)$ is called a *convex risk measure* if it satisfies the following properties:[15]

$$\text{Monotonicity: If } x \leq y, \text{ then } \rho(x) \geq \rho(y). \tag{50}$$

$$\text{Translation invariance: If } z \in \mathbb{R}, \text{ then } \rho(x+z) = \rho(x) - z. \tag{51}$$

$$\text{Convexity: } \rho(\lambda x + (1-\lambda)y) \leq \lambda\rho(x) + (1-\lambda)\rho(y) \text{ for any } \lambda \in [0,1]. \tag{52}$$

It is well-known that any convex risk measure $\rho(x)$ has a dual representation:

$$\rho(x) = \sup_{\mathbb{Q}\in\mathcal{M}} \Big\{\mathbb{E}^{\mathbb{Q}}[-x] - \alpha(\mathbb{Q})\Big\}, \tag{53}$$

where $\mathcal{M}$ is the set of probability measures on Ω and $\alpha(\cdot)$ is a penalty function on $\mathcal{M}$. It is also known that for entropic risk measure $\rho_\theta(x)$, which is a convex risk measure, the penalty function is given by $H(\mathbb{Q}|\mathbb{P})/\theta$ (e.g., Föllmer and Schied (2002, 2016)):[16]

$$\rho_\theta(x) = \sup_{\mathbb{Q}\in\mathcal{M}(\mathbb{P})} \Big\{\mathbb{E}^{\mathbb{Q}}[-x] - \frac{1}{\theta}H(\mathbb{Q}|\mathbb{P})\Big\}. \tag{54}$$

Thus, the certainty equivalent of time-to-build in (49) can be represented as follows:

$$\bar{\tau}_c = -\sup_{\mathbb{Q}\in\mathcal{M}(\mathbb{P})} \Big\{\mathbb{E}^{\mathbb{Q}}[-\tau] - \frac{1}{r-\mu}H(\mathbb{Q}|\mathbb{P})\Big\} = \inf_{\mathbb{Q}\in\mathcal{M}(\mathbb{P})} \Big\{\mathbb{E}^{\mathbb{Q}}[\tau] + \frac{1}{r-\mu}H(\mathbb{Q}|\mathbb{P})\Big\}. \tag{55}$$

[15] A risk measure $\rho(x)$ is called a *coherent risk measure* if it satisfies subadditivity (i.e., $\rho(x+y) \leq \rho(x) + \rho(y)$) and positive homogeneity (i.e., if $\lambda \geq 0$, then $\rho(\lambda x) = \lambda\rho(x)$) instead of convexity in (52) (e.g., Artzner et al. (1999)).

[16] See Föllmer and Knispel (2011), Ahmadi-Javid (2012), and Pichler and Schlotter (2020) for the detailed illustration regarding entropy-based risk measures.

A.11 Proof of Proposition 6

Suppose τ_1 has possible outcomes of 0 and 2 with equal likelihood while τ_2 has possible outcomes of $1-\epsilon$, 1, and $1+\epsilon$ with equal likelihood for small $\epsilon > 0$. It is obvious that entropy of τ_2 is higher than that of τ_1 but the certainty equivalent of τ_2 is smaller than that of τ_1.

A.12 Proof of Proposition 7

By Lemma 1, $f(\bar{\tau}) = f(\mathbb{E}[\bar{\tau}+\epsilon]) < \mathbb{E}[f(\bar{\tau}+\epsilon)]$ where $\mathbb{E}[\epsilon] = 0$. Since $f(\tau)$ strictly decreases with τ, $\mathbb{E}[f(\bar{\tau}+\epsilon+u)] < \mathbb{E}[f(\bar{\tau}+\epsilon)]$ holds for a constant $u > 0$. Therefore, there always exists $\tau_u := \bar{\tau}+\epsilon+u$ such that $f(\bar{\tau}) = \mathbb{E}[f(\tau_u)]$, or equivalently, $\delta(\bar{\tau}) = \delta(\tau_u)$. This, combined with the definitions of $\delta(\bar{\tau})$ and the moment-generating function, amounts to (20).

A.13 Proof of Corollary 5

Suppose τ_u is the uncertainty equivalent of $\bar{\tau}$. That is, $\delta(\bar{\tau}) = \delta(\tau_u)$, or equivalently, $f(\bar{\tau}) = \mathbb{E}[f(\tau_u)] = \mathbb{E}[f(\bar{\tau}+\epsilon+u)]$ where $\mathbb{E}[\epsilon] = 0$ and $u > 0$ is a constant.

Meanwhile, suppose τ_u' is a mean-preserving spread of τ_u. That is, $\tau_u' = \tau_u + \epsilon'$ where $\mathbb{E}[\epsilon'|\tau_u] = 0$. By Lemma 1, $\mathbb{E}[f(\tau_u)] < \mathbb{E}[f(\tau_u')]$ always holds. Since $f(\tau)$ strictly decreases with τ, there always exists a constant $u' > 0$ such that $\mathbb{E}[f(\tau_u)] = \mathbb{E}[f(\tau_u'+u')]$ holds, which implies $\delta(\bar{\tau}) = \delta(\tau_u) = \delta(\hat{\tau}_u)$ where $\hat{\tau}_u := \tau_u' + u'$. Namely, both τ_u and $\hat{\tau}_u$ are the uncertainty equivalents of $\bar{\tau}$. By the definition, $\mathbb{E}[\tau_u] = \bar{\tau}+u$ and $\mathbb{E}[\hat{\tau}_u] = \bar{\tau}+u+u'$, which completes the proof.

A.14 Proof of Corollary 6

By Proposition 7, for any $\bar{\tau} \geq 0$, there always exists $\tau_u := \bar{\tau}+\epsilon+u$ where $\mathbb{E}[\epsilon] = 0$ and $u > 0$ is a constant such that $f(\bar{\tau}) = \mathbb{E}[f(\tau_u)]$. Because $f(\tau)$ strictly decreases with τ, there always exists a constant $w \in (0, u)$ such that $\mathbb{E}[f(\tau_u)] < \mathbb{E}[f(\tau_w)]$ where $\tau_w := \bar{\tau}+\epsilon+w$, which implies $\delta(\bar{\tau}) = \delta(\tau_u) < \delta(\tau_w)$ and $\mathbb{E}[\tau_w] = \bar{\tau}+w$.

A.15 Proof of Proposition 8

Suppose τ follows a uniform distribution on $[a-c, b+c]$ where $0 < c < a$. As c increases, its variance increases while its mean remains the same. With (22), it is straightforward to show that $\partial M_\tau(t)/\partial c > 0$. This, combined with (8), implies that an increase of c, which increases the variance, leads to a decrease of $\bar{\tau}_c$, and thus, an increase of $\mathbb{E}[\tau] - \bar{\tau}_c$.

As can be seen from Appendix B, differential entropy of uniform distribution increases if and only if its variance increases, which completes the proof.

A.16 Proof of Proposition 9

Suppose τ follows a symmetric triangular distribution on $[a-d, b+d]$ with its mode $c = (a+b)/2$. It is obvious that its variance increases while its mean remains the same when d increases. With (24), one can easily show that $\partial M_\tau(t)/\partial d > 0$. That is, an increase of d, which increases its variance, results in a decrease of $\bar{\tau}_c$, and thus, an increase of $\mathbb{E}[\tau] - \bar{\tau}_c$.

It is straightforward to show from Appendix B that differential entropy of symmetric triangular distribution increases if and only if its variance increases, which completes the proof.

B The characteristics of probability distributions

Distribution	Uniform	Triangular	Log-normal	Exponential	Gamma	Scaled beta
Parameters	$0 \leq a < b$	$0 \leq a \leq c \leq b$ and $a < b$	μ_τ and $\sigma_\tau > 0$	$\lambda \in (0, \infty)$	$k, \theta > 0$	$0 \leq a < c$ and $\alpha, \beta > 0$
Support	$[a, b]$	$[a, b]$	$(0, \infty)$	$[0, \infty)$	$(0, \infty)$	$[a, c]$
Mean	$\frac{a+b}{2}$	$\frac{a+b+c}{3}$	$e^{\mu_\tau + \sigma_\tau^2/2}$	$\frac{1}{\lambda}$	$k\theta$	$\frac{\alpha c+\beta a}{\alpha+\beta}$
Variance	$\frac{(b-a)^2}{12}$	$\frac{a^2+b^2+c^2-ab-bc-ca}{18}$	$e^{2\mu_\tau+\sigma_\tau^2}(e^{\sigma_\tau^2}-1)$	$\frac{1}{\lambda^2}$	$k\theta^2$	$\frac{\alpha\beta(c-a)^2}{(\alpha+\beta)^2(\alpha+\beta+1)}$
Skewness	0	$\frac{\sqrt{2}(a+b-2c)(2a-b-c)(a-2b+c)}{5(a^2+b^2+c^2-ab-bc-ca)^{\frac{3}{2}}}$	$\sqrt{e^{\sigma_\tau^2}-1}(e^{\sigma_\tau^2}+2)$	2	$\frac{2}{\sqrt{k}}$	$\frac{2(\beta-\alpha)\sqrt{\alpha+\beta+1}}{(\alpha+\beta+2)\sqrt{\alpha\beta}}$
Excess kurtosis	$-\frac{6}{5}$	$\frac{3}{5}$	$e^{4\sigma_\tau^2}+2e^{3\sigma_\tau^2}$ $+3e^{2\sigma_\tau^2}-6$	6	$\frac{6}{k}$	$\frac{6[(\alpha-\beta)^2(\alpha+\beta+1)-\alpha\beta(\alpha+\beta+2)]}{\alpha\beta(\alpha+\beta+2)(\alpha+\beta+3)}$
Differential entropy	$\ln(b-a)$	$\frac{1}{2}+\ln\frac{b-a}{2}$	$\mu_\tau + \ln(2\pi e \sigma_\tau^2)/2$	$1-\ln\lambda$	$\ln(\theta\Gamma(k))$ $+(1-k)\psi(k)+k$ where $\psi(\cdot)$ is a digamma function	$\ln((c-a)B(\alpha,\beta))$ $-(\alpha-1)\psi(\alpha)$ $-(\beta-1)\psi(\beta)$ $+(\alpha+\beta-2)\psi(\alpha+\beta)$

Table 2: Moments and differential entropy of representative probability distributions

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