Can we have a sensible definition of collective risk aversion?

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Abstract

This paper shows that two natural requirements on collective decision under risk, one that when individuals are more risk-averse so should be the society, the other that if everybody prefers one risky prospect over another so should the society, lead to an unpleasant property: the social ranking over risky prospects has to be always identical with one individual's risk preference, in each equivalence class of risk preferences which yield the same profile of ordinal preferences over deterministic outcomes.

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1 Introduction

Collective decision under risk, such as what policy to take under the risk of pandemic, requires us to think of how the society should be "risk-averse." We examine if such concept of collective risk aversion can be a sensible one.

The key issue is what risk aversion is about at a social level. Even when each individual has one dimensional space of own individual outcomes (consumption or wealth), the set of social outcomes (allocations) is already a multi-dimensional space. As we know from the studies on risk aversion with many commodities (such as Kihlstrom and Mirman [12]), defining a degree of risk aversion is a non-obvious issue when the set of outcomes is multi-dimensional. It appears immediately that measurement of the degree of risk aversion in any sense will depend on which dimension to look at.

An intuitive way to resolve the problem would be first to measure social outcomes in the form of some one-dimensional object, such as monetary measure of welfare, aggregate wealth or inequality index, and then to consider aversion to riskiness about such object. Can it be a sensible way, though?

We take an axiomatic approach to answer this question. We propose a natural requirement on collective decision criteria that when individuals are more risk-averse so should be the society, where the definition of comparative risk aversion follows the traditonal one by Arrow [1] and Pratt [19]. This is what we call Comparative Risk Aversion Monotonicity. Another natural requirement is Ex-ante Pareto: if everybody prefers one risky prospect over another so should the society. We assume that both individuals' risk preferences and the social ranking over risky prospects follow the expected utility theory due to von-Neumann and Morgenstern [18].

We show that the two natural requirements lead to an unpleasant property: the social ranking over risky prospects is always identical with one individual's risk preference, as far as the individuals' risk preferences induce the same profile of rankings over deterministic individual consumptions. Thus we have a dictatorship conditional on each equivalence class of risk preferences which yield the same profile of ordinal preferences.

The key argument is as follows. From Ex-ante Pareto, the Harsanyi theorem (Harsanyi [10]) delivers that the society's von Neumann/Morgenstern index is a weighted sum of the individual ones. But this means that the society's *ordinal* ranking over deterministic consumption allocations must depend on *cardinal* (in the sense of curvature) properties of the individuals' vNM indices. On the other hand, Comparative Risk Aversion Monotonicity implies that the society's ordinal ranking over deterministic social outcomes must depend only on the individuals' *ordinal* preferences over own deterministic consumpitons (we call this property Ordinal Invariance). The two assertions are compatible only under the conditional disctatorship.

The result suggests that we have to choose between two options, in order to avoid the conditional dictatorship. One is to accept that even in evaluating deterministic social outcomes we have to take the individuals' risk attitudes into account. The other is to give up the presumption that everybody is responsible for his/her risk attitude, in other words, to give up the presumption that risk attitudes are a "matter of taste."

The paper proceeds as follows. In Section 2 we present the basic model and the axioms. In Section 3 we present the impossibility result in the domain in which each individual cares only about marginal distributions over his/her consumptions. Section 4 discusses policy implications and concludes.

2 The setting and axioms

Let X be the set of social outcomes, which will be specified later on and we leave it to be abstract at this point. Let $\Delta(X)$ denote the set of simple lotteries over X.

Let $\mathcal{R}$ be the set of preferences over $\Delta(X)$ which satisfy the non-Neumann/Morgenstern expected utility theory, that is, satisfy four axioms: completeness, transitivity, mixture continuity and mixture independence. Such preference allows representation in the expected utility form

$$U(p) = E_p[u(x)]$$

where $u : X \to \mathbb{R}$ is called von-Neumann/Morgenstern index throughout.

Let $\mathcal{R}_i \subset \mathcal{R}$ denote the domain for individual i's risk preferences, which will be specified later on.

An aggregation rule is a function from $\prod_{i\in I} \mathcal{R}_i$ into $\mathcal{R}$. We rather describe it as a graph $G \subset \prod_{i\in I\cup\{0\}} \mathcal{R}_i$, such that for all $(\succsim_i)_{i\in I} \in \prod_{i\in I} \mathcal{R}_i$ there exists a unique $\succsim_0 \in \mathcal{R}$ such that $(\succsim_i)_{i\in I\cup\{0\}} \in G$.

The most natural axiom will be the ex-ante Pareto condition, which states that if everyone prefers one risky prospect over another so should the society.

Ex-ante Pareto: For all $(\succsim_i)_{i\in I\cup\{0\}} \in G$, if $p \succ q$ for all $i \in I$ then $p \succ_0 q$.

The second axiom is a mimimal necessary condition for the concept of collective risk aversion to make sense. It states that when the individuals are more risk averse so should be the society.

To formulate the axiom, first we state the definition of comparative risk aversion, following Arrow [1] and Pratt [19].

Definition 1 For each $i \in I \cup \{0\}$, say that $\succsim_i'$ is *more risk-averse* than $\succsim_i$ if

$$p \succsim_i' \delta(x) \quad \Longrightarrow \quad p \succsim_i \delta(x)$$

and

$$p \succ_i' \delta(x) \quad \Longrightarrow \quad p \succ_i \delta(x)$$

for all $p \in \Delta(X)$ and $x \in X$, where $\delta(x)$ denotes the lottery degenerate on x.

Notice that being more risk-averse implies having the same preference over deterministic outcomes, otherwise risk attitudes cannot be comparable.

The assertion below is standard.

Lemma 1 Let u_i' and u_i be von-Neumann/Morgenstern indices which form representations of $\succsim_i'$ and $\succsim_i$ respectively in the expected utility form.

Suppose that for all $x, y \in X$ and $\lambda \in [0,1]$ there exists $z \in X$ such that $\delta(z) \sim_i \lambda\delta(x) + (1-\lambda)\delta(y)$.

Then, $\succsim_i'$ is more risk-averse than $\succsim_i$ if and only if there exists a monotone and concave function $\phi : u_i(X) \to \mathbb{R}$ such that $u_i'(x) = \phi(u_i(x))$ for all $x \in X$.

Now we state the second axiom.

Comparative Risk Aversion Monotonicity: For all $(\succsim_i)_{i \in I \cup \{0\}}, (\succsim_i')_{i \in I \cup \{0\}} \in G$, if $\succsim_i'$ is more risk-averse than $\succsim_i$ for all $i \in I$, then $\succsim_0'$ is more risk-averse than $\succsim_0$.

We can show that Comparative Risk Aversion Monotonicity implies the condition what we call Ordinal Invariance, under certain domain richness condition. Given $\succsim_i$, for $i \in I \cup \{0\}$, let $\succsim_i |_X$ denote the restriction of $\succsim_i$ on X, the set of deterministic outcomes.

Ordinal Invariance: For all $(\succsim_i)_{i \in I \cup \{0\}}, (\succsim_i')_{i \in I \cup \{0\}} \in G$, if $\succsim_i |_X = \succsim_i' |_X$ for all $i \in I$ then $\succsim_0 |_X = \succsim_0' |_X$.

Ordinal Invariance is by itself an appealing requirement, whcih states that social ranking between deterministic outcomes should depend only on how the individuals rank between deterministic outcomes.

Say that $\mathcal{R}_i$ is *rich* if for all $\succsim_i, \succsim_i' \in \mathcal{R}_i$with $\succsim_i |_X = \succsim_i' |_X$ there exists $\succsim_i'' \in \mathcal{R}_i$ which is more or less risk-averse than both $\succsim_i$ and $\succsim_i'$.

Lemma 2 Suppose $\mathcal{R}_i$ is rich for all $i \in I$. Then Comparative Risk Aversion Monotonicity implies Ordinal Invariance.

Proof. Suppose $\succsim_i |_X = \succsim_i' |_X$ for all $i \in I$. Let $(\succsim_i'')_{i \in I}$ be a preferece profile such that $\succsim_i''$ is more/less risk-averse than both $\succsim_i$ and $\succsim_i'$, for all $i \in I$. Then $\succsim_0''$ is more/less risk-averse than both $\succsim_0$ and $\succsim_0'$. This implies $\succsim_0'' |_X = \succsim_0 |_X$ and $\succsim_0'' |_X = \succsim_0' |_X$. Hence $\succsim_0' |_X = \succsim_0 |_X$. ■

3 Economic domain

Let L be the set of commodities and consider that each individual has consumption set $\mathbb{R}_+^L$. Thus the set of deterministic social outcomes is taken as $X = \mathbb{R}_+^{I \times L}$.

Let $\mathcal{U}$ be the set of strictly increasing and concave functions from $\mathbb{R}_+^L$ to $\mathbb{R}$. For each $i \in I$, given $u \in \mathcal{U}$, let $\succsim_{i,u}$ denote i's expected utility preference over $\Delta(X)$ represented in the form

$$U_{i,u}(p) = E_{p_i}[u(x_i)]$$

where p_i denotes the marginal of p over i's consumptions. Then let

$$\mathcal{R}_i = \{\succsim_{i,u} \in \mathcal{R} : u \in \mathcal{U}\}.$$

Note that duplication should already taken into account in the sense that $\succsim_{i,u'} = \succsim_{i,u}$ if and only if $u' = au + b$ for some constants $a > 0$ and b. Note also that for all $\succsim_i \in \mathcal{R}_i$ it holds $p \sim_i q$ whenever $p_i = q_i$ for all $p, q \in \Delta(X)$. That is, each individual cares only about marginals over own consumptions. This is standard in the context of individual consumption/investment.[1]

[1]The argument readily extends to include public goods as far as there is at least one private good.

Note that such $\mathcal{R}_i$ is rich because the set of all concave functions on $\mathbb{R}^L_+$ representing the same ordinal preference over $\mathbb{R}^L_+$ has a least concave element (see Debreu [4]).

Note that in the domain $\prod_{i\in I}\mathcal{R}_i$ so called the minimal agreement condition (De Meyer and Mongin [5]) is met : for all $(\succsim_i)_{i\in I}\in\prod_{i\in I}\mathcal{R}_i$ there exist $p,q\in\Delta(X)$ such that $p\succ_i q$ for all $i\in I$.

The domain of individual risk preferences $\mathcal{R}_i$ is classified into equivalence classes according to preferences over deterministic outcomes. Let Θ_i denote the set of preferences over deterministic consumtions $\mathbb{R}^L_+$ induced by the elements of $\mathcal{R}_i$. Then we can write $\mathcal{R}_i=\bigcup_{\Theta_i}\mathcal{R}_i(\theta_i)$, where $\mathcal{R}_i(\theta_i)$ is the equivalence class of risk preferences which induce the same preference θ_i over own determinsitic consumptions.

Theorem 1 Aggregation rule G satisfies Ex-ante Pareto and Ordinal Invariance if and only if for every profile of equivalence classes $\theta=(\theta_i)_{i\in I}$ there is $i(\theta)\in I$ such that for all $(\succsim_i)_{i\in I\cup\{0\}}\in G$ with $(\succsim_i)_{i\in I}\in\prod_{i\in I}\mathcal{R}_i(\theta_i)$ it holds

$$p\succsim_0 q \quad\Longleftrightarrow\quad p_i\succsim_{i(\theta)} q_i$$

for all $p,q\in\Delta(X)$.

Proof. Pick any two $(\succsim_i)_{i\in I\cup\{0\}},(\succsim'_i)_{i\in I\cup\{0\}}\in G$, where $(\succsim_i)_{i\in I}$ and $(\succsim'_i)_{i\in I}$ induce the same profile of ordinal rankings over deterministic outcomes, and fix the profiles of vNM indices $(u_i)_{i\in I\cup\{0\}}$ and $(u'_i)_{i\in I\cup\{0\}}$, which form represenions of them respectively in the expected utility form.

Then, because the minimal agreement condition is met, by the Harsanyi theorem (Harsanyi [10], De Meyer and Mongin [5]) there exist $(\alpha_i)_{i\in I},(\alpha'_i)_{i\in I}\in\mathbb{R}^I_+\setminus\{\mathbf{0}\}$ and $\beta,\beta'\in\mathbb{R}$ such that

$$u_0(x)=\sum_{i\in I}\alpha_i u_i(x_i)+\beta$$

and

$$u_0'(x) = \sum_{i \in I} \alpha_i' u_i'(x_i) + \beta'.$$

Now suppose $\alpha_i, \alpha_j > 0$ for different $i, j \in I$, then u_0 is strictly increasing in x_i and x_j. In view of Ordinal Invariance, in order that u_0' preserves strong monotonicity in x_i and x_j we must have $\alpha_i', \alpha_j' > 0$. However, then, the rankings over deterministic (x_i, x_j), while other things remain equal, cannot be the same between under $\alpha_i u_i(x_i) + \alpha_j u_j(x_j)$ and under $\alpha_i' u_i(x_i) + \alpha_j' u_j(x_j)$, which contradicts to Ordinal Invariance.[2]

Hence $\alpha_i > 0$ can be true only for one $i \in I$.

In view of Ordinal Invariance again, in order that $u_0'(x)$ depends only on x_i, we have to have $\alpha_j' = 0$ for all $j \neq i$, for any profile $(\succsim_i')_{i \in I \cup \{0\}}$ which yield the same profile of ordinal rankings over deterministic consumptions. ■

Since the economic domain is rich, we obtain the following corollary.

Corollary 1 Aggregation rule G satisfies Ex-ante Pareto and Comparative Risk Aversion Monotonicity if and only if for every profile of equivalence classes $\theta = (\theta_i)_{i \in I}$ there is $i(\theta) \in I$ such that for all $(\succsim_i)_{i \in I \cup \{0\}} \in G$ with $(\succsim_i)_{i \in I} \in \prod_{i \in I} \mathcal{R}_i(\theta_i)$ it holds

$$p \succsim_0 q \quad \Longleftrightarrow \quad p_i \succsim_{i(\theta)} q_i$$

for all $p, q \in \Delta(X)$.

Is the negative result due to a *mere* violation of Ordinal Invariance?

One might have the above question. Our answer is No. Violation of Ordinal Invariance leads to a significant violation of Comparative Risk Aversion Monotonicity.

[2]To understand, the reader may consider that $|L| = 1$ and $u_i(x_i) = x_i$ for every $i \in I$. Then $u_0(x) = \sum_{i \in I} \alpha_i x_i + \beta_i$ cannot represent the same ranking as $\sum_{i \in I} \alpha_i' u_i'(x_i) + \beta'$ when u_i' is generally non-linear, except under dictatorship within the equivalence class.

To illustrate, consider that $I = \{1, 2\}$ and $|L| = 1$, and consider two preference profiles $(\succsim_1, \succsim_2)$ and $(\succsim_1', \succsim_2')$. Consider that the individuals are risk-neutral at $(\succsim_1, \succsim_2)$ and the vNM index for each preference is given by $u_i(x_i) = x_i$. Also consider that they exhibit risk aversion at $(\succsim_1', \succsim_2')$ and the vNM index for each preference is given by $u_i(x_i) = x_i^\rho$ with $0 < \rho < 1$. Note that there is no heterogeneity in risk attitude or in ordinal preference.

For simplicity we put symmetric weights at both profiles, then the vNM index for $\succsim_0$ takes the form $u_0(x) = x_1 + x_2$, where we omit the constant term without loss of generality. On the other hand, the vNM index for $\succsim_0'$ takes the form $u_0'(x) = x_1^\rho + x_2^\rho$.

The ordinal ranking over deterministic allocations induced by u_0 exhibits no inequality aversion, while the ordinal ranking induced by u_0' exhibits significant inequality aversion. Therefore, by picking any $x \in \mathbb{R}^2_+$ with $x_1 \neq x_2$, the set

$$A(x) = \{y \in \mathbb{R}^2_+ : \delta(y) \succ_0' \delta(x) \text{ and } \delta(y) \prec_0' \delta(x)\}$$

is non-empty and has a positive measure. $A(x)$ is larger when x is more unequal.

Then any $p \in \Delta(X)$ with its support being included to $A(x)$ leads to

$$p \succ_0' \delta(x) \text{ and } p \prec_0' \delta(x).$$

Now pick an arbitrary $q \in \Delta(X)$, then for sufficiently small $\lambda \in [0, 1]$ we obtain

$$(1 - \lambda)p + \lambda q \succ_0' \delta(x) \text{ and } (1 - \lambda)p + \lambda q \prec_0' \delta(x).$$

What is happening is that the society with more risk-averse individuals can be occasionally less risk-averse when it comes to comparison between a risky prospect being likely to generate less unequal outcomes and a sure prospect with a more unequal outcome, because the ordinal ranking over deterministic allocations exhibits more inequality aversion. That's why the requirement that risk attitudes are comparable only when the ordinal rankings agree over deterministic outcomes is significant at a social level as well.

4 Discussions

We have shown that two natural requirements on collective decision under risk, one that when individuals are more risk-averse so should be the society, the other that if everybody prefers one risky prospect over another so should the society, lead to conditional discatorship: the social ranking over risky prospects has to be always identical with one individual's risk preference, in each equivalence class of risk preferences which yield the same profile of ordinal preferences.

The result suggests that we have to choose between two options, in order to avoid the conditional dictatorship. One is to accept that even in evaluating deterministic social outcomes we have to take the individuals' risk attitudes into account. This is indeed the Benthamite-type idea in the optimal taxation context (Mirlees [15]).

The other is to give up the presumption that everybody is responsible for his/her risk attitude, in other words, to give up the presumption that risk attitudes are a "matter of taste." The ex-ante Pareto condition has been regarded as problematic when there is disagreements in beliefs (Mongin [16, 17], Gilboa, Samet and Schmeidler [7]). Our result suggests that even when there is no belief disagreement the ex-ante Pareto condition may be problematic.

The reader might be uncomfortable with the nature of the adopted definition of comparative risk aversion, since it includes the condition that being more risk-averse implies having the same ranking over deterministic outcomes. One might want to compare risk attitudes even when the rankings induced over deterministic outcomes differ. However, as we discussed above the basic requirement that risk aversion can be comparable only under the same ranking over deterministic outcomes seems uncompromisable.

One may wonder the relationship with the long literature of risk torelance of representative agent (see Arrow and Lind [2], Malinvaud [14], Mazzocco [13], Hara, Huang and Kuzmics [9], Jouini, Napp and Nocetti [11], Chambers and Echenique [3], among many). These studies focus on the representative agent's risk attitude over random *aggregate* resource, assuming that the random aggregate is allocated among the individuals in an efficient way along with the ex-ante Pareto condition. Thus, in general the ranking over risky *allocations* falls in the Harsanyi theorem argument, which cannot allow a consistent definition of collective risk aversion as we saw, although they show that in certain class of individual risk preferences the representative agent's ranking over risky aggregate resources inherits reasonable natures of risk attitude, or that it converges to be risk-neutral when the society tends to be larger.

The final point is that we have adopted expected utility theory, not only as a descriptive assumption on individuals' preferences but also as a normative requirement for the social ranking. It is known in the literature of consumption-investment that risk aversion and elasticity of intertemporal substitution are mutually entangled under expected utility theory (Hall [8], Epstein and Zin [6]). Taking an analogoue of this suggests that imposing expected utility theory on social ranking forces risk aversion and inequality aversion to be entangled with each other.

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