Intransitive indifference with direction-dependent sensitivity

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Abstract

Much of the literature has argued that intransitive indifference is more likely to occur when alternatives have conflicting criteria than when one alternative dominates the other. To study such a phenomenon, we first axiomatize the essentially unique *expected utility with direction-dependent sensitivity* (EUDS) representation on the set of lotteries which extend the classic models of imperfect discrimination (e.g., Fishburn, 1970a; Luce, 1956) to enable a direction-dependent just-noticeable difference function. The key axioms in this characterization are *irresolute independence*, wherein mixing alternatives with another may (or may not) alter a strict preference for indifference while preserving indifference, and *strict preference convexity*, which obtains the convexity of strict upper and lower contour sets. Thereafter, we indicate that intransitive indifference in EUDS can be divided into that caused by *imperfect discrimination* (Fishburn, 1970a; Luce, 1956) and that caused by *uncertainty about tastes* (Dubra et al., 2004) by considering the transitive core (Nishimura, 2018) of EUDS. We also obtain two special cases of our model, i.e., one-directional and categorical sensitivity.

Keywords: intransitive indifference, direction-dependence, incomplete preference, transitive core

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1 Introduction

1.1 Background and outline

Decision theory usually assumes the *transitivity* of the indifference relation, as well as that of strict preference. However, many authors have argued that the transitivity of indifference may be systematically *violated* (e.g., Armstrong, 1939, 1948, 1950; Fishburn, 1970a, 1970b; Luce, 1956) because the decision maker (DM) is often *insensitive* to small differences between utilities generated by compared alternatives.[1] The *semiorder* model (Luce, 1956) and the *interval order* model (Fishburn, 1970a) attempt to explain such an *intransitive indifference* phenomenon by the following representation of a strict preference $\succ$ on the set X of prizes,

$$x \succ y \Longleftrightarrow u(x) > u(y) + \delta(y) \tag{1}$$

for all x, $y \in X$. Here, u denotes the *utility function*, such that $x \succ y$ implies that $u(x) > u(y)$, but not necessarily the converse, and δ denotes the *just-noticeable difference (JND) function* that determines the threshold of utility difference between alternatives within which the alternatives are indistinguishable.

As (1) indicates, the semiorder/interval order approach assumes that the JND function δ only depends on the inferior (or reference) alternative y and is independent of the superior alternative x. However, many studies have reported that intransitive indifference may be reasonably *direction-dependent*, i.e., a dominating alternative is strictly preferred to a dominated one, even if the alternatives differ only by a small margin, whereas alternatives are more likely to be indifferent if neither of them dominates the other (Armstrong, 1939, 1948, 1950; Fishburn, 1968, 1970b). (We present the motivating examples in the next section.) These results suggest that δ should depend on *both* the superior and inferior alternatives (i.e., x and y in (1), respectively) rather than only the inferior alternative y.

The objective of this study is to extend the semiorder/interval order approach to accommodate the latter case. By considering the set $\mathcal{P}(X)$ of lotteries (i.e., the Borel probability measures on X) as the domain of choice, our main theorems axiomatize an essentially unique preference representation, which is referred to as *expected utility with direction-dependent sensitivity* (EUDS),

$$p \succ q \Longleftrightarrow \int_X u(x)dp > \int_X u(x)dq + \delta(p, q)$$

for all p, $q \in \mathcal{P}(X)$ (Theorems 1 and 2). As anticipated, δ is a direction-dependent JND function, i.e., $\delta(p, q)$ is constant with respect to any simultaneous shift of p and q.

[1] Intransitive indifference is also relevant to the more recent literature on choice under risk (Rubinstein, 1988), choice under uncertainty (Lehrer and Teper, 2011; Giarlotta and Greco, 2013), and critical cost-efficiency index (Dziewulski, 2020).

The key axioms in this characterization are twofold: first, *irresolute independence* allows a strict preference to be altered into indifference when two alternatives (p and q) are mixed with another (r), whereas it disallows the alteration of indifference, which is the driving force to derive a direction-dependent JND function. Second, *strict preference convexity* entails that the mixture of superior alternatives is strictly preferred to the mixture of inferior alternatives, and it obtains convex strict upper and lower contour sets.

Next, readers may surmise that our intransitive indifference model could only be a relabeling of indecisiveness (i.e., inability to rank alternatives), which has been extensively studied in the recent literature on incomplete preference (Bewley, 1986; Dubra et al., 2004; Ok et al., 2012), because the indecisiveness relation in the latter approach is generally intransitive.[2] This conjecture is only half true: Theorem 3 indicates that EUDS has a strong connection to an incomplete preference model (i.e., an indecisiveness model), referred to as *expected multi-utility* (EMU) (Dubra et al., 2004), by considering the transitive core (Nishimura, 2018). However, Theorem 4 associates indifference between the alternatives p and q in EUDS with indecisiveness in the corresponding EMU, *only if* $\delta(p, q)$ takes an *infinite* value, whereas such a connection does not exist if $\delta(p, q)$ is finite.

Put differently, EUDS admits two distinct types of intransitive indifference, i.e., those caused by *imperfect discrimination* (Fishburn, 1970a; Luce, 1956) and *uncertainty about tastes* (Dubra et al., 2004), which are associated with the finite and infinite values of the JND function, respectively. One of our contributions is to resolve the abovementioned confusion between intransitive indifference and indecisiveness, by constructing a model that can distinguish between both types of intransitive indifference, thereby accurately characterizing what type of intransitive indifference can be relevant to indecisiveness.

Finally, we focus on the following two special cases of EUDS to clarify the dichotomy of intransitive indifference: the first is *one-directional sensitivity*, which obtains a constant JND function δ and the counterpart of the semiorder approach (Theorem 5). The second special case is *categorical sensitivity*, i.e., the JND function δ only takes a value of zero or infinity, which obtains strict preference or indifference depending on the direction of comparison (Theorem 6). The latter case is also reminiscent of Lehrer and Teper's (2011) justifiability model, and we discuss the connection between this special case and their model (Corollary 2).

The remainder of this paper is organized as follows: in the next subsection, we present motivating examples. The basic framework and axioms that we focus on in this study are

[2] A similar view is offered in the literature. For example, Kreps (2013, p. 20) concludes that the indifference relation can possibly be intransitive if it is identified with indecisiveness. Galaabaatar and Karni (2013) also argue that defining the indifference relation by the absence of strict preference may overlook the distinction between intransitive indifference and indecisiveness.

proposed in Section 2. In Section 3, the main representation theorem and the uniqueness result are stated. We relate EUDS to its counterpart—EMU—in Section 4 and discuss special cases of EUDS in Section 5. The related literature is reviewed in Section 6 and concluding remarks are made in Section 7.

1.2 Motivating examples

In this section, we examine the following two examples, both of which are motivated by the literature and highlight the main focus of this study.

Example 1 (Choice over multiattribute goods) Consider the following model of choice over multiattribute goods.[3] Let $X = X_1 \times X_2$ be the set of goods, where X_1, $X_2 \subseteq \Re$ be the sets of attribute values. Assume that for $x = (x_1, x_2)$, $x' = (x'_1, x'_2)$, $y = (y_1, y_2) \in X$, good x dominates x' by a small margin (i.e., $x_1 = x'_1 + \epsilon_1$ and $x_2 = x'_2 + \epsilon_2$ for sufficiently small ϵ_1 and ϵ_2) and neither x and y nor x' and y, dominate the other (i.e., $x_1 > y_1$, $x_2 < y_2$, $x'_1 > y_1$ and $x'_2 < y_2$, without loss of generality). The DM may readily exhibit a strict preference for good x over x' because she recognizes the former alternative dominating the latter; however, she may still be indifferent both between x and y as well as between x' and y because she finds it difficult to discriminate between the alternatives owing to conflicting attribute values.

Example 2 (Choice over lotteries) Consider the following model of choice over lotteries.[4] Let $X \subset \Re$ be the set of prizes and $\mathcal{P}(X)$ be the set of lotteries (Borel probability measures on X), endowed with the standard mixture operation. For some p, q, $r \in \mathcal{P}(X)$ and $x \in X$, assume that $p(x+1) = 1$, $q(x) = 1$, $r(x-50) = .5$, and $r(x+100) = .5$. The DM may readily exhibit a strict preference for p over q because p first-order stochastically dominates q, despite a small difference (which is equal to one) between the resulting prizes; however, she may still be indifferent both between p and r, as well as between q and r, which obtains intransitive indifference, because there is no evident rationale to discriminate between alternatives in each pair.

There are two crucial observations from the examples above: first, the DM's inability to discriminate between alternatives generally depends on the *direction* of comparison between

[3]This example is motivated by several studies, including Armstrong (1948), Fishburn (1970b), and May (1954), which associate intransitive preference or intransitive indifference with conflicting attribute values. A similar discussion can be found in Tversky and Shafir (1992), in the context of choice deferral.

[4]A similar example was first considered by Fishburn (1968). First-order stochastic dominance is also used as a criterion of "unambiguously better" probability distribution in the literature on risky choice (for an introductory discussion, see, e.g., Mas-Colell et al., 1995).

alternatives, which cannot be explained by the semiorder/interval order approach. In particular, intransitive indifference would seem less likely to occur for choice between dominating and dominated alternatives (in an appropriate sense), whereas it would be more likely for choice between mutually undominated alternatives.[5] This view conforms with many theoretical and experimental studies (Armstrong, 1939, 1948, 1950; Fishburn, 1968, 1970b; Tversky, 1969; Tversky and Shafir, 1992) and is reminiscent of incomplete preferences resulting from conflicting beliefs or tastes (Bewley, 1986; Dubra et al., 2004; Ok et al., 2012). Second, the strict upper and strict lower contour sets in these examples can naturally be assumed as *convex*, which is often assumed in the classical demand theory and decision theory.[6] Suppose in the multiattribute good choice example that both alternatives z and z' dominate w. Then, the DM would readily exhibit a strict preference for a mixture of z and z' to w because the former alternative also dominates the latter. Similarly, if both p and p' first-order stochastically dominate q, in the second example, a mixture of p and p' would also first-order stochastically dominate q, which renders the resulting lottery strictly preferred to q. Accordingly, both examples are consistent with convex strict upper and strict lower contour sets.

2 Preliminaries and axioms

Let X be a compact metric space of prizes. The set of lotteries (Borel probability measures) on X endowed with the weak convergence topology is denoted by $\mathcal{P}(X)$, and the set of all continuous functions on X, which is topologized by the sup-norm, is denoted by $C(X)$. Let $\mathrm{ca}(X)$ be the set of all Borel signed measures on X.[7] Because X is compact, $\mathrm{ca}(X)$ (endowed with the total variation norm) is isometrically isomorphic to the topological dual of $C(X)$ (endowed with the sup norm). Accordingly, we regard $\mathrm{ca}(X)$ as being endowed with the weak*-topology, which generates the standard weak convergence topology when it is restricted to $\mathcal{P}(X)$.

We refer to generic elements $p, q, r, \ldots \in \mathcal{P}(X)$ as *alternatives* and assume a binary relation $\succ$ over $\mathcal{P}(X)$, which we refer to as the *strict preference*, i.e., $p \succ q$ implies that alternative p is *definitely* preferred to q. We also define the indifference relation $\sim$ such that for all $p, q \in \mathcal{P}(X)$, $p \sim q$ if neither $p \succ q$ nor $q \succ p$. Accordingly, we interpret $p \sim q$ as alternatives p and q that are unable to be discriminated, rather than being found equally desirable. We define the

[5] Indeed, May (1954) suggests that "... it [his experiment] does not prove that individual patterns are always intransitive. It does, however, suggest that where choice depends on conflicting criteria, preference patterns *may* be intransitive unless one criterion dominates" (p. 7).

[6] For an introductory discussion on convexity in the classical demand theory, see, e.g., Mas-Colell et al. (1995).

[7] Note that $\mathrm{ca}(X) = \mathrm{span}(\mathcal{P}(X))$.

weak preference $\succsim$, so that for all p, $q \in \mathcal{P}(X)$, $p \succsim q$ if $p \succ q$ or $p \sim q$, and thus, $\succsim$ is complete. In the following analysis, we use the symbols $\succsim$ and $(\succ, \sim)$ interchangeably. Next, for all p, $q \in \mathcal{P}(X)$ and $\alpha \in [0,1]$, $\alpha p + (1-\alpha)q$ denotes the mixture of p and q with probability α, i.e., a randomization between probability distributions generated by p and q. Finally, we define the strict upper and lower contour sets $U_p(\succ)$ and $L_p(\succ)$, given $p \in \mathcal{P}(X)$, respectively, by $U_p(\succ) \equiv \{q \in \mathcal{P}(X) : q \succ p\}$ and $L_p(\succ) \equiv \{q \in \mathcal{P}(X) : p \succ q\}$.

We impose the following axioms on the pair of binary relations $(\succ, \sim)$.

Axiom 1 (Strict partial order) $\succ$ is irreflexive and transitive.

This axiom states that $\succ$ is a *strict partial order* (Suppes, 1957), which is more general than the classes of intransitive indifference models considered in the literature, such as a semiorder (Luce, 1956) and an interval order (Fishburn, 1970a). Intuitively, strict preference $\succ$ satisfies transitivity because it implies the definite preference of the DM, as we mentioned earlier. Axiom 1 also implies that indifference $\sim$ is reflexive, i.e., $p \sim p$ for all $p \in \mathcal{P}(X)$; however, $\sim$ is *not* necessarily transitive because indifference in our model denotes the absence of a strict preference rather than equal desirability. Accordingly, the DM may still strictly prefer alternative p to r even when she is indifferent between p and q, as well as between q and r.

Our second axiom is the standard form of continuity, which is stated with respect to the strict preference $\succ$.

Axiom 2 (Archimedean continuity) For all p, q, $r \in \mathcal{P}(X)$, $\{\alpha \in [0,1] : \alpha p + (1-\alpha)q \succ r\}$ and $\{\alpha \in [0,1] : \alpha p + (1-\alpha)q \succ r\}$ are open.

The continuity axiom assumed here is weaker than Dubra et al.'s (2004), although the proof of our main theorem applies an argument similar to theirs.[8] This is because our utility function u only carries a one-directional implication (i.e., for all p, $q \in \mathcal{P}(X)$, $p \succ q$ implies that $\int_X u(x)dp > \int_X u(x)dq$, but $\int_X u(x)dp > \int_X u(x)dq$ does *not* imply that $p \succ q$), and its existence can be proven without assuming a stronger continuity axiom. Put differently, Axiom 2 is mainly used to construct the JND function δ.

Next, we need certain forms of independence to derive an expected utility function. However, it is known that the standard independence axiom may be too strong to retain intransitive indifference.[9] Accordingly, we consider the following three forms of independence weakening:

[8]Their continuity axiom states that for all sequences of lotteries $(p_n)_{n=1}^{\infty}$ and $(q_n)_{n=1}^{\infty}$, $p_n \succsim q_n$ for all n implies that $\lim_{n\to\infty} p_n \succsim \lim_{n\to\infty} q_n$.

[9]As Fishburn (1968) indicates, the standard independence axiom imposed on a semiorder on the set of

the first one allows a strict preference to be altered into indifference by mixing two alternatives with another one, while it preserves indifference.

Axiom 3 (Irresolute independence) For all p, q, $r \in \mathcal{P}(X)$ and $\alpha \in (0,1)$, the following statements hold:

(a) $p \succ q$ implies $\alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)r$;

(b) $p \sim q$ implies $\alpha p + (1-\alpha)r \sim \alpha q + (1-\alpha)r$.

Axiom 3 is equivalent to the standard independence axiom, except that a *weak* preference $\alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)r$, instead of a strict preference, is assumed in the latter half of condition (a). To explain its implication, suppose that alternative p is strictly preferred to q and these alternatives are mixed with a third alternative r. The first case covered by Axiom 3(a) admits a strict preference for alternative $p' \equiv \alpha p + (1-\alpha)r$ to $q' \equiv \alpha q + (1-\alpha)r$ because the DM is entirely definite about the strict preference for alternative p to q, even after mixing with r. However, the second case assumes *indifference* between p' and q' because mixing alternatives with alternative r may obscure the strict preference for p to q. In contrast, condition (b) confirms indifference between p' and q' once indifference between p and q has occurred, because the latter condition assumes the indistinguishability of the alternatives.

Note that we obtain $p' - q' = \alpha(p - q)$ by definition, i.e., mixing alternatives p and q with alternative r retains the direction of the signed measure (i.e., $p - q$) generated by them, while reducing the distance between the alternatives. Accordingly, Axiom 3 offers a behavioral implication for direction-dependent sensitivity by characterizing the discrimination between alternatives in a certain direction.

The second weakening of independence elaborates the structure of strict preference.

Axiom 4 (Strict preference convexity) For all p, q, p', $q' \in \mathcal{P}(X)$ and $\alpha \in [0,1]$, $p \succ q$ and $p' \succ q'$ imply $\alpha p + (1-\alpha)p' \succ \alpha q + (1-\alpha)q'$.

Axiom 4 renders the mixture of strictly preferred alternatives (i.e., p and p') strictly preferred to the mixture of strictly less preferred alternatives (i.e., q and q'), which implies that the DM regards hedging between the former alternatives as more valuable than that between the latter. Letting $q = q'$ in Axiom 4 derives the convexity of strict upper contour sets $U_p(\succ)$ for all $p \in \mathcal{P}(X)$, which carries an implication similar to that of convex indifference curves in the

lotteries eliminates intransitivity of indifference, which results in a weak order.

classical demand theory wherein diversification is valuable.[10] Likewise, letting $p = q'$ obtains the convexity of strict lower contour sets $L_p(\succ)$ for all $p \in \mathcal{P}(X)$.

Axiom 4 is implied by the standard independence axiom. However, we still need this axiom because our irresolute independence axiom only obtains a *weak* preference between the mixtures, i.e., we may have $\alpha p + (1-\alpha)p' \sim \alpha q + (1-\alpha)q'$, even if $p \succ q$ and $p' \succ q'$, and thus, we need to clarify that the strict preference holds in this case. Through a motivation similar to ours, Fishburn (1968) and Nakamura (1988) also consider Axiom 4 as a weakening of independence to characterize intransitive indifference.

The final weakening of independence establishes a more elaborate structure for the indifference relation.

Axiom 5 (Balanced indifference) For all p, q, p', $q' \in \mathcal{P}(X)$, if $p \sim q$ and $\frac{1}{2}p + \frac{1}{2}p' = \frac{1}{2}q + \frac{1}{2}q'$, then $p' \sim q'$.

The intuition behind this axiom is as follows: suppose that the DM is unable to discriminate between p and q, i.e., $p \sim q$. Then, $\frac{1}{2}p + \frac{1}{2}p' = \frac{1}{2}q + \frac{1}{2}q'$ implies that the change from p to q can be precisely compensated by the change from p' to q'. Axiom 5 states that for such p, q, p', and q', the DM should also naturally regard p' and q' as indistinguishable, which obtains $p' \sim q'$.

Because $\frac{1}{2}p + \frac{1}{2}p' = \frac{1}{2}q + \frac{1}{2}q'$ is equivalent to $p - q = p' - q'$ (the latter equation is between the signed measures, i.e., $p - q$, $p' - q' \in \mathrm{ca}(X)$), Axiom 5 renders the indifference relation invariant with respect to a parallel shift, i.e., $p \sim q$ and $p - q = p' - q'$ imply that $p' \sim q'$. The latter condition also entails that strict upper and strict lower contour sets given alternative p, i.e., $U_p(\succ)$ and $L_p(\succ)$, are symmetric.[11]

This axiom is similar to those assumed in the literature to obtain an additively separable preference representation (e.g., Hyogo, 2007; Karni, 2004); however, our approach differs from the latter approach in the following two respects: first, we use Axiom 5 to specify the indifference relation (and thus, the JND function) rather than the additive separability of preference representation as they do. Second, unlike their approach, we require *equivalence*, not indifference, between $\frac{1}{2}p + \frac{1}{2}q'$ and $\frac{1}{2}q + \frac{1}{2}p'$ because our model generally admits thick indifference curves, and thus, we need to precisely measure the compensation effect on the indifference relation.

Finally, our final axiom is nontriviality, i.e., there exists at least one alternative pair, one of

[10] We should note, however, that Axiom 4 only implies the convexity of *strict* upper contour sets whereas the classical demand theory usually assumes the convexity of *weak* upper contour sets, which can be translated into our framework that $p \succsim r$ and $q \succsim r$ entails that $\alpha p + (1-\alpha)q \succsim r$ for all p, q, $r \in \mathcal{P}(X)$ and $\alpha \in (0,1)$. This is because our model allows for $r \succ \alpha p + (1-\alpha)q$, even if $p \sim r$ and $q \sim r$, to accommodate intransitive indifference.

[11] By the definition of indifference, $p \succ q$ and $p - q = q - r$ imply that $q \succ r$, for all p, q, $r \in \mathcal{P}(X)$.

which is strictly preferred to the other. We omit the explanation for this axiom because it is standard.

Axiom 6 (Nontriviality) There exist p, $q \in \mathcal{P}(X)$ such that $p \succ q$.

3 Main theorem

In this section, we state our main theorem. Thus, we first define two components of our preference representation—the *utility function* and the *JND function*.

First, a utility function partly represents the strict preference as follows:

Definition 1 (Utility function) For the strict preference $\succ$, a function $u \in C(X)$ is referred to as the *utility function of* $\succ$ if for all p, $q \in \mathcal{P}(X)$, $p \succ q$ implies that $\int_X u(x)dp > \int_X u(x)dq$.

Definition 1 is an extension of Luce's (1956) utility function to risky alternatives. In particular, we do *not* assume that if $\int_X u(x)dp > \int_X u(x)dq$, $p \succ q$, unlike a utility function representing a weak order.

Next, a JND function denotes the minimum utility difference that allows the DM to discriminate between alternatives. Let $H(u) = \{(p, q) : p, q \in \mathcal{P}(X), \int_X u(x)dp \geq \int_X u(x)dq\}$, i.e., $H(u)$ is the set of alternative pairs (p, q), wherein p has a higher expected utility than q, given a utility function u.

Definition 2 (JND function) For a utility function $u \in C(X)$, a function $\delta : \mathcal{P}(X) \times \mathcal{P}(X) \to \Re_+$ is referred to as the *JND function given* u if it satisfies the following conditions:

(a) $\delta(p, q) = \delta(p', q')$ for all p, q, p', $q' \in \mathcal{P}(X)$ and $\lambda > 0$ such that $p - q = \lambda(p' - q')$;

(b) $\delta(\alpha p + (1 - \alpha)p', q) \leq \alpha\delta(p, q) + (1 - \alpha)\delta(p', q)$ for all p, p', $q \in \mathcal{P}(X)$ and $\alpha \in [0, 1]$;

(c) $\delta(p, q) = +\infty$, if $(p, q) \notin H(u)$ or $\delta(p, q) \geq \sup_{p,q \in \mathcal{P}(X)} \left(\int_X u(x)dp - \int_X u(x)dq\right)$.

Condition (a) denotes the *direction-dependence* of the JND function δ, i.e., for all p, q, p', q' $\in \mathcal{P}(X)$, $\delta(p, q) = \delta(p', q')$ whenever the signed measure $p - q$ parallels $p' - q'$. Next, condition (b) implies that for an arbitrary $q \in \mathcal{P}(X)$, $\delta(\cdot, q)$ is concave, which follows from Definition 3 below that strict upper contour sets are convex. Finally, condition (c) states that δ takes an infinite value for alternative pairs (p, q) if p's expected utility is lower than q's or δ exceeds the supremum of the possible utility difference; accordingly, the JND function is well-defined.

Now, integrating the ideas explained above, we indicate our preference representation.

Definition 3 (Expected utility with direction-dependent sensitivity) For a utility function u and a JND function δ, the function pair (u, δ) is referred to as the EUDS representation of $(\succ, \sim)$ if it satisfies the following conditions:

(a) for all p, $q \in \mathcal{P}(X)$, $p \succ q$ if and only if $\int_X u(x)dp > \int_X u(x)dq + \delta(p, q)$;

(b) for all p, $q \in \mathcal{P}(X)$, $p \sim q$ if and only if $\int_X u(x)dq - \delta(q, p) \leq \int_X u(x)dp \leq \int_X u(x)dq + \delta(p, q)$.

Definition 3 implies that both weak and strict preferences can be specified by the *combination* of utility and sensitivity functions: condition (a) ensures that the strict preference for p over q holds if and only if the difference in expected utility, $\int_X u(x)dp - \int_X u(x)dq$, is greater than the value $\delta(p, q)$ of the JND function, in which case the DM can successfully discriminate between alternatives p and q. In contrast, condition (b) implies that p and q are indifferent if and only if $\int_X u(x)dp - \int_X u(x)dq$ is within the range of $-\delta(q, p)$ and $\delta(p, q)$.

This definition evidently indicates that EUDS is direction-dependent, i.e., for alternatives p and q, whether a strict preference or indifference holds depends not only on the difference in the expected utility between p and q but also on the direction of the signed measure $p - q$. The latter property embodies intransitive indifference that we discuss in the motivating examples in the Introduction.

The next theorem is the main result of this study.

Theorem 1 The following statements are equivalent.

(a) Preference pair $(\succ, \sim)$ satisfies Axioms 1–6.

(b) Preference pair $(\succ, \sim)$ admits an EUDS representation (u, δ).

A proof sketch is presented as follows (for detailed proof, see the Appendix): first, we define $D(\succ) \equiv \mathrm{cl}(\{\lambda(p - q) : p, q \in \mathcal{P}(X),\ \lambda > 0,\ p \succ q\}) \subseteq \mathrm{ca}(X)$, i.e., $D(\succ)$ is the closure of the cone generated by signed measures $p - q$, wherein p is strictly preferred to q. Irresolute independence, strict preference convexity, and other axioms follow that $D(\succ)$ is a closed convex cone, $p \succ q$ implies that $p - q \in D(\succ)$, and $p - q \in D(\succ)$ implies that $p \succsim q$, i.e., all the conditions run parallel (Lemma 1). Next, we define $\mathcal{U} \equiv \left\{u \in C(X) : \int_X u(x)d\mu \geq 0 \text{ for all } \mu \in D(\succ)\right\}$, which is a nonempty closed convex cone in $C(X)$. Based on the Hahn–Banach separation theorem, $\int_X u(x)dp \geq \int_X u(x)dq$ for all $u \in \mathcal{U}$ implies that $p - q \in D(\succ)$, and thus, $p \succsim q$ (Lemma 2). Now, take any $u \in \mathcal{U}$, then u serves as a utility function because $\int_X u(x)dp > \int_X u(x)dq$ implies that $p \succsim q$. Finally, we derive the JND function δ with the help of irresolute independence and Archimedean continuity (Lemma 3), which concludes the proof.

Next, we state our uniqueness result. To this end, we define an operator $\langle \cdot \rangle : 2^{C(X)} \to 2^{C(X)}$ as $\langle \mathcal{V} \rangle \equiv \mathrm{cl}(\mathrm{cone}(\mathcal{V}) + \{\theta \mathbf{1}_X\}_{\theta \in \Re})$ for all $\mathcal{V} \subseteq C(X)$, following Dubra et al.'s (2004) approach. The next theorem indicates that the set of possible utility functions, which we refer to as $\mathcal{U} \subseteq C(X)$, is unique under this operation. We write δ_u to denote the JND function given a $u \in C(X)$, wherein EUDS (u, δ_u) represents $\succsim$.

Theorem 2 There exists a closed and convex $\mathcal{U} \subseteq C(X)$, such that $(\succ, \sim)$ admits an EUDS representation (u, δ) if and only if $u \in \langle \mathcal{U} \rangle$. Moreover, for each $u \in \langle \mathcal{U} \rangle$, δ_u is unique.

This theorem indicates that there may be multiple utility functions u (and the corresponding JND functions δ_u) that represent the preference, but the set $\mathcal{U}$ of such utility functions is essentially unique. This is because the indifference curves in our model are generally thick, and they admit multiple hyperplanes that separate strict upper and lower contour sets, and each $u \in \mathcal{U}$ corresponds to such hyperplanes.

Note that if $\mathcal{U}$ is a singleton, i.e., $\mathcal{U} = \{u\}$ for some $u \in C(X)$, the uniqueness result is reduced to that of the standard expected utility, i.e., uniqueness up to a positive affine transformation. Similarly, we can also obtain the uniqueness in the usual sense for a non-singleton $\mathcal{U}$ by restricting our attention to a $u \in \mathcal{U}$ with a specific property, such as the centroid of convex cone $\mathcal{U}$, in the case of X being a Euclidean space.

In the remainder of this section, we explain the examples considered in the Introduction by using EUDS.

Example 1' (Choice over multiattribute goods, revisited) Assume, for simplicity, an additive separable and linear utility function, i.e., there exist some a_1, $a_2 > 0$ such that $u(z_1, z_2) = a_1 z_1 + a_2 z_2$ for all $z = (z_1, z_2) \in X$. For all $v = (v_1, v_2)$, $w = (w_1, w_2) \in X$ (which can be considered as degenerate lotteries), define $\delta(v, w) = 0$ if v dominates w (i.e., $v_1 > w_1$ and $v_2 > w_2$) and $\delta(v, w) = +\infty$ if neither v nor w dominates the other. Then, for x, x', $y \in X$ defined in Example 1, we obtain $x \succ x'$, $x \sim y$, and $x' \sim y$. We can also extend δ to $\mathcal{P}(X) \times \mathcal{P}(X)$, so that all the conditions in Definition 2 are satisfied; in particular, $\delta(\cdot, q)$ is naturally assumed to be a concave function.

Example 2' (Choice over lotteries, revisited) Assume that $u(x)$ is increasing with respect to $x \in X$ (note that $X \subset \Re$) and that for all p', $q' \in \mathcal{P}(X)$, $\delta(p', q') = 0$ if lottery p' first-order stochastically dominates q' and $\delta(p', q') = +\infty$ otherwise. Then, for p, q, $r \in \mathcal{P}(X)$ defined in Example 2, we obtain $p \succ q$, whereas $p \sim r$ and $q \sim r$. Again, this configuration is compatible with the conditions in Definition 2, especially, the concavity of δ, because the set of lotteries p that first-order stochastically dominate a given lottery q is the intersection of upper half-spaces, which is convex.[12]

4 Relating EUDS to incomplete preferences

Up to this point, we have axiomatized EUDS, a representation of intransitive indifference, and obtained its uniqueness. In this section, however, we relate EUDS to an incomplete preference model, which is significant for the following two reasons:

First, a certain type of intransitive indifference in EUDS can be relevant to the indecisiveness between alternatives, which an incomplete preference model assumes: as we mention in the Introduction, intransitive indifference is more likely to occur under conflicting criteria because the DM cannot clearly discriminate between alternatives. However, a similar argument also motivates incomplete preference, wherein conflict among multiple beliefs and utility functions prevent the DM from making a decisive choice (Bewley, 1986; Dubra et al., 2004; Ok et al., 2012). Moreover, the indecisiveness relation in an incomplete preference model is generally intransitive, which bears another similarity to intransitive indifference.

Second, EUDS hardly admits welfare measurement because intransitive indifference may cause a cycle.[13] For example, we have $x \succ x' \sim y \sim x$ in Example 1' of the previous section. Such a property can be interpreted as a "mistaken" choice and complicates welfare comparison among the alternatives. Accordingly, it is desirable to infer the "true" preference, i.e., the welfare-relevant preference without a cycle. One way to conduct such an exercise is by systematically relating intransitive indifference to an incomplete and transitive preference.

4.1 Transitive core of EUDS

To relate EUDS to an incomplete preference, we focus on the *transitive core* (Nishimura, 2018) of preference pair $(\succ, \sim)$ in this section.[14] Readers may suspect that there are other possible ways to relate EUDS to an incomplete preference; in particular, indecisiveness can typically

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[12]Let $|X| = n$ and F_p and F_q be the cumulative distribution functions generated by lotteries $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ (p_i and q_i for $i = 1, \dots, n$ denote the probability assigned to prize x_i by p and q), respectively. Then, p first-order stochastically dominating q implies that $F_p(x) \le F_q(x)$ for all $x \in X$, which follows that $\sum_{i=1}^{j} p_i \le \sum_{i=1}^{j} q_i$, i.e., $\sum_{i=1}^{j}(p_i - q_i) \le 0$, for all $j = 1, \dots, n$. Accordingly, $\{p : p$ first-order stochastic dominates $q\}$ is convex because it is the intersection of half-spaces in $\mathrm{ca}(X)$. A similar argument also holds for an infinite X.

[13]A sequence of alternatives $(p_i)_{i=1}^n$ is referred to as a *cycle* if $p_1 \succsim \dots \succsim p_n \succsim p_1$ with at least one strict preference.

[14]The transitive core is also referred to as the *trace* in the theory of semiorders and interval orders (for an extensive survey, see, e.g., Bouyssou and Pirlot, 2005). A similar concept is also considered in the context of incomplete preference (e.g., Galaabaatar and Karni, 2013).

be defined by the absence of (weak or strict) preference. However, the latter definition cannot generally distinguish between indifference and indecisiveness because we also define indifference by the absence of strict preference.[15] Moreover, as Nishimura (2018) demonstrates, the transitive core describes the welfare ranking inferred from intransitive indifference, which establishes a foundation for the welfare evaluation of EUDS.

Now, the transitive core of $(\succ, \sim)$ can be defined as follows:

Definition 4 (Transitive core) For any preference pair $(\succ, \sim)$, we refer to a binary relation $\succsim^{\text{TC}}$ as the *transitive core of* $(\succ, \sim)$ if for all p, $q \in \mathcal{P}(X)$,

$$p \succsim^{\text{TC}} q \text{ if and only if } \begin{cases} r \succsim p \text{ implies } r \succsim q \\ q \succsim r \text{ implies } p \succsim r \end{cases} \quad \text{for all } r \in \mathcal{P}(X).^{16}$$

Nishimura (2018) indicates that the transitive core is generally transitive but possibly incomplete. Accordingly, for all p, $q \in \mathcal{P}(X)$, we write $p \bowtie^{\text{TC}} q$ if neither $p \succsim^{\text{TC}} q$ nor $q \succsim^{\text{TC}} p$.

As a candidate for the transitive core of EUDS, we consider the following incomplete preference model, which is referred to as expected multi-utility (Dubra et al., 2004).

Definition 5 (Expected multi-utility) Given a set $\mathcal{V} \subseteq C(X)$, we refer to $\succsim^{\text{EMU}}_{\mathcal{V}}$ as the *EMU preference with* $\mathcal{V}$, if for all p, $q \in \mathcal{P}(X)$,

$$p \succsim^{\text{EMU}}_{\mathcal{V}} q \text{ if and only if } \int_X v(x)dp \geq \int_X v(x)dq \text{ for all } v \in \mathcal{V}.$$

We also denote for all p, $q \in \mathcal{P}(X)$, $p \bowtie^{\text{EMU}}_{\mathcal{V}} q$ if neither $p \succsim^{\text{EMU}}_{\mathcal{V}} q$ nor $q \succsim^{\text{EMU}}_{\mathcal{V}} p$.

This model characterizes choice under uncertainty, wherein alternative p is preferred to q only if the utility functions in $\mathcal{V}$ *unanimously* rank p higher than q, whereas the ranking between p and q is indecisive if such unanimity is not achieved.

The next theorem indicates that the transitive core of an EUDS preference is equivalent to EMU with the set $\mathcal{U}$ of utility functions characterized in Theorem 2.

[15] Defining an indecisiveness relation $\hat{\bowtie}$ by an absence of strict preference (i.e., for all p, $q \in \mathcal{P}(X)$, $p \,\hat{\bowtie}\, q$ if and only if neither $p \succ q$ nor $q \succ p$) in our model obtains the equivalence of $\hat{\bowtie}$ to the intransitive indifference relation $\sim$, which renders indecisiveness and indifference indistinguishable. Alternatively, if we define an indecisiveness relation $\tilde{\bowtie}$ by weak preference (i.e., for all p, $q \in \mathcal{P}(X)$, $p \,\tilde{\bowtie}\, q$ if and only if neither $p \succsim q$ nor $q \succsim p$), $\tilde{\bowtie}$ is empty because $\succsim$ is complete.

[16] Recall that $p \succsim q$ indicates $p \succ q$ or $p \sim q$.

Theorem 3 Assume that $(\succ, \sim)$ admits an EUDS representation (u, δ). Then, $\succsim^{\mathrm{TC}} = \succsim_{\mathcal{U}}^{\mathrm{EMU}}$, wherein $\mathcal{U}$ is the set of utility functions defined in Theorem 2.

The proof can be found in the Appendix. Theorem 3 equates the transitive core of EUDS with EMU, wherein the set of utility functions in the latter is the set $\mathcal{U}$ of utility functions characterized in Theorem 2. In other words, if our welfare evaluation relies on the transitive core as Nishimura (2018) does, the welfare ranking can be determined by the unanimity rule of all the possible utility functions u in $\mathcal{U}$.

4.2 Separating two types of intransitive indifference

As previously discussed, intransitive indifference and indecisiveness are often identified in the literature. However, in this section, we clarify that indecisiveness can be associated with intransitive indifference only in *specific* cases, and intransitive indifference in our framework can in fact be divided into two distinct types, depending on the value of δ.

Consequently, we define $I \equiv \{(p, q) : (p, q) \in H(u) \text{ and } \delta_u(p, q) = +\infty \text{ for some } u \in \mathcal{U}\}$, i.e., I is the set of nontrivial alternative pairs that can never be discriminated, given some utility function u. Apparently, I is a closed set and we denote the boundary of I (in the product topology generated by $\mathcal{P}(X) \times \mathcal{P}(X)$) by $\partial(I)$.

The following theorem relates a strict preference and indecisiveness in the transitive core to finite and infinite values of the JND function.

Theorem 4 Assume that $(\succ, \sim)$ admits an EUDS representation (u, δ) and $\mathcal{U}$ is the set defined in Theorem 2. Let $(p, q) \in H(u)$ be such that $(p, q) \notin \partial(I)$. Then, the following statements hold:

(a) $(p, q) \notin I$ if and only if $p \succ^{\mathrm{TC}} q$ (or equivalently, $p \succ_{\mathcal{U}}^{\mathrm{EMU}} q$);
(b) $(p, q) \in I$ if and only if $p \bowtie^{\mathrm{TC}} q$ (or equivalently, $p \bowtie_{\mathcal{U}}^{\mathrm{EMU}} q$).

The proof can be found in the Appendix. Theorem 4 classifies two distinct types of intransitive independence in EUDS: case (a) associates a finite value of JND function δ with decisiveness in the transitive core. This type of intransitive indifference can be interpreted as a result of *imperfect discrimination* because a finite value of δ denotes inability to detect a small difference in the expected utility of alternatives. Moreover, intransitive indifference in this case can be completely *eliminated* by the transitive core, i.e., the transitive core ranking between alternatives p and q is *decisive*, even if the alternatives are indifferent in EUDS. In contrast, case (b) associates an infinite value of the JND function with indecisiveness in the transitive core. The intransitive indifference of this type can be considered as *uncertainty about tastes* because we have $\succ^{\mathrm{TC}} = \succ^{\mathrm{EMU}}_{\mathcal{U}}$ from Theorem 3, and thus, indecisiveness in the transitive core entails conflicting evaluations of alternatives (i.e., $\int_X u(x)dp > \int_X u(x)dq$ and $\int_X u'(x)dp < \int_X u'(x)dq$ for some u, $u' \in \mathcal{U}$). Unlike the previous case, intransitive indifference in case (b) *remains* as indecisiveness in the transitive core, i.e., the indifference between alternatives p and p (i.e., $p \sim q$) *implies* the indecisiveness of the transitive core ranking between them (i.e., $p \bowtie^{\mathrm{TC}} q$). Put differently, Theorem 4 can successfully derive the transitive core (i.e., welfare) ranking from the first type of intransitive indifference, but not necessarily from the second. The latter result is intuitive because conflict between tastes will persist, regardless of how far apart the alternatives becomes, whereas imperfect discrimination will readily disappear as the distance between alternatives increases.

Much of the literature on intransitive indifference assumes a finite-valued JND function in their preference representations (e.g., Fishburn, 1968, 1970a; Luce, 1956; Vincke, 1980). Theorem 4 suggests that they mainly focus on the first type of intransitive indifference discussed in the previous paragraph (i.e., that caused by imperfect discrimination). As we mentioned in the Introduction, however, many studies argue that intransitive indifference is more likely to occur if alternatives are mutually undominated than if one of the alternatives dominates the other (Fishburn, 1970b; May, 1954; Tversky, 1969). Theorem 4 associates the latter behavioral implication with the second type of intransitive indifference, i.e., that caused by uncertainty about tastes, and thus, an infinite value of the JND function because mutually undominated alternatives readily generate conflicting evaluations. One of our contributions is to establish a framework that allows us to discuss the different types of intransitive indifference, which has been impossible in existing approaches.

Finally, we exclude the possibility of alternative pair (p, q) being on the boundary $\partial(I)$ of I. This is because the boundary of I consists of alternatives p and q, such that $\int_X u(x)dp > \int_X u(x)dq$ for some $u \in \mathcal{U}$ and $\int_X u'(x)dp = \int_X u'(x)dq$ for all $u' \in \mathcal{U}$, with $u' \neq u$. It follows that $p \succ^{\mathrm{TC}} q$ because $\succ^{\mathrm{TC}} = \succ^{\mathrm{EMU}}_{\mathcal{U}}$, whereas $p \sim q$ because $(p, q) \in I$, which may be counterintuitive.

One possible settlement of this boundary issue is to define another binary relation $\widehat{\succ}^{\mathrm{TC}}$ on $\mathcal{P}(X)$ as follows: $p \mathrel{\widehat{\succ}^{\mathrm{TC}}} q$ for some p, $q \in \mathcal{P}(X)$ if and only if for all r, $s \in \mathcal{P}(X)$, there is $\epsilon > 0$, such that $(1-\alpha)p + \alpha r \succ^{\mathrm{TC}} (1-\alpha)q + \alpha s$ for all $\alpha \in [0, \epsilon]$. It is easily shown that $p \mathrel{\widehat{\succ}^{\mathrm{TC}}} q$ if and only if

$$\int_X u(x)dp > \int_X u(x)dq \text{ for all } u \in \mathcal{U}.$$

A similar technique has been developed by Cerreia-Vioglio et al. (2020) to derive a preference stronger than an incomplete preference under uncertainty. The derived ranking $\widehat{\succ}^{\mathrm{TC}}$ is the algebraic interior of $\widehat{\succ}^{\mathrm{TC}}$, which is equivalent to the interior of $\succ^{\mathrm{TC}}$ because $\succ^{\mathrm{TC}}$ evidently has a nonempty interior. We also denote the indecisiveness relation derived from $\widehat{\succ}^{\mathrm{TC}}$ by $\widehat{\bowtie}^{\mathrm{TC}}$. Now, we obtain the following corollary, which completely characterizes the induced indecisiveness relation $\widehat{\bowtie}^{\mathrm{TC}}$ by infinite values of δ.

Corollary 1 Assume that $(\succ, \sim)$ admits an EUDS representation (u, δ) and $\mathcal{U}$ is the set defined in Theorem 2. Let $(p, q) \in H(u)$ be such that $p \neq q$. Then, the following statements hold:

(a) $(p, q) \notin I$ if and only if $p \mathrel{\widehat{\succ}^{\mathrm{TC}}} q$;
(b) $(p, q) \in I$ if and only if $p \mathrel{\widehat{\bowtie}^{\mathrm{TC}}} q$.

5 Special cases

In this section, we explore two important special cases of EUDS.

5.1 One-directional sensitivity

Here, we consider an extension of Luce's (1956) semiorder approach to choice under risk, which assumes a constant JND function.

To this end, we consider the following axiom.

Axiom 7 (Indifference convexity) For all p, p', $q \in \mathcal{P}(X)$ and $\alpha \in [0, 1]$, $p \sim q$ and $p' \sim q$ imply that $\alpha p + (1 - \alpha)p' \sim q$.

Axiom 7, which has an implication similar to the strict preference convexity axiom, obtains indifference curves that are convex (and generally thick) sets, except that a hedge between indifferent alternatives no longer creates any value, i.e., the mixture of alternatives that are indifferent to q is also indifferent to q.

Readers may suspect that indifference convexity is implied by irresolute independence because the former axiom is implied by the standard independence axiom. However, we still need to impose Axiom 7 because irresolute independence is *not* sufficiently strong to obtain the indifference between alternatives $\alpha p + (1 - \alpha)p'$ and q, even if p and p' are indifferent to q.

We consider the following special case of EUDS in this section.

Definition 6 (Expected utility with one-directional sensitivity) An EUDS (u, δ) is referred to as the *expected utility with one-directional sensitivity (EUOS)* if there exists $k > 0$

such that $\delta(p,q) = k$ for all $(p,q) \in H(u)$.

The next theorem indicates that this special case can be derived from Axiom 7, along with other axioms.

Theorem 5 The following statements are equivalent.

(a) Preference pair $(\succ, \sim)$ satisfies Axioms 1–7.

(b) Preference pair $(\succ, \sim)$ admits an EUOS representation (u, δ). Moreover, u in the EUOS representation is unique up to a positive affine transformation (and $\delta = \delta_u$ is uniquely defined by each u).

The proof is presented in the Appendix, whose intuition is as follows: first, along with strict preference convexity, indifference convexity implies that, for a given alternative, the intersection between the closure of the strict upper contour set and the indifference curve forms a hyperplane. Moreover, balanced indifference obtains symmetry between strict upper and strict lower contour sets, which eliminates direction-dependence and derives a constant JND function.

Notably, EUOS is the counterpart of the semiorder model (Dalkiran et al., 2018; Luce, 1956; Vincke, 1980) in our setting. In particular, Dalkiran et al. (2018) and Vincke (1980) derive a preference representation over lotteries, which is comparable to EUOS. However, there are two major differences between their approach and ours: first, they axiomatize the counterpart of EUOS from a semiorder, whereas we start with a strict partial order, a more general class of preferences than a semiorder, to axiomatize EUDS and then derive the EUOS model as a special case. Second and more importantly, the existing studies impose the independence axiom on the *transitive core* of the preference rather than on the preference itself, whereas our independence axiom is *directly* imposed on the preference pair $(\succ, \sim)$.[17] Accordingly, our axiomatization of a constant JND function is more compelling because the preference pair $(\succ, \sim)$ is readily observable from choice, unlike its transitive core.

Note that the finiteness of δ in EUOS (except on the indifference curve) implies that this special case only involves intransitive indifference caused by imperfect discrimination, i.e., the first type of intransitive indifference discussed in Section 4. This finiteness property also leads to a stronger uniqueness result (i.e., uniqueness of u up to a positive affine transformation) for EUOS. As Theorem 3 indicates, intransitive indifference in this model can be completely eliminated, i.e., the standard expected utility can be obtained by the transitive core of EUOS;

[17] In our notation, the independence axiom that they assume states that "$p \sim^{\mathrm{TC}} q$ implies $\frac{1}{2}p + \frac{1}{2}r \sim^{\mathrm{TC}} \frac{1}{2}q + \frac{1}{2}r$ for all p, q, $r \in \mathcal{P}(X)$."

that is, for all p, $q \in \mathcal{P}(X)$, we have $p \succsim^{\mathrm{TC}} q$ if and only if $\int_X u(x)dp \geq \int_X u(x)dq$.

5.2 Categorical sensitivity

In this section, we focus on categorical sensitivity, wherein only complete sensitivity or complete insensitivity is allowed for all alternative pairs.

First, consider the following strengthening of the irresolute independence axiom.

Axiom 3' (Independence) For all p, q, $r \in \mathcal{P}(X)$ and $\alpha \in (0,1)$, the following statements hold:

(a) $p \succ q$ implies $\alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)r$;

(b) $p \sim q$ implies $\alpha p + (1-\alpha)r \sim \alpha q + (1-\alpha)r$.

The only difference between Axioms 3 and 3' is that a strict, rather than a weak, preference is assumed (i.e., indifference is ruled out) in the latter half of Axiom 3'(a). In other words, once the DM has exhibited a strict preference for alternative p to q, mixing the alternatives with the third alternative r never alters the strict preference; consequently, a strict preference holds between any alternatives p' and q', such that a signed measure $p' - q'$ is parallel with $p - q$.

Now, we consider the following special case of EUDS.

Definition 7 (Expected utility with categorical sensitivity) An EUDS (u, δ) is referred to as the *expected utility with categorical sensitivity (EUCS)* if either $\delta(p,q) = 0$ or $\delta(p,q) = +\infty$ for all $(p,q) \in H(u)$.

This definition states that only complete sensitivity (i.e., $\delta(p,q) = 0$) or complete insensitivity (i.e., $\delta(p,q) = +\infty$) is exhibited for all p, $q \in \mathcal{P}(X)$. The next theorem axiomatizes this special case.

Theorem 6 The following statements are equivalent.

(a) Preference pair $(\succ, \sim)$ satisfies Axioms 1, 2, 3', and 6.

(b) Preference pair $(\succ, \sim)$ admits an EUCS representation (u, δ).

The proof is presented in the Appendix. Intuitively, independence retains a strict preference between alternatives after mixing the two alternatives with another one. This property excludes a nonzero and finite value of the JND function, which obtains an EUCS preference representation. It follows from Theorem 4 that this special case only admits intransitive indifference caused by uncertainty about tastes, which cannot be eliminated by the transitive core. Therefore, the set $\mathcal{U}$ of utility functions is generally non-singleton, and thus, its uniqueness result follows from Theorem 2.

The latter property establishes an intimate relationship between EUCS and the incomplete preference model, which can be stated as a corollary of Theorem 4 as follows.

Corollary 2 Assume that $(\succ, \sim)$ admits an EUCS representation (u, δ) and $\mathcal{U}$ is the set defined in Theorem 2. Let $(p, q) \in H(u)$ be such that $(p, q) \notin \partial(I)$. Then, the following statements hold:

(a) $p \succ q$ if and only if $p \succ^{\mathrm{TC}} q$ (or equivalently, $p \succ_{\mathcal{U}}^{\mathrm{EMU}} q$);

(b) $p \sim q$ if and only if $p \bowtie^{\mathrm{TC}} q$ (or equivalently, $p \bowtie_{\mathcal{U}}^{\mathrm{EMU}} q$).

Because EUCS admits no alternative pair (p, q) such that $0 < \delta(p, q) < +\infty$, preference and indifference in EUCS can directly be linked to preference and indecisiveness in EMU (or equivalently, the transitive core of the EUCS preference); specifically, indifference in the former framework can be relabeled as indecisive in the latter for any alternative pair $(p, q) \in \mathcal{P}(X) \times \mathcal{P}(X)$, except for those on the boundary of I. (The boundary issue can be addressed by considering $\widehat{\succ}^{\mathrm{TC}}$, the algebraic interior of $\succ^{\mathrm{TC}}$, in place of $\succ^{\mathrm{TC}}$, as in Section 4.)

In EUCS, $p \succsim q$ for some p, $q \in \mathcal{P}(X)$ assures the existence of *at least one* utility function u in $\mathcal{U}$, such that alternative p gives higher expected utility than q, i.e., $\int_X u(x)dp \geq \int_X u(x)dq$. This result is reminiscent of the justifiability model (Lehrer and Teper, 2011), wherein one alternative is weakly preferred to the other whenever the former alternative gives higher subjective expected utility than the latter for at least one belief. The justifiability model is generally complete and intransitive, and, as Nishimura (2018) indicates, its transitive core is equivalent to Bewley's (1986) Knightian uncertainty model, which is incomplete and transitive. Accordingly, Corollary 2 parallels Nishimura's result in the sense that it relates EUCS, a complete and intransitive ranking, to EMU, an incomplete and transitive ranking, by the transitive core. A major difference between the two approaches, however, is that we focus on uncertainty about *tastes* (which can be captured using multiple utility functions), whereas the justifiability and Bewley's models consider uncertainty about *beliefs*.

6 Related literature

In this section, we review the literature relevant to this study.

First, intransitive indifference has been studied by many authors. Semiorder, which is one of the most renowned models on intransitive indifference, has been initially defined by Luce (1956) for the preference over prizes, and it has been extended by many studies (e.g., Dalkiran et al., 2018; Gilboa and Lapson, 1995; Vincke, 1980). Interval order (Fishburn, 1970a) and strict partial order (Suppes, 1957) have also been proposed to generalize the semiorder approach.

Among others, Vincke (1980) and Dalkiran et al. (2018) are relevant to this study because they characterize a semiorder over *lotteries* rather than over prizes, unlike Luce's (1956) classic study. As noted in Section 5.1, however, major differences between our approach and theirs are that our primitive is a strict partial order, which is more general than a semiorder, and our independence axiom directly refers to preference pair $(\succ, \sim)$ rather than its transitive core, both of which are crucial assumptions to derive a direction-dependent JND function. Moreover, Gilboa and Lapson (1995) consider a semiorder in a product space, which is relevant to our model, wherein X is the set of multiattribute goods. However, their model also assumes a direction-*in*dependent JND function as in Luce (1956), whereas our main focus is a direction-dependent JND function.

Second, a finite value of the JND function is reminiscent of preference intensity (Fishburn, 1970c; Gerasimou, 2019) and grades of indecisiveness (Minardi and Savochkin, 2015) because the value of the JND function can be interpreted as a degree of indifference, i.e., how robust the indifference between alternatives is.[18] Whereas these approaches measure preference intensity by using a binary relation over *alternative pairs*, we axiomatize EUDS, and thus, obtain a JND function from the preference, i.e., the standard binary relation over *alternatives*.

7 Concluding remarks

In this study, we axiomatize the EUDS model to analyze intransitive indifference, wherein the DM exhibits direction-dependent sensitivity to utility difference between alternatives. We also relate EUDS to an incomplete preference model, i.e., EMU, by considering the transitive core. A possible future study will be to explore further connections between EUDS and other models, such as incomplete preference and preference intensity, by generalizing this framework.

[18] Another relevant approach, which associates the difficulty of choice between two alternatives with the *decision time* (response time), has been propounded by Echenique and Saito (2017), He and Natenzon (2019), and Koida (2017). In particular, Koida relates the decision time not only to the utility difference between alternatives but also to the *angle* formed by the utility vector generated by alternatives and the indifference curve. The latter property is comparable to the direction-dependent intransitive indifference in this study.

Appendix

Proof of Theorem 1 The necessity part of the theorem is straightforward.

The sufficiency part is as follows: first, we define $D(\succ) \equiv \mathrm{cl}(\{\lambda(p-q) : p, q \in \mathcal{P}(X),\ \lambda > 0,\ p \succ q\}) \subseteq \mathrm{ca}(X)$; that is, $D(\succ)$ is the closure of the cone generated by signed measures $p-q$ (the closure is taken in the weak*-topology), wherein p is strictly preferred to q.

The following lemma characterizes these sets.

Lemma 1 The following statements hold:
(a) $D(\succ)$ is an closed convex cone.
(b) For all p, $q \in \mathcal{P}(X)$, $p \succ q$ implies $p - q \in D(\succ)$.
(c) For all p, $q \in \mathcal{P}(X)$, $p - q \in D(\succ)$ implies $p \succsim q$.

Proof To prove statement (a), we first note that $D(\succ)$ is a closed cone by definition. Next, we also indicate that $D(\succ)$ is convex. Assume that for some p, q, p', $q' \in \mathcal{P}(X)$ such that $p-q$, $p'-q' \in D(\succ)$. Then, strict preference convexity implies that for all $\alpha \in [0,1]$, $\alpha p + (1-\alpha)p'$ $\succ \alpha q + (1-\alpha)q'$, which indicates that $\alpha(p-q) + (1-\alpha)(p'-q') \in D(\succ)$; accordingly, $D(\succ)$ is convex.

Next, statement (b) is implied by definition. To indicate statement (c), assume that $p - q \in D(\succ)$ and $q \succ p$ for some p, $q \in \mathcal{P}(X)$. It follows from the definition of $D(\succ)$ that there exist p', $q' \in \mathcal{P}(X)$ and $\lambda > 0$ such that $p' \succ q'$ and $p - q = \lambda(p' - q')$, which entails that $\frac{q}{1+\lambda} + \frac{\lambda p'}{1+\lambda}$ $= \frac{p}{1+\lambda} + \frac{\lambda q'}{1+\lambda}$. However, strict preference convexity implies that $\alpha q + (1-\alpha)p' \succ \alpha p + (1-\alpha)q'$ for all $\alpha \in [0,1]$, which is a contradiction. Accordingly, we obtain $p \succsim q$. **Q.E.D.**

Here, we define

$$\mathcal{U} \equiv \left\{ u \in C(X) : \int_X u(x) d\mu \geq 0 \text{ for all } \mu \in D(\succ) \right\}, \tag{2}$$

which is a nonempty closed convex cone in $C(X)$. The following lemma relates $\mathcal{U}$ to $D(\succ)$, in a way similar to that of Dubra et al. (2004).

Lemma 2 $\int_X u(x)dp \geq \int_X u(x)dq$ for all $u \in \mathcal{U}$ if and only if $p - q \in D(\succ)$.

Proof The "if" part holds by definition.

Consider the "only if" part. Suppose, contrariwise, that, for any p, $q \in \mathcal{P}(X)$ such that

$$\int_X u(x)dp \geq \int_X u(x)dq \tag{3}$$

for all $u \in \mathcal{U}$, the sets $\{p - q\}$ and $D(\succ)$ are disjoint. Because $D(\succ)$ is a closed convex cone (under the weak*-topology), the Hahn–Banach separation theorem (Aliprantis and Border, 2006) entails that there exist a continuous linear functional L on $\mathrm{ca}(X)$ and a $\alpha \in \Re$ such that $L(\mu) \geq \alpha > L(p - q)$ for all $\mu \in D(\succ)$.

Because $0 \in D(\succ)$, we have $0 = L(0) \geq \alpha > L(p - q)$. However, because $D(\succ)$ is a cone, $kL(\mu) = L(k\mu) \geq \alpha$ for all $\mu \in \mathrm{ca}(X)$ and $k > 0$, which implies that $L(\mu) \geq 0$ for all $\mu \in \mathrm{ca}(X)$. Accordingly, we have $L(\mu) \geq 0 > L(p - q)$ for all $\mu \in \mathrm{ca}(X)$. Next, the duality between $C(X)$ and $\mathrm{ca}(X)$ implies that there exists $v \in C(X)$ such that $L(\mu) = \int_X v(x)d\mu$ for all $\mu \in \mathrm{ca}(X)$. Thus, $\int_X v(x)d\mu \geq 0 > \int_X v(x)d(p - q)$ for all $\mu \in D(\succ)$. It follows that $v \in \mathcal{U}$ and $\int_X v(x)dp < \int_X v(x)dq$, which contradicts (3). **Q.E.D.**

Now, it follows from Lemmas 1 and 2 that for all $u \in \mathcal{U}$ and p, $q \in \mathcal{P}(X)$, $p \succ q$ implies $\int_X u(x)dp \geq \int_X u(x)dq$ and $\int_X u(x)dp \geq \int_X u(x)dq$ implies $p \succsim q$. Accordingly, any $u \in \mathcal{U}$ is a utility function of $\succ$.

We fix an arbitrary utility function $u \in \mathcal{U}$. The following lemma constructs a JND function.

Lemma 3 There exists a JND function δ given u.

Proof First, assume that for some $(\hat{p}, \hat{q}) \in H(u)$, $\hat{p} \succ \hat{q}$. Irresolute independence (a) implies that there exists $\alpha \in [0, 1]$ such that $\alpha\hat{p} + (1 - \alpha)\hat{q} \sim \hat{q}$. Let $\bar{\alpha}(\hat{p}, \hat{q})$ be the least upper bound of α satisfying the latter condition; that is, for all p, $q \in \mathcal{P}(X)$, $\bar{\alpha}(p, q) \equiv \sup\{\alpha \in [0, 1] : \alpha p + (1 - \alpha)q \sim q\}$. It follows from Archimedean continuity that $\bar{\alpha}(\hat{p}, \hat{q})\hat{p} + (1 - \bar{\alpha}(\hat{p}, \hat{q}))\hat{q} \sim \hat{q}$. Strict preference convexity implies that $\alpha\hat{p} + (1 - \alpha)\hat{q} \succ \hat{q}$ for all $\alpha > \bar{\alpha}(\hat{p}, \hat{q})$, whereas irresolute independence (b) implies that $\alpha\hat{p} + (1 - \alpha)\hat{q} \sim \hat{q}$ for all $\alpha \leq \bar{\alpha}(\hat{p}, \hat{q})$. Now, let $r_{p,q} \equiv \bar{\alpha}(p, q)p + (1 - \bar{\alpha}(p, q))q$ and $\delta(p, q) \equiv \int_X u(x)dr_{p,q} - \int_X u(x)dq$. For all p, $q \in \mathcal{P}(X)$ and $\lambda > 0$ such that $p - q = \lambda(\hat{p} - \hat{q})$, balanced indifference entails that $r_{p,q} - q = r_{\hat{p},\hat{q}} - \hat{q}$, which obtains $\delta(p, q) = \delta(\hat{p}, \hat{q})$. Because the choice of such a $(\hat{p}, \hat{q})$ is arbitrary, the latter argument guarantees condition (a) in Definition 2.

Second, assume that for some $(\hat{p}, \hat{q}) \in H(u)$, $\hat{p} \sim \hat{q}$. If there exists $p' \in \mathcal{P}(X)$ such that $p' \succ \hat{q}$ and $\hat{p} = \alpha p' + (1 - \alpha)\hat{q}$ for some $\alpha \in (0, 1]$, use p' in place of $\hat{p}$ and apply the argument in the previous paragraph. Further, if there exists no such p', irresolute independence (b) implies that $p'' \sim \hat{q}$ for all $p'' \in \mathcal{P}(X)$ such that $\hat{p} = \alpha p'' + (1 - \alpha)\hat{q}$ for some $\alpha \in (0, 1]$. We can naturally define $\delta(\hat{p}, \hat{q}) = +\infty$ in this case, which also entails the second half of condition (c) in Definition 2.

Next, to indicate condition (b) in Definition 2, let p, p', $q \in \mathcal{P}(X)$ be such that $p \succ q$ and p'

$\succ q$. Then, strict preference convexity implies that $\alpha p + (1-\alpha)p' \succ q$ for all $\alpha \in [0,1]$, which implies that $\delta(\alpha p + (1-\alpha)p', q) \leq \alpha\delta(p,q) + (1-\alpha)\delta(p',q)$.

Finally, for all $(p,q) \in (\mathcal{P}(X) \times \mathcal{P}(X)) \setminus H(u)$, we can consistently define $\delta(p,q) = +\infty$, which obtains the first half of condition (c) in Definition 2. **Q.E.D.**

Proof of Theorem 2 We define $\mathcal{U}$ as in (2) in the proof of the previous theorem, which is closed and convex by definition.

First, $u \in \langle\mathcal{U}\rangle$ implies that u is a utility function of $\succ$, by construction. (Note that $u \in \langle\mathcal{U}\rangle$ if and only if $\alpha u + \beta \in \langle\mathcal{U}\rangle$ for all $\alpha > 0$ and $\beta \in \Re$.) Conversely, assume that $(\succ, \sim)$ admits an EUDS representation (u, δ) and suppose, contrariwise, that $u \notin \langle\mathcal{U}\rangle$. Because $u \notin \langle\mathcal{U}\rangle$, $\int_X u(x)dp < \int_X u(x)dq$ for some p, $q \in \mathcal{P}(X)$ such that $p - q \in D(\succ)$. By the definition of $D(\succ)$, there exist p', $q' \in \mathcal{P}(X)$ and $\lambda > 0$ such that $p' - q' = \lambda(p-q)$ and $p' \succ q'$. However, $\int_X u(x)dp < \int_X u(x)dq$ and $p' - q' = \lambda(p-q)$ imply that $\int_X u(x)dp' < \int_X u(x)dq'$, which contradicts the fact that u is a utility function of $\succ$.

Finally, Lemma 3 entails that the JND function δ can uniquely be defined once a utility function $u \in \langle\mathcal{U}\rangle$ has been determined. **Q.E.D.**

Proof of Theorem 3 First, because $\succsim$ is complete, Definition 4 can be restated as follows, using the strict, rather than the weak, preference: for all p, $q \in \mathcal{P}(X)$,

$$p \succsim^{\mathrm{TC}} q \text{ if and only if } \begin{cases} r \succ p \text{ implies } r \succ q \\ q \succ r \text{ implies } p \succ r \end{cases} \quad \text{for all } r \in \mathcal{P}(X).$$

Now, assume that $p \succsim_{\mathcal{U}}^{\mathrm{EMU}} q$ for some p, $q \in \mathcal{P}(X)$. Noting that $\mathcal{U}$ is the set of utility functions of $\succ$, and $\delta(p', q') = \delta(p'', q'')$ for all p', p'', q', $q'' \in \mathcal{P}(X)$ and $\lambda > 0$, such that $p'' - q''$ $= \lambda(p' - q')$, we obtain $U_p(\succ) \subseteq U_q(\succ)$ and $L_p(\succ) \supseteq L_q(\succ)$. Accordingly, for all $r \in \mathcal{P}(X)$, $r \succ p$ implies that $r \succ q$, while $q \succ r$ implies that $p \succ r$, which follows that $p \succsim^{\mathrm{TC}} q$.

Conversely, assume that $p \succsim^{\mathrm{TC}} q$ and $p \not\succsim_{\mathcal{U}}^{\mathrm{EMU}} q$. The latter condition obtains $p - q \notin D(\succ)$ ($D(\succ)$ is the set defined in the proof of Theorem 1) or equivalently, $p \notin D(\succ) + q$, which implies that $(D(\succ)+p) \setminus (D(\succ)+q) \neq \phi$. (Note that $0 \in D(\succ)$.) It follows from the definition of $D(\succ)$ that there exists $s \in \mathcal{P}(X)$ such that $s \succ p$ and $s \notin D(\succ) + q$. However, the latter condition implies that $q \succsim s \succ p$, which contradicts the statement that $p \succsim^{\mathrm{TC}} q$. **Q.E.D.**

Proof of Theorem 4 Let $\mathcal{U}$ be the set defined in Theorem 2 and $D(\succ)$ be the set defined in the proof of Theorem 1.

First, we prove statement (a). Because $(p,q) \in H(u)$ and $(p,q) \notin I$, there exist p', $q' \in \mathcal{P}(X)$ and $\lambda > 0$ such that $p' - q' = \lambda(p-q)$ and $p' \succ q'$, which entail that $p - q \in D(\succ)$.

Accordingly, we have $\int_X u'(x)dp > \int_X u'(x)dq$ for all $u' \in \mathcal{U}$; that is, we obtain $p \succ_{\mathcal{U}}^{\text{EMU}} q$.

Next, to indicate statement (b), we first prove the following lemma.

Lemma 4 For all $u \in \mathcal{U}$ and $(p,q) \in H(u)$, $(p,q) \in I$ implies that there exists $\hat{u} \in \mathcal{U}$ such that $\int_X \hat{u}(x)dp = \int_X \hat{u}(x)dq$.

Proof For a given $u \in \mathcal{U}$ and $(p,q) \in H(u)$, let $\delta_u(p,q) = +\infty$. If $\int_X u(x)dp = \int_X u(x)dq$, the desired result can trivially be obtained. Accordingly, we assume that $\int_X u(x)dp \neq \int_X u(x)dq$, which implies one of the following two cases:

First, assume that $\int_X u'(x)dp > \int_X u'(x)dq$ and $\int_X u''(x)dp < \int_X u''(x)dq$ for some u', $u'' \in \mathcal{U}$. Then, because $\mathcal{U}$ is convex, there exists $\alpha \in [0,1]$ such that $\int_X \hat{u}(x)dp = \int_X \hat{u}(x)dq$ with $\hat{u} \equiv \alpha u' + (1-\alpha)u''$, which obtains the desired result.

Second, let $\int_X u'(x)dp > \int_X u'(x)dq$ for all $u' \in \mathcal{U}$ or $\int_X u'(x)dp < \int_X u'(x)dq$ for all $u' \in \mathcal{U}$. We assume the former condition without loss of generality. It follows from the definition of $\mathcal{U}$ that $p - q \in D(\succ)$, which implies that $p' \succ q'$ for some p', $q' \in \mathcal{P}(X)$ and $\lambda > 0$, such that $p' - q' = \lambda(p-q)$. However, the latter argument entails that $\delta(p,q) < +\infty$, which is a contradiction. **Q.E.D.**

Now, statement (b) trivially holds if $I \setminus \partial(I)$ is empty. Accordingly, we assume that there exists some $(p,q) \in I \setminus \partial(I)$. It follows from Lemma 4 that there exists $\hat{u} \in \mathcal{U}$ such that $\int_X \hat{u}(x)dp = \int_X \hat{u}(x)dq$. Because (p,q) is in the interior of I (which is equal to $I \setminus \partial(I)$), there exist $p' \in \mathcal{P}(X)$ and $\epsilon > 0$, such that $(p',q) \in I$, $p' \in N_\epsilon(p)$, and $\int_X \hat{u}(x)dp' > \int_X \hat{u}(x)dq$. However, Lemma 4 also guarantees the existence of $u' \in \mathcal{U}$ such that $\int_X u'(x)dp' = \int_X u'(x)dq$, which also obtains $\int_X u'(x)dp < \int_X u'(x)dq$ by construction. Similarly, $(p,q) \in I \setminus \partial(I)$ implies that there exist $p'' \in \mathcal{P}(X)$ and $\epsilon > 0$, such that $(p'',q) \in I$, $p'' \in N_\epsilon(p)$, and $\int_X \hat{u}(x)dp'' < \int_X \hat{u}(x)dq$. Again, Lemma 4 guarantees the existence of $u'' \in \mathcal{U}$, such that $\int_X u''(x)dp'' = \int_X u''(x)dq$ and $\int_X u''(x)dp > \int_X u''(x)dq$. Accordingly, we obtain $p \bowtie_{\mathcal{U}}^{\text{EMU}}$ because $\int_X u'(x)dp < \int_X u'(x)dq$ and $\int_X u''(x)dp > \int_X u''(x)dq$. **Q.E.D.**

Proof of Theorem 5 The sufficiency part is straightforward.

Conversely, assume that Axioms 1 to 7 are satisfied. It follows from Axioms 1 to 6 that preference $(\succ, \sim)$ admits an EUDS representation (u, δ). Now, for all $q \in \mathcal{P}(X)$, let B_q be the intersection between the closure of the strict upper contour set and the indifference curve, given q; that is, $B_q \equiv \text{cl}(U_q) \cap \{\hat{p} \in \mathcal{P}(X) : \hat{p} \sim q\}$, which is apparently nonempty. It follows from indifference convexity that for some p, $p' \in B_q$, $p_\alpha \equiv \alpha p + (1-\alpha)p'$ is also in B_q for all $\alpha \in [0,1]$;

that is, $p_\alpha \sim q$ for all $\alpha \in [0,1]$. Accordingly, by defining $u \in \mathcal{U}$ so that $\int_X u(x)dp = \int_X u(x)dp'$, we obtain $\delta_u(p,q) = \delta_u(p',q) = \delta_u(p_\alpha,q)$ for all $\alpha \in [0,1]$. Because the latter argument holds for all such p, p', $q \in \mathcal{P}(X)$, we obtain an EUOS representation (u, δ_u). By construction, u in the EUOS representation is unique up to a positive affine transformation. **Q.E.D.**

Proof of Theorem 6 The sufficiency part is straightforward.

Conversely, assume that Axioms 1, 2, 3', and 6 are satisfied. Because independence implies irresolute independence, strict preference convexity, and balanced indifference, preference $(\succ, \sim)$ admits an EUDS representation (u, δ).

Now, suppose, conversely, that $0 < \delta(p,q) < +\infty$ for some p, $q \in \mathcal{P}(X)$. This entails that there exist p', q', p'', $q'' \in \mathcal{P}(X)$ and $\lambda > 0$ such that $p'' - q'' = \lambda(p' - q')$ and $p' \sim q'$ and $p'' \succ q''$. However, it follows from Axiom 3' that

$$\frac{\lambda}{1+\lambda}q' + \frac{1}{1+\lambda}q'' \sim \frac{\lambda}{1+\lambda}p' + \frac{1}{1+\lambda}q'' = \frac{\lambda}{1+\lambda}q' + \frac{1}{1+\lambda}p'' \succ \frac{\lambda}{1+\lambda}q' + \frac{1}{1+\lambda}q'',$$

(wherein the equality is derived from the construction), which is a contradiction. Accordingly, we obtain $\delta(p,q) = 0$ or $\delta(p,q) = +\infty$ for all p, $q \in \mathcal{P}(X)$. **Q.E.D.**

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