Optimal Funded Pension for Consumers with Heterogeneous Self-Control

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Abstract

Though funded pensions have a role as forced saving, it is known that the introduction of the funded pensions does not affect to the amount of saving in standard models. On the other hand, from a practical viewpoint, consumers with disposable income often face temptations that make them overspend. They cannot help feeling psychological costs to resist this. Under this assumption, the forced saving is significant since it leads consumers to optimal saving level with lower self-control cost. The government maximizes the sum of consumers' utilities taking account of self-control cost by making each consumer choose a plan, a pair of premium and payout, from a list. Problems are that the degree of temptation is differ among consumers, that a government cannot observe the degree and that consumers can borrow money with collateralizing their future payouts. We show that the optimal list consists of plans that return just as managed premium paid by a consumer that chose the plan. Furthermore, the consumer with smaller self-control cost chooses a smaller premium in an equilibrium. If he pays the large amount of premium, he faces larger temptation of borrowing since the premium brings him a large borrowing facility. So he chooses a small premium to avoid it. We also show that the government does not need to know the distribution of types. This is a beneficial feature from a practical viewpoint.

JEL Classification: D03; D82; H55;
Key Words: Self-Control; Overconsumption; Funded Pension

> *(One of) the justification(s) for social security is that many individuals will not save enough for retirement if left to their own devices. ⋯ One could peg the argument on the difficulty in obtaining suitable information for informed judgement on the need for savings for retirement. ⋯ One could base the argument on efficiency in decision making rather than in information gathering. Third one might simply fall back on the factors that lead people to spend more now and less later than seems sensible.* (Diamond, 1977: pp.281-282).

1 Introduction

We consider a public funded pension scheme. Social security policies including a funded pension have some objectives. One of them is income redistribution. This is common objective in almost all social security policies such as funded pensions, social aid by taxation

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and pay-as-you-go pensions. Other important role of social securities is that as forced saving devices, which is specific to funded pensions. This is because funded pensions assure subscribers that they can receive retirement incomes at least their contribution. However, it is well known that for consumers in standard economic models, funded pensions do not improve social welfare (e.g. Samuelson, 1975). For example, suppose that a consumer found that his optimal amount of saving is \$1000 per month. When there is no pension policy, he privately saves \$1000. When a pension is introduced and the premium that he pays is \$500 per month, he pays it and privately saves \$500 because his optimal saving is \$1000. Since his total amount of saving does not change between before and after the introducing of the pension, so does his lifetime utility. Of course, if the premium is larger than \$1000, he goes into oversaving that decreases his lifetime utility. Thus the pension does not improve social welfare.

One of factors to justify funded pensions is temptation that leads consumers to overconsumption (Diamond, 1977). Even if a consumer knows how much to save his income, he often cannot help but overspending more than the amount when he has disposable income. Once he faces this temptation, he feels psychological cost to self-control his wasteful spending. So it is ideal for him not to face temptation by a valid way for commitment. This is why we need funded pensions as a forced saving policy.

However, in the context of temptation, most of pension schemes presently used are not suitable. It is natural that the degrees of temptation are not the same for all consumers. While there are consumers that spend much of their income as soon as they earns them and regret their myopic behavior, there are also consumers that can spend their income farsightedly. Since pension schemes affect the behavior of consumers, they should be designed with thinking of this point. However, in present pension schemes, a premium that each consumer pays is determined mainly by his income level, not by his degree of temptation. Our purpose is to design a pension scheme that maximizes social welfare considering the difference in the degree of temptation.

In our model, the government proposes a pension schedule to consumers, where pension schedule is a list of pension plans, a pair of pension premium and its payout. Each consumer choose a plan from the schedule and pay the premium according to the plan before his consumption, that is, before he faces to temptation. This is why the pension works as forced saving [1]. However, we allow the consumer to borrow money by collateralizing his future payout.

Some literature studied policies as a way to resist temptation. It was shown that funded pensions improve social welfare (Gul and Pesendorfer, 2004). This is because, as stated above, they enable consumers to avoid temptation with less self-control cost. In addition, pay-as-you-go pension is also able to improve social welfare if temptation is sufficiently large (Kumru and Thanopoulos, 2008). Other way to relieve self-control cost is to make consumption relatively unattractive. This works because the cost arises from the gap between the attractiveness of normatively desirable alternatives and that of tempting alternative. Krusell et al. (2010) shows that subsidies for savings improve social welfare for this reason. Though these studies are important, their research objects are economies with consumers whose degrees of temptation are the same for all of them. On the other hand, our model can analyze the situation in which the degrees are different for different consumers.

Galperti (2015) investigated an optimal contract as a commitment device. The agent is either consistent or inconsistent, which is randomly determined and is private information

[1] We focus on sophisticated consumers, that is, each consumer knows that he will face to temptation at the decision of consumption.

of the agent. If he is inconsistent, he values his consumption in the second period smaller than that in the first period. The characteristic of the model is that it takes in the preference for flexibility. It is an important stuff especially when the problem has long period. On the other hand, it is assumed that no third party can offer the agent contracts that might interfere with the provider's devices (Galperti, 2015:p.1431). It is the advantage of our model that agents (consumers) are allowed to make debt after they choose their devices.

Since we need a model that describes decisions with temptation and self-control, we apply the model proposed by Gul and Pesendorfer (2001), henceforth GP. They considered two kinds of preferences over alternatives, normative preferences and temptation preferences, which are represented by functions u and v, respectively. Using u and v, the valuation of menu (a set of alternatives) M is defined by

$$W(M) \equiv \max_{x \in M}[u(x) + v(x)] - \max_{y \in M} v(y). \tag{1}$$

GP assumes that, for a given menu M, a decision maker chooses an alternative $x \in M$ that maximizes $u(x) + v(x)$. The intuition is that he chooses a compromise between normative desirability and temptation desirability after facing temptation. Let $x^* \in M$ be an alternative that maximizes $u(x) + v(x)$. Then (1) is rewritten as

$$W(M) = u(x^*) - \left\{ \max_{y \in M} v(y) - v(x^*) \right\}.$$

The term $\max_{y \in M} v(y) - v(x^*)$ represents self-control cost. For instance, suppose that a consumer faces a menu $M = \{s, h\}$, where s and h represents salad and hamburger, respectively. And suppose that he wants to diet but he likes meat. In GP model, u corresponds to a desirability of health and v does to a desirability of having what he likes. Thus $u(s) > u(h)$ and $v(h) > u(s)$ holds. If $u(s)+v(s) > u(h)+v(h)$, he chooses salad for his lunch. Note that salad does not maximize v, thus he gives up satisfaction of temptation as much as $v(h) - v(s)$. GP defined self-control cost by this difference. Generalizing this concept, we have the expression above.

Menus correspond to budget sets in consumption choice problems. If a pension premium can shrink the budget sets, it works as a commitment device and decreases self-control cost. Formally, assume that a pension premium makes a menu M shrink to $M' \subset M$. Then we have $\max_{y \in M'} v(y) \leqslant \max_{y \in M} v(y)$, thus $\max_{y \in M'} v(y) - v(x^*) \leqslant \max_{y \in M} v(y) - v(x^*)$ follows. Especially, if the inequality holds strictly, the self-control cost decreases strictly.

As an alternative approach to analyze myopic behaviors, we may use (quasi) hyperbolic discounting model provided by Laibson (1997). Actually, some literature on economic policies employ this approach (e.g. Roeder, 2014). The model describes the situation in which preferences are time inconsistent. However, the model cannot describe self-control cost explicitly. Since we want to treat effects of temptation and self-control separately, we employ GP model that is proper for our objective.

Our main contribution is presenting a concrete way to design the optimal pension scheme under the assumption that normative utility is $\log(c)$. This scheme has some interesting character. First, the optimal schedule consists of plans that return just as managed premium paid by a consumer that chose the plan. This implies that no income redistribution is made. Second, the consumer with smaller self-control cost chooses a smaller premium in an equilibrium. If he pays the large amount of premium, he faces larger temptation of borrowing since the premium brings him a large borrowing facility.

So he chooses a small premium to avoid it. If consumers cannot borrow money, the optimal schedule has unique plan whose premium is equal to the amount of saving that consumers choose when there is no temptation and no pension scheme. Third, the optimal schedule does not depend on the information of types such as its distribution and even what types there are. This means that the government does not need to know these information. This is a beneficial feature from a practical viewpoint.

Section 2 introduces notations and assumptions used in the study and looks at the consumption-saving decision of a consumer with a self-control preference. Section 3 studies identical-type and two-type economies as benchmarks and shows the monotonicity of an optimal pension. In section 4, we generalize the results in section 3 to finitely many types and continuous-types model. We show that the results are robust for generalization. In section 5, we discuss the effect of a borrowing constraint and extend the model to analyze income diversity.

2 Model

The only difference between standard models of pension and our model is that consumers have preference with temptation and self-control. There are consumers and an government. At first, the government offers a set of pairs of pension payout and pension premium to each consumer before his consumption decision in working age. Then the consumers choose one of pairs from the set. After a payment of premium the consumer has chosen, he decides how much to consume in working age.

2.1 Budget Constraint

We standardize the population of consumer to be 1. Each consumer is endowed with an identical income of $I \in \mathbb{R}_{++}$. Let $R \in \mathbb{R}_{+}$ and $P \in [0, I]$ be a pension payout and a pension premium, respectively. We assume the upper bound of P to be I to rule out situations in which consumers borrow money to pay the premiums. We call a pair of R and P *pension plan*. For the simplicity of notation, we define the set of possible pension plans as $T \equiv \mathbb{R}_{+} \times [0, 1]$. A *pension schedule* is a set of pension plans.

In period 0, each consumer chooses a pension plan $\tau \in T$, from a pension schedule $S \subseteq T$. At the same time, he has to pay the government the pension premium he has chosen. In period 1, he decides the amounts of consumption $c_1 \in \mathbb{R}_{+}$ and saving $I - P - c_1$ in working age. We assume that consumers can also borrow money in period 1 with putting up their pension income as collateral. Thus, if $I - P - c_1$ is strictly greater (less) than 0, it represents the amount of saving (borrowing). Let $r,\ \rho \in \mathbb{R}_{++}$ be the interest rate for saving and borrowing, respectively. Thus if a consumer chose $\tau = (R, P)$, he can borrow up to $\frac{R}{1+\rho}$. About interest rates, we assume that $\rho > r > 0$. This assumption is natural from a practical standpoint. In period 2, he receives a pension income R and decides consumption in old age $c_2 \in \mathbb{R}_{+}$.

The budget set for consumers having chosen $\tau = (R, P)$ is summarized as follows.

$$B(\tau) \equiv \left\{ (c_1, c_2) \in \mathbb{R}_{+}^2 : \begin{array}{c} c_1 \leqslant I - P + \frac{R}{1+\rho}, \\ I - P - c_1 \geqslant 0 \Rightarrow c_2 \leqslant (1+r)(I - P - c_1) + R, \\ I - P - c_1 < 0 \Rightarrow c_2 \leqslant (1+\rho)(I - P - c_1) + R \end{array} \right\}.$$

Denote the set of all possible $B(\tau)$ by $\mathscr{B}$. Figure shows this budget constraint. As you see, the budget line kinks at $(c_1, c_2) = (I - P, R)$.

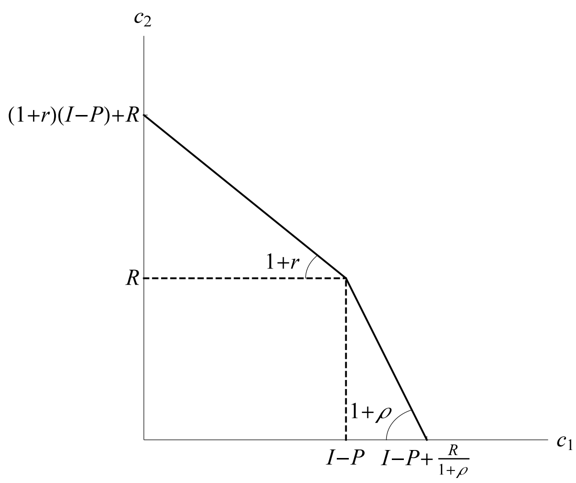

Figure 1: Budget constraint

2.2 Preference

Consumers have self-control preference introduced by GP. We begin with defining two kinds of utility functions representing preference over consumption. One is normative utility function $u\colon \mathbb{R}_+ \to \mathbb{R}$ and the other is temptation utility function $v\colon \mathbb{R}_+ \to \mathbb{R}$. To express the temptation, v is specialized as $v(\cdot,\cdot) \equiv \lambda u(\cdot,\cdot)$, where $\lambda \in \Lambda \subseteq \mathbb{R}_+$ denotes the strength of the temptation. This form of temptation utility is also used in Gul and Pesendorfer (2004). We assume the value λ differs across agent and is private information. For any $\lambda \in \Lambda$, let n_λ be a proportion of consumer, that is, n_λ satisfies $0 \leqslant n_\lambda \leqslant 1$ and $\sum_{\lambda\in\Lambda} n_\lambda = 1$. In section 4.2, we consider continuous Λ. Then the proportion is represented by a distribution function $F\colon \Lambda \to [0,1]$ with density function $f\colon \Lambda \to \mathbb{R}_+$.

We impose a few assumptions about the normative utility function.

Assumption 2.1. *u is twice continuously differentiable.*

Assumption 2.2. *u is strictly concave and satisfies $u'(c) > 0$ and $u''(c) < 0$.*

Assumption 2.3. $\lim_{c\to 0} u'(c) = \infty$.

For example, these assumption is satisfied if normative utility functions is $u(c) = \log(c)$ or $u(c) = c^\alpha$ with $\alpha \in (0,1)$. In the latter half of this chapter, we assume $u(c) \equiv \log(c)$ to derive sharper results. Applying the utility function by Gul and Pesendorfer (2001), the preference over budget sets is represented by the function $W\colon \mathscr{B} \times \mathbb{R}_+ \to \mathbb{R}$ satisfies that

$$\begin{aligned} W(B(\tau);\lambda) &= \max_{(c_1,c_2)\in B(\tau)} \big[u(c_1) + \delta u(c_2) + \lambda u(c_1)\big] - \max_{(c_1,c_2)\in B(\tau)} \lambda u(c_1) \\ &= \max_{(c_1,c_2)\in B(\tau)} \big[(1+\lambda)u(c_1) + \delta u(c_2)\big] - \max_{(c_1,c_2)\in B(\tau)} \lambda u(c_1), \end{aligned}$$

where $\delta \in (0,1)$ is a discount factor.

Using assumption 2.2, we have the following claim.

Claim 2.1. $\max_{(c_1,c_2)\in B(\tau)} \lambda u(c_1) = \lambda u\left(I - P + \frac{R}{1+\rho}\right)$.

This is obvious by $u'(c) > 0$ (assumption 2.2) and the constraint for c_1 is $c_1 \leqslant I - P + \frac{R}{1+\rho}$.

2.3 Consumption

In this section, we consider the consumption problem contained in $W(B(\tau);\lambda)$, that is, $\max_{(c_1,c_2)\in B(\tau)}\big[(1+\lambda)u(c_1)+\delta u(c_2)\big]$. When the consumer of type $\lambda \in \Lambda$ chooses a pension plan $\tau \in T$, let his optimal consumption in period t be $c_t^\lambda(\tau)$.

There can be the three types of consumption: positive saving, zero saving (or balanced) and negative saving (or borrowing). The type of optimal consumption depends on the pension plan that a consumer has chosen. Intuitively, since a large premium brings too little disposable income at a working age, it may be necessary for him to borrow to achieve his desired level of consumption. Formally, we separate T as

$$\begin{aligned}T_s^\lambda &\equiv \{\tau \in T\colon c_1^\lambda(\tau) < I - P\}\\ T_b^\lambda &\equiv \{\tau \in T\colon c_1^\lambda(\tau) = I - P\}\\ T_d^\lambda &\equiv \{\tau \in T\colon c_1^\lambda(\tau) > I - P\}.\end{aligned}$$

τ_s^λ, τ_b^λ and τ_d^λ denote the typical elements of T_s^λ, T_b^λ and T_d^λ, respectively. λ in superscript is omitted if there is no threat of confusion. Note that if $\tau \in T_b$, $c_2^\lambda(\tau) = R$, since $u'(c) > 0$ and income in period 2 is only pension income R.

We have the necessary conditions for optimality in different form for τ_s and τ_d since the budget constraint in period 2 differs. In this consumption optimization problem, an objective function is $U(c_1, c_2) \equiv (1+\lambda)u(c_1) + \delta u(c_2)$. So (the absolute value of) the marginal rate of substitution of c_1 for c_2 is

$$\frac{\frac{\partial U(c_1,c_2)}{\partial c_1}}{\frac{\partial U(c_1,c_2)}{\partial c_2}} = \frac{(1+\lambda)u'(c_1)}{\delta u'(c_2)}.$$

If $\tau \in T_s$, the budget line has the slope of $1 + r$. So the necessary condition for τ_s is [2]

$$\begin{aligned}&\frac{(1+\lambda)u'(c_1(\tau_s))}{\delta u'(c_2(\tau_s))} = 1 + r\\ \Leftrightarrow&(1+\lambda)u'(c_1(\tau_s)) = (1+r)\delta u'(c_2(\tau_s)).\end{aligned} \tag{2}$$

On the other hand, if $\tau \in T_d$, the budget line has a slope of $1 + \rho$. So the necessary condition for τ_d is

$$\begin{aligned}&\frac{(1+\lambda)u'(c_1(\tau_d))}{\delta u'(c_2(\tau_d))} = 1 + \rho\\ \Leftrightarrow&(1+\lambda)u'(c_1(\tau_d)) = (1+\rho)\delta u'(c_2(\tau_d)).\end{aligned} \tag{3}$$

Since a budget line kinks at the consumption vector for τ_b, the slope of the line there is not defined. Thus we cannot derive a condition as in other cases. However, we have at least the following inequality:

$$\begin{aligned}&1 + r < \frac{(1+\lambda)u'(c_1(\tau_b))}{\delta u'(c_2(\tau_b))} < 1 + \rho\\ \Leftrightarrow&(1+r)\delta u'(c_2(\tau_b)) < (1+\lambda)u'(c_1(\tau_b)) < (1+\rho)\delta u'(c_2(\tau_b)).\end{aligned}$$

[2] There is not any corner solution because of assumption 2.3.

2.4 Government

The government offers consumers a pension schedule $S \equiv \{\tau_\lambda \in T : \lambda \in \Lambda\} \subset T$, a set of pension plans. The objective of the government is to maximize the (expected) aggregated welfare of consumers while controlling S to satisfy the following conditions:

$$W(B(\tau_\lambda);\lambda) \geqslant W(B(\tau');\lambda),\ \forall\lambda \in \Lambda,\ \forall\tau' \in S \tag{IC}$$

$$(1+r)\sum_{\lambda\in\Lambda} n_\lambda P_\lambda \geqslant \sum_{\lambda\in\Lambda} n_\lambda R_\lambda. \tag{FB}$$

The first condition states that the consumers of type $\lambda \in \Lambda$ will choose a plan $\tau(\lambda)$ at their own initiative; in other words, this is the condition for not giving consumers any incentive to report their types untruthfully. Note that this is not necessary when types are public information since then government can force any plan on consumers.

The second condition states that S is feasible. We assume that pension management interest rate r is the same as that of the private saving. This means that there is no difference between the government and private banks in the ability to manage assets.

We also assume that we can ignore an individual rationality condition since the pension is managed by the government, which has the power of consumers' participation.

3 Benchmark

3.1 Identical-type consumers

In this section, we consider the simple case in which all consumers have identical type, that is, $\Lambda = \{\lambda\}$ and this is common knowledge. Then the condition (IC) can be ignored. Moreover, since λ is identical, the condition (FB) is rewritten as private condition, that is, $(1+r)P(\lambda) \geqslant R(\lambda)$.

Consider the individual welfare function $W(\tau;\lambda)$. Define $\sigma_\lambda\colon T \to \mathbb{R}$ to be the marginal rate of substitution of R for P on $\tau \in T$, that is,

$$\sigma_\lambda(\tau) = -\frac{\frac{\partial W(\tau;\lambda)}{\partial R}}{\frac{\partial W(\tau;\lambda)}{\partial P}}.$$

Since we cannot represent the optimal consumption in a general form, we have to consider the optimal pension plan separately. However, the following lemma makes it easy to analyze.

Lemma 3.1. *For any individual welfare level, an indifference curve corresponding to the level in R-P plane satisfies following properties.*

(i) $\sigma_\lambda(\tau_s)$ *is greater than or equal to* $\frac{1}{1+r}$*, where equality holds if and only if* $\lambda = 0$*.*

(ii) $\sigma_\lambda(\tau_d)$ *is equal to* $\frac{1}{1+\rho} < \frac{1}{1+r}$*.*

(iii) Every indifference curve is differentiable.

Proof of property (i). Consider arbitrary $\tau \in T_s$. Then we have

$$W(B(\tau);\lambda) = (1+\lambda)u(c_1(\tau)) + \delta u\big(c_2(\tau)\big) - \lambda u\left(I - P + \frac{R}{1+\rho}\right).$$

The marginal welfare of R is calculated as follows:

$$\begin{aligned}\frac{\partial W(B(\tau);\lambda)}{\partial R} =&(1+\lambda)u'(c_1(\tau))\frac{\partial c_1(\tau)}{\partial R}+\delta u'\big(c_2(\tau)\big))\frac{\partial c_2(\tau)}{\partial R}\\&-\frac{\lambda}{1+\rho}u'\left(I-P+\frac{R}{1+\rho}\right)\end{aligned}$$

Since $c_2(\tau_s)=(1+r)(I-P-c_1(\tau_s))+R$, $\frac{\partial c_2(\tau_s)}{\partial R}=-(1+r)\frac{\partial c_1(\tau_s)}{\partial R}+1$. Hence we have

$$\begin{aligned}\frac{\partial W(B(\tau);\lambda)}{\partial R}&=\Big\{(1+\lambda)u'(c_1(\tau))-(1+r)\delta u'(c_2(\tau))\Big\}\frac{\partial c_1(\tau)}{\partial R}\\&+\delta u'(c_2(\tau))-\frac{\lambda}{1+\rho}u'\left(I-P+\frac{R}{1+\rho}\right)\\&=\delta u'(c_2(\tau))-\frac{\lambda}{1+\rho}u'\left(I-P+\frac{R}{1+\rho}\right).\end{aligned}$$

The second equality follows by the first order condition for optimal consumption, that is, $(1+\lambda)u'(c_1(\tau))=(1+r)\delta u'(c_2(\tau))$. Moreover, the last line can be rewritten as

$$\frac{1+\lambda}{1+r}u'(c_1(\tau))-\frac{\lambda}{1+\rho}u'\left(I-P+\frac{R}{1+\rho}\right),$$

and this is strictly positive. To see this, use assumption 2.2. Since $c_1(\tau)\leqslant I-P+\frac{R}{1+\rho}$ and $u''(c)<0$, it holds that $u'(c_1(\tau))\geqslant u'\left(I-P+\frac{R}{1+\rho}\right)$. By $\rho>r$ and $\lambda\geqslant 0$, we have $\frac{\partial W(\tau;\lambda)}{\partial R}>0$.

Next, the marginal welfare of P is

$$\begin{aligned}\frac{\partial W(B(\tau);\lambda)}{\partial P}&=(1+\lambda)u'(c_1(\tau))\frac{\partial c_1(\tau)}{\partial P}-\delta u'(c_2(\tau))(1+r)\left(1+\frac{\partial c_1(\tau)}{\partial P}\right)\\&+\lambda u'\left(I-P+\frac{R}{1+\rho}\right)\\&=\Big\{(1+\lambda)u'(c_1(\tau))-(1+r)\delta u'(c_2(\tau))\Big\}\frac{\partial c_1(\tau)}{\partial P}\\&-(1+r)\delta u'(c_2(\tau))+\lambda u'\left(I-P+\frac{R}{1+\rho}\right)\\&=-(1+\lambda)u'(c_1(\tau))+\lambda u'\left(I-P+\frac{R}{1+\rho}\right).\end{aligned}$$

By the same ways as in the previous paragraph, the equalities follow and the last line is strictly negative.

Thus, $\sigma_\lambda(\tau)$ for $\tau\in T_s$ is

$$\sigma_\lambda(\tau)=-\frac{\frac{\partial W(B(\tau);\lambda)}{\partial R}}{\frac{\partial W(B(\tau);\lambda)}{\partial P}}=\frac{-\delta u'(c_2(\tau))+\frac{\lambda}{1+\rho}u'\left(I-P+\frac{R}{1+\rho}\right)}{-(1+\lambda)u'(c_1(\tau))+\lambda u'\left(I-P+\frac{R}{1+\rho}\right)}.$$

We show that this is greater than or equal to $\frac{1}{1+r}$. Since $\rho>r$, we have

$$\begin{aligned}&-(1+\lambda)u'(c_1(\tau))+\frac{1+r}{1+\rho}\lambda u'\left(I-P+\frac{R}{1+\rho}\right)\\&\qquad\leqslant-(1+\lambda)u'(c_1(\tau))+\lambda u'\left(I-P+\frac{R}{1+\rho}\right),\end{aligned}$$

where equality holds if $\lambda = 0$. Furthermore, since $u'\left(I - P + \frac{R}{1+\rho}\right) > 0$ it holds only if $\lambda = 0$. Using the first order condition,

$$\begin{aligned}
&-(1+r)\delta u'(c_2(\tau)) + \frac{1+r}{1+\rho}\lambda u'\left(I - P + \frac{R}{1+\rho}\right) \\
&\qquad \leqslant -(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right).
\end{aligned}$$

Note that the right-hand side is strictly negative. Thus this inequality is rewritten as,

$$\begin{aligned}
&\frac{-\delta u'(c_1(\tau)) + \frac{\lambda}{1+\rho}u'\left(I - P + \frac{R}{1+\rho}\right)}{-(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)} \geqslant \frac{1}{1+r} \\
\Leftrightarrow &\sigma_\lambda(\tau) \geqslant \frac{1}{1+r}.
\end{aligned}$$

□

Proof of Property (ii). Calculating $\sigma_\lambda(\tau)$ for $\tau \in T_d$, we have

$$\begin{aligned}
\sigma_\lambda(\tau) &= \frac{-\delta u'(c_2(\tau)) + \frac{\lambda}{1+\rho}u'\left(I - P + \frac{R}{1+\rho}\right)}{-(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)} \\
&= \frac{-(1+\rho)\delta u'(c_2(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)}{(1+\rho)\left\{-(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)\right\}}.
\end{aligned}$$

By the first order condition for $\tau \in T_d$ in the consumption problem, we have

$$(1+\rho)\delta u'(c_2(\tau)) = (1+\lambda)u'(c_1(\tau)).$$

Therefore $\sigma_\lambda(\tau) = \frac{1}{1+\rho} < \frac{1}{1+r}$. □

Proof of Property (iii). It is enough to show that there exist $\tau \in T_s$ and $\tau \in T_d$ such that $\sigma_\lambda(\tau)$ is equal to that for $\tau \in T_b$, respectively. We have found that the slopes are

$$\frac{-\delta u'(c_2(\tau)) + \frac{\lambda}{1+\rho}u'\left(I - P + \frac{R}{1+\rho}\right)}{-(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)}, \text{ if } \tau \in T_s \tag{4}$$

$$\frac{-\delta u'(R) + \frac{\lambda}{1+\rho}u'\left(I - P + \frac{R}{1+\rho}\right)}{-(1+\lambda)u'(I-P) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)}, \text{ if } \tau \in T_b \tag{5}$$

$$\frac{-\delta u'(c_2(\tau)) + \frac{\lambda}{1+\rho}u'\left(I - P + \frac{R}{1+\rho}\right)}{-(1+\lambda)u'(c_1(\tau)) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)}, \text{ if } \tau \in T_d. \tag{6}$$

Fix arbitrary $\hat{P} \in [0, I]$ and choose any pair $(R, \hat{P})$ such that $(R, \hat{P}) \in T_s$. Similarly, choose any pair $(R', \hat{P})$ such that $(R', \hat{P}) \in T_b$. Denote $\sup_{(R,\hat{P})\in T_s} R$ by $\overline{R}(\hat{P})$. Note that

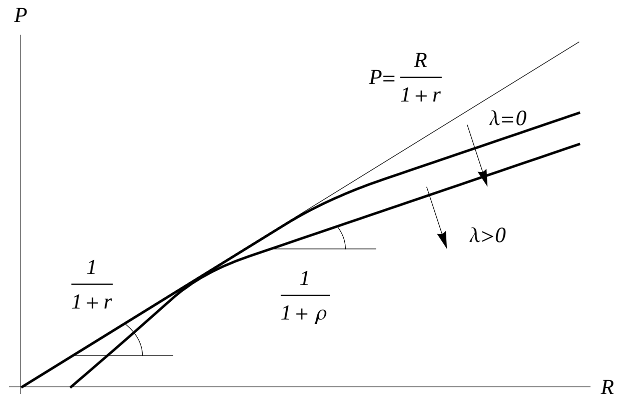

Figure 2: Indifference curves

consumption function is continuous. According to the definitions of T_s and T_b, it follows that

$$\begin{aligned}&c_1(R,\hat{P}) \to I-\hat{P},\ c_2(R,\hat{P}) \to \overline{R}(\hat{P})\\&c_1(R'\hat{P}) \to I-\hat{P},\ c_2(R',\hat{P}) \to \overline{R}(\hat{P})\\&\text{as } R \to \overline{R}(\hat{P}) \text{ and } R' \to \overline{R}(\hat{P}).\end{aligned}$$

Then we have

$$\lim_{R\to\overline{R}(\hat{P})} \sigma_\lambda(R,\hat{P}) = \frac{-\delta u'(\overline{R}(\hat{P})) + \frac{\lambda}{1+\rho}u'\left(I-\hat{P}+\frac{\overline{R}(\hat{P})}{1+\rho}\right)}{-(1+\lambda)u'(I-\hat{P}) + \lambda u'\left(I-\hat{P}+\frac{\overline{R}(\hat{P})}{1+\rho}\right)}$$

$$\lim_{R'\to\overline{R}(\hat{P})} \sigma_\lambda(R',\hat{P}) = \frac{-\delta u'(\overline{R}(\hat{P})) + \frac{\lambda}{1+\rho}u'\left(I-\hat{P}+\frac{\overline{R}(\hat{P})}{1+\rho}\right)}{-(1+\lambda)u'(I-\hat{P}) + \lambda u'\left(I-\hat{P}+\frac{\overline{R}(\hat{P})}{1+\rho}\right)},$$

that is, $\lim_{R\to\overline{R}(\hat{P})} \sigma_\lambda(R,\hat{P}) = \lim_{R'\to\overline{R}(\hat{P})} \sigma_\lambda(R',\hat{P})$. Therefore the indifference curve is smoothly continuous at $(\overline{R}(\hat{P}),\hat{P})$. Similarly we can prove the latter half of property (iii). □

Even though the consumption patterns are different for τ, this lemma states that indifference curves for fixed welfare are smooth. Thus we can use the standard method of welfare maximization. The indifference curve is shown in Figure 2. Here, the pension plan that satisfies feasibility is in the upper-left area of the line $P = \frac{R}{1+r}$. Hence we find that optimal plans are determined at the tangent point of the indifference curve and the line of $P = \frac{R}{1+r}$. According to the lemma 3.1, the indifference curve does not have any tangent to the line $P = \frac{R}{1+r}$ at $\tau \in T_s$ and $\tau \in T_d$ if $\lambda > 0$. Thus, if τ is optimal, it is included in T_b except the case of $\lambda = 0$. If $\lambda = 0$, since $\sigma_\lambda(\tau_s) = \frac{1}{1+r}$, all $\tau_s \in T_s$ are optimal.

Let us consider optimal τ. First, we assume that $\lambda > 0$. As we discussed, it is enough to obtain optimal plans to consider $\tau \in T_b$. The optimal plan is determined at the tangent point of the indifference curve and $P = \frac{R}{1+r}$. Here, we have

$$\sigma_\lambda(\tau_b) = \frac{-\delta u'(R) + \frac{\lambda}{1+\rho}u'\left(I-P+\frac{R}{1+\rho}\right)}{-(1+\lambda)u'(I-P) + \lambda u'\left(I-P+\frac{R}{1+\rho}\right)}.$$

Thus the necessary condition is

$$\frac{-\delta u'(R) + \frac{\lambda}{1+\rho} u'\left(I - P + \frac{R}{1+\rho}\right)}{-(1+\lambda)u'(I-P) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)} = \frac{1}{1+r}.$$

This can be rearranged as

$$\lambda = \frac{(1+r)\delta u'(R) - u'(I-P)}{u'(I-P) - \frac{\rho - r}{1+\rho} u'\left(I - P + \frac{R}{1+\rho}\right)}.$$

The denominator on the right-hand side is obviously positive since $I - P \leqslant I - \frac{\rho - r}{1+\rho} P$ and$\frac{\rho-r}{1+\rho} < 1$. However, we cannot state whether the numerator is positive or negative. Indeed, $R = (1+r)P$ in the solution then the numerator is $(1+r)\delta u'((1+r)P) - u'(I-P)$. This is positive for a sufficiently small P but negative for a sufficiently large P. Since λ is greater than 0, in order to characterize the optimal plan, we consider only P such that $(1+r)\delta u'((1+r)P) - u'(I-P) > 0$. Define $Q \colon \mathscr{P} \to \Lambda$ as

$$Q(P) \equiv \frac{(1+r)\delta u'((1+r)P) - u'(I-P)}{u'(I-P) - \frac{\rho - r}{1+\rho} u'\left(I - \frac{\rho - r}{1+\rho} P\right)}, \tag{7}$$

where $\mathscr{P} \equiv \{P \in [0, I] \colon (1+r)\delta u'((1+r)P) - u'(I-P) > 0\}$.

Let us see what properties Q has. Importantly, the following lemma says that the optimal τ is unique for each $\lambda > 0$, that is, there exists an inverse function for Q.

Lemma 3.2. *Q satisfies the following properties.*

(i) Q is strictly decreasing.

(ii) $Q(P) \to \infty$ as $P \to 0$.

(iii) $Q(P) = 0$ if P satisfies $(1+r)\delta u'((1+r)P) - u'(I-P) = 0$.

Proof of property (i). Let P and P' be arbitrary premiums such that $P' > P$. By the assumptions of $u'(c) > 0$ and $u''(c) < 0$, we have

$$u'(I-P') - \frac{\rho - r}{1+\rho} u'\left(I - \frac{\rho - r}{1+\rho} P'\right) > u'(I-P) - \frac{\rho - r}{1+\rho} u'\left(I - \frac{\rho - r}{1+\rho} P\right)$$

and

$$(1+r)\delta u'((1+r)P') - u'(I-P') < (1+r)\delta u'((1+r)P) - u'(I-P).$$

So it follows that

$$\begin{aligned} Q(P) &= \frac{(1+r)\delta u'((1+r)P) - u'(I-P)}{u'(I-P) - \frac{\rho - r}{1+\rho} u'\left(I - \frac{\rho - r}{1+\rho} P\right)} > \frac{(1+r)\delta u'((1+r)P) - u'(I-P)}{u'(I-P') - \frac{\rho - r}{1+\rho} u'\left(I - \frac{\rho - r}{1+\rho} P'\right)} \\ &> \frac{(1+r)\delta u'((1+r)P') - u'(I-P')}{u'(I-P') - \frac{\rho - r}{1+\rho} u'\left(I - \frac{\rho - r}{1+\rho} P'\right)} = Q(P'). \end{aligned}$$

Therefore, Q is strictly decreasing. □

Proof of property (ii). By assumption 2.3, we have

$$\lim_{P\to 0} Q(P) = \lim_{P\to 0} \left[\frac{(1+r)\delta u'((1+r)P) - u'(I-P)}{u'(I-P) - \frac{\rho-r}{1+\rho} u'\left(I - \frac{\rho-r}{1+\rho}P\right)} \right] = \infty.$$

□

Proof of property (iii). Since $u'(I-P) - \frac{\rho-r}{1+\rho} u'\left(I - \frac{\rho-r}{1+\rho}P\right)$ is positive for any $P \leqslant I < \infty$, the property is obvious. □

The following theorem characterizes the optimal pension plan.

Theorem 3.1. *The optimal pension plan is determined by the inverse function of Q.*

Proof. Q corresponds to an arbitrary pension premium $P \in \mathscr{P}$ to type $\lambda \in \Lambda$ whose optimal pension premium is P. By the lemma 3.2, Q is one to one function. In addition, properties (i) and (ii) say that Q is onto function. Therefore, for Q, there exists an inverse function that determines the optimal pension premium for any $\lambda \in \Lambda$. □

Theorem 3.1 together with property (i) in the lemma 3.2 says that it is optimal to set a lower premium for consumers that feel greater temptation. This result can be understood by the following discussion. Consider the situation when a consumer does not save and not borrow at all. His welfare function is rewritten as

$$\begin{aligned} W(B(\tau);\lambda) &= (1+\lambda)u(I-P) + \delta u((1+r)P) - \lambda u\left(I - \frac{\rho-r}{1+\rho}P\right) \\ &= u(I-P) + \delta u((1+r)P) - \lambda \left[u\left(I - \frac{\rho-r}{1+\rho}P\right) - u(I-P) \right]. \end{aligned}$$

The summation of the first and second terms represents the utility of consumption and the third term represents the cost of self-control. Differentiate both sides with P and we have the marginal utility minus the marginal cost of P,

$$\begin{aligned} \frac{\partial W(B(\tau);\lambda)}{\partial P} &= -u'(I-P) + (1+r)\delta u'((1+r)P) \\ &\quad - \lambda \left[-u'\left(I - \frac{\rho-r}{1+\rho}P\right)\left(\frac{\rho-r}{1+\rho}\right) + u'(I-P) \right]. \end{aligned}$$

The first-order condition says that the marginal utility equals the marginal cost at optimal P. Differentiate this again with P and we have

$$\begin{aligned} \frac{\partial^2 W(B(\tau);\lambda)}{\partial P^2} =& u''(I-P) + (1+r)^2\delta u''((1+r)P) \\ &- \lambda \left[u''\left(I - \frac{\rho-r}{1+\rho}P\right)\left(\frac{\rho-r}{1+\rho}\right)^2 - u''(I-P) \right]. \end{aligned}$$

By assumption 2.2, the marginal utility decreases as P increases. And in the proof of the lemma 3.2, we have seen that the marginal cost increases as P does. Note that the increase in λ raises the marginal cost and we can see that P, at the point where marginal utility equals marginal cost, moves lower. Therefore, higher λ implies lower $P(\lambda)$. Intuitively, the increase in P has two effects. One effect reduces the amount of money the consumer can use when she is young. For the optimal plan, it holds that $R = (1+r)P$. Hence

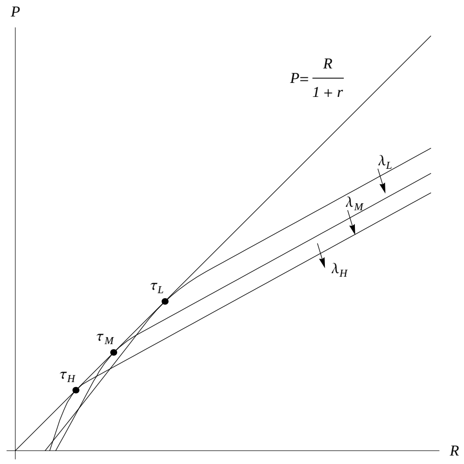

Figure 3: Optimal plans

the increase of P means the increase of R. Thus another effect enlarges the amount of money that the consumer can borrow when she is young and it increases the temptation of borrowing. For consumers with larger λ, the latter effect is stronger. Therefore to avoid a high self-control cost, it is not proper to apply a large pension premium to a consumer with large λ. This result can be understood also by a picture. In Figure 3, each of three indifference curves for τ_L, τ_M, τ_H, where ,$\tau_L < \tau_M < \tau_H$, is a tangent to line $P = \frac{R}{1+r}$. The tangent points are the optimal plans. Obviously, as λ increases, the optimal plan is determined in the lower-left area of the plane.

At the end of this subsection, we consider the value of $(1+r)\delta u'((1+r)P) - u'(I-P)$ in property (iii). It is equivalent to the amount of saving when $\lambda = 0$ and there is no pension policy. We have already shown that any $\tau \in T_s^0$ is optimal for a consumer with $\lambda = 0$. That is, when a consumer does not feel any temptation, it is optimal to impose an arbitrary premium between 0 and the optimal amount of saving. This is a well-known result: if the pension earns the same interest rate as private savings, these two ways of saving are indifferent and so the pension does not improve welfare (For example, see Samuelson, 1975). Interestingly, however, when a consumer has $\lambda > 0$, this result does not hold. Even if the interest rates are equal to each other, the private saving and the pension are not equivalent.

3.2 Two types of consumers

In this section, we consider two types of heterogeneous consumers. The assumption of complete information is a useful benchmark. So we begin with the situation where the government knows what type each consumer has. Denote the types as λ_L and λ_H, where $0 < \lambda_L < \lambda_H$. And the population of λ_L and λ_H are n_L and n_H, respectively, where we assume that $n_L > 0$, $n_H > 0$ and $n_L + n_H = 1$. Note that, unlike in the identical-type case, we cannot state that $R = (1+r)P$ is always satisfied at optimal schedules since there may be transfer between the types.

Remember that the marginal welfare of P for each agent is negative. Hence the pension schedule $S \equiv \{\tau_L, \tau_H\}$ such that $P_L > \frac{R_L}{1+r}$ and $P_H > \frac{R_H}{1+r}$ is not optimal since the schedule satisfies the feasibility constraint slackly, so that a sufficiently small decrease in P is feasible and improves social welfare. It is also clear that the pension schedule such that $P_L < \frac{R_L}{1+r}$ and $P_H < \frac{R_H}{1+r}$ is not feasible.

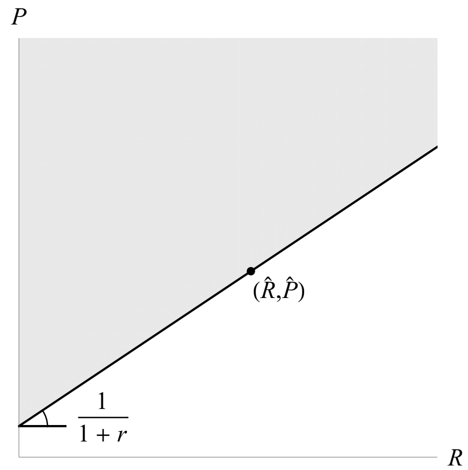

Figure 4: Individually feasible schedules for $(\hat{R}, \hat{P})$

In addition, the following lemma holds.

Lemma 3.3. *If a pension schedule $\{\hat{\tau}_L, \hat{\tau}_H\}$ is feasible, then the schedules $\{\tau_L, \tau_H\}$ such that*

$$\left[P_L \geqslant \frac{R_L - \hat{R}_L}{1+r} + \hat{P}_L, \tau_H = \hat{\tau}_H\right] \tag{8}$$

or

$$\left[P_H \geqslant \frac{R_H - \hat{R}_H}{1+r} + \hat{P}_H, \tau_L = \hat{\tau}_L\right] \tag{9}$$

are also feasible.

Proof. Without loss of generality, we show that the schedule that satisfies (8) is feasible. Suppose that $\{\hat{\tau}_L, \hat{\tau}_H\}$ is feasible; that is,

$$\begin{aligned} &(1+r)(n_L\hat{P}_L + n_H\hat{P}_H) \geqslant n_L\hat{R}_L + n_H\hat{R}_H \\ \Leftrightarrow &(1+r)n_L\hat{P}_L \geqslant -(1+r)n_H\hat{P}_H + n_L\hat{R}_L + n_H\hat{R}_H. \end{aligned} \tag{10}$$

Choose a schedule such that $P_L \geqslant \frac{R_L - \hat{R}_L}{1+r} + \hat{P}_L, \tau_H = \hat{\tau}_H$ arbitrarily. Then it follows that

$$\begin{aligned} &P_L \geqslant \frac{R_L - \hat{R}_L}{1+r} + \hat{P}_L \\ \Leftrightarrow &(1+r)n_L P_L - n_L(R_L - \hat{R}_L) \geqslant (1+r)n_L\hat{P}_L. \end{aligned}$$

Together with (10), we have

$$\begin{aligned} &(1+r)n_L P_L - n_L(R_L - \hat{R}_L) + (1+r)n_H\hat{P}_H \geqslant n_L\hat{R}_L + n_H\hat{R}_H \\ \Leftrightarrow &(1+r)(n_L P_L + n_H\hat{P}_H) \geqslant n_L R_L + n_H\hat{R}_H. \end{aligned}$$

By $\tau_H = \hat{\tau}_H$, it follows that

$$(1+r)(n_L P_L + n_H P_H) \geqslant n_L R_L + n_H R_H.$$

Thus, the schedule $\{\tau_L, \tau_H\}$ is feasible. We can show the feasibility of the schedule that satisfies (9). □

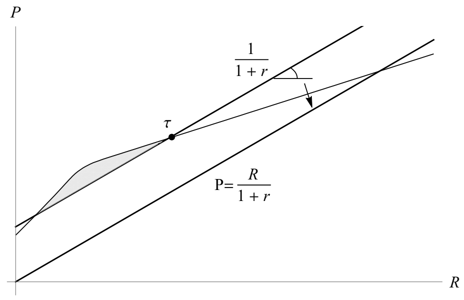

Figure 5: τ is not optimal plan

For a given plan $(\hat{\tau}_\lambda)$, we call the plans τ_λ that satisfy $P_\lambda \geqslant \frac{R_\lambda - \hat{R}_\lambda}{1+r} + \hat{P}_\lambda$ individually feasible. Figure 4 depicts the set of schedules that is individually feasible for $(\hat{R}, \hat{P})$. The lemma 3.3 is useful for searching for solutions. For example, in Figure 5, $\sigma_\lambda(\tau)$ is lower than $\frac{1}{1+r}$. Note that with complete information, we can ignore the incentive compatibility conditions. Hence a pension plan for another type can be fixed arbitrarily. Then there is room for the improvement. Indeed, obviously τ_λ is not optimal since we can find individually feasible and more preferable plans in the hatched area in the Figure. The point is that if this improvement is available, it works individually; that is, it does not need transfer between types. Thus this is a Pareto improvement.

The next lemma is important for proving the lemma 3.5.

Lemma 3.4. *For any $\lambda \in \Lambda \subseteq \mathbb{R}_+$, τ such that $\sigma_\lambda(\tau) = \frac{1}{1+r}$ satisfies $R > 0$.*

Proof. Fix arbitrary $\lambda \in \Lambda \subseteq \mathbb{R}_+$ and choose τ such that $\sigma_\lambda(\tau) = \frac{1}{1+r}$. Note that $\sigma_\lambda(\tau) = \frac{1}{1+r}$ implies $\tau \in T_b$. $\sigma_\lambda(\tau)$ is equal to

$$\frac{-\delta u'(R) + \frac{\lambda}{1+\rho} u'\left(I - P + \frac{R}{1+\rho}\right)}{-(1+\lambda)u'(I-P) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)}.$$

Then, by assumption 2.3,

$$\begin{aligned}
&\lim_{R\to 0} \frac{-\delta u'(R) + \frac{\lambda}{1+\rho} u'\left(I - P + \frac{R}{1+\rho}\right)}{-(1+\lambda)u'(I-P) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)} \\
=&\frac{-\lim_{R\to 0} \delta u'(R) + \frac{\lambda}{1+\rho} u'(I-P)}{-u'(I-P)} \\
=&\frac{-\infty}{-u'(I-P)} = \infty.
\end{aligned}$$

Since $\sigma_\lambda(\tau)$ is decreasing in R, τ such that $\sigma_\lambda(\tau) = \frac{1}{1+r}$ satisfies $R > 0$. □

The following lemma, derived from the previous lemma gives the condition for optimality.

Lemma 3.5. *In the optimal schedule with complete information $\{\tau_L, \tau_H\}$, both $\sigma_L(\tau_L)$ and $\sigma_H(\tau_H)$ are equal to $\frac{1}{1+r}$.*

Proof. Suppose that either $\sigma_L(\tau_L)$ or $\sigma_H(\tau_H)$ is not equal to $\frac{1}{1+r}$. Without loss of generality, suppose that $\sigma_L(\lambda_L) \neq \frac{1}{1+r}$.

(i) $\sigma_L(\tau_L) > \frac{1}{1+r}$

Consider ε_1, $\varepsilon_2 > 0$ such that $\varepsilon_2 = \frac{\varepsilon_1}{1+r}$. For such ε_1 and ε_2, let $R' \equiv R_L + \varepsilon_1$ and $P' \equiv P_L + \varepsilon_2$. Then it follows that

$$\begin{aligned}\varepsilon_2 &= \frac{\varepsilon_1}{1+r}\\ \Leftrightarrow P_L + \varepsilon_2 &= \frac{R_L + \varepsilon_1 - R_L}{1+r} + P_L\\ \Leftrightarrow P' &= \frac{R' - R_L}{1+r} + P_L,\end{aligned}$$

that is, $\tau' = (R', P')$ is individually feasible by lemma 3.3. The line represented by this equation has the slope of $\frac{1}{1+r}$ on R-P plane and passes through (R_L, P_L). Furthermore, the marginal rate of substitution at τ_L is $\sigma_L(\tau_L) > \frac{1}{1+r}$. Thus for sufficiently small ε_1 and ε_2, τ' brings more individual welfare. Since there exists an another schedule that is feasible and preferable to $\{\tau_L, \tau_H\}$, $\{\tau_L, \tau_H\}$ is not optimal.

(ii) $\sigma_L(\tau_L) < \frac{1}{1+r}$

Consider ε_1, $\varepsilon_2 > 0$, which satisfies that $\varepsilon_2 = \frac{\varepsilon_1}{1+r}$. For such ε_1 and ε_2, let $R' \equiv R_L - \varepsilon_1$ and $P' \equiv P_L - \varepsilon_2$. Here, for τ_L such that $\sigma_L(\tau_L) < \frac{1}{1+r}$, it is satisfied that $R > 0$ by the lemma 3.4. Then, similar to the previous case, it is satisfied that

$$\begin{aligned}\varepsilon_2 &= \frac{\varepsilon_1}{1+r}\\ \Leftrightarrow P_L - \varepsilon_2 &= \frac{R_L - \varepsilon_1 - R_L}{1+r} + P_L\\ \Leftrightarrow P' &= \frac{R' - R_L}{1+r} + P_L,\end{aligned}$$

that is, $\tau' = (R', P')$ is individually feasible by the lemma 3.3. This line has the slope of $\frac{1}{1+r}$ and passes through (R_L, P_L). Furthermore, the marginal rate of substitution at τ_L is $\sigma_L(\tau_L) < \frac{1}{1+r}$. Thus, for sufficiently small ε_1 and ε_2, τ' brings more individual welfare. Since there exists another schedule that is feasible and preferable to $\{\tau_L, \tau_H\}$, $\{\tau_L, \tau_H\}$ is not optimal. □

By the lemma 3.5, if a pension schedule is optimal, it holds that $\sigma_\lambda(\tau) = \frac{1}{1+r}$ for all $\lambda \in \Lambda$. Together with the lemma 3.4, if a pension schedule is optimal, we have $R_\lambda > 0$ for all $\lambda \in \Lambda$. That is, if a consumer feels temptation, the funded pension improves social welfare.

Here, we have the following result.

Theorem 3.2. *If $\lambda \neq \lambda'$, it is not optimal that $\tau_\lambda = \tau_{\lambda'}$.*

Proof. Consider an arbitrary λ, λ'. By the lemma 3.5, if the singleton schedule of $\{\tau\}$ is optimal, it is satisfied that

$$\frac{-\delta u'(R) + \frac{\lambda}{1+\rho} u'\left(I - P + \frac{R}{1+\rho}\right)}{(1+\lambda)u'(I-P) + \lambda u'\left(I - P + \frac{R}{1+\rho}\right)} = \frac{1}{1+r}$$

and

$$\frac{-\delta u'(R)+\frac{\lambda'}{1+\rho}u'\left(I-P+\frac{R}{1+\rho}\right)}{(1+\lambda')u'(I-P)+\lambda' u'\left(I-P+\frac{R}{1+\rho}\right)}=\frac{1}{1+r},$$

then it must be follow that

$$\frac{-\delta u'(R)+\frac{\lambda}{1+\rho}u'\left(I-P+\frac{R}{1+\rho}\right)}{(1+\lambda)u'(I-P)+\lambda u'\left(I-P+\frac{R}{1+\rho}\right)}=\frac{-\delta u'(R)+\frac{\lambda'}{1+\rho}u'\left(I-P+\frac{R}{1+\rho}\right)}{(1+\lambda')u'(I-P)+\lambda' u'\left(I-P+\frac{R}{1+\rho}\right)}. \tag{11}$$

This is a necessary condition for τ is optimal. For simplicity, we abbreviate some parts of (11) as follows:

$$\begin{aligned}
&A \equiv -\delta u'(R)\\
&B \equiv \frac{1}{1+\rho}u'\left(I-P+\frac{R}{1+\rho}\right)\\
&C \equiv u'(I-P)\\
&D \equiv u'(I-P)+u'\left(I-P+\frac{R}{1+\rho}\right).
\end{aligned}$$

Then (11) is rewritten to,

$$\begin{aligned}
&\frac{A+\lambda B}{C+\lambda D}=\frac{A+\lambda' B}{C+\lambda' D}\\
\Leftrightarrow&(A+\lambda B)(C+\lambda' D)=(A+\lambda' B)(C+\lambda D)\\
\Leftrightarrow&(\lambda-\lambda')(AD-BC)=0.
\end{aligned}$$

By assumption 2.2, $AD=-\delta u'(R)\left[u'(I-P)+u'\left(I-P+\frac{R}{1+\rho}\right)\right]$ is strictly negative and $BC=\frac{1}{1+\rho}u'\left(I-P+\frac{R}{1+\rho}\right)u'(I-P)$ is strictly positive, it must be that $\lambda=\lambda'$. This implies (11), necessary condition for singleton schedule to be optimal, is satisfied only if $\lambda=\lambda'$. □

That is, if consumers have different strength of temptation, applying a common pension plan is not optimal. This is consistent with our first intuition.

Next we construct the optimal schedule in this situation. To obtain a precise result, henceforth we specialize in the normative utility function as $u(c)=\log c$ for all $c\in\mathbb{R}_+$. $\log c$ satisfies assumptions 2.2 and 2.3. For this specialized utility function, we have the following consumptions:

$$c_1(\tau)=\begin{cases}\frac{(1+\lambda)\left((1+r)(I-P)+R\right)}{(1+r)(1+\delta+\lambda)} & \text{if } 0\leqslant P< I-\frac{(1+\lambda)R}{(1+r)\delta}\\ I-P & \text{if } I-\frac{(1+\lambda)R}{(1+r)\delta}\leqslant P\leqslant I-\frac{(1+\lambda)R}{(1+\rho)\delta}\\ \frac{(1+\lambda)\left((1+\rho)(I-P)+R\right)}{(1+\rho)(1+\delta+\lambda)} & \text{otherwise,}\end{cases}$$

$$c_2(\tau)=\begin{cases}\frac{\delta\left((1+r)(I-P)+R\right)}{1+\delta+\lambda} & \text{if } 0\leqslant P< I-\frac{(1+\lambda)R}{(1+r)\delta}\\ R & \text{if } I-\frac{(1+\lambda)R}{(1+r)\delta}\leqslant P\leqslant I-\frac{(1+\lambda)R}{(1+\rho)\delta}\\ \frac{\delta\left((1+\rho)(I-P)+R\right)}{1+\delta+\lambda} & \text{otherwise.}\end{cases}$$

Next we consider the locus of points at which $\sigma_\lambda(\tau)$ equals $\frac{1}{1+r}$. According to the lemma 3.1, it is enough to obtain the locus to consider τ such that $I - \frac{(1+\lambda)R}{(1+r)\delta} \leqslant P \leqslant I - \frac{(1+\lambda)R}{(1+\rho)\delta}$. Consider an arbitrary $\lambda \in \Lambda$. Then the individual welfare for such τ is

$$W(B(\tau);\lambda) = (1+\lambda)\log(I-P) + \delta\log(R) - \lambda\log\left(I - P + \frac{R}{1+\rho}\right).$$

We have

$$\sigma_\lambda(\tau) = \frac{-\frac{\delta}{R} + \frac{\lambda}{(1+\rho)(I-P)+R}}{-\frac{1+\lambda}{I-P} + \frac{(1+\rho)\lambda}{(1+\rho)(I-P)+R}}.$$

The relation between P and R such that $\sigma_\lambda(\tau) = \frac{1}{1+r}$ is as follows:

$$\begin{aligned} P &= I - K_\lambda R, \\ K_\lambda &\equiv \frac{1+\rho+(1+r)(\lambda-\delta)+\sqrt{(1+r)^2\delta^2+(1+\rho+(1+r)\lambda)^2+2\delta(1+r)(1+(1-r)\lambda+(1+2\lambda)\rho)}}{2\delta(1+r)(1+\rho)}. \end{aligned}$$

Proposition 3.1. *For any $\lambda > 0$ and $\lambda' > 0$ such that $\lambda \neq \lambda'$, if $\sigma_\lambda(\tau) = \sigma_{\lambda'}(\tau') = \frac{1}{1+r}$, then $\tau \neq \tau'$.*

Proof. Fix arbitrarily $\lambda > 0$ and $\lambda' > 0$ such that $\lambda \neq \lambda'$. Suppose that for τ and τ', $\sigma_\lambda(\tau) = \sigma_{\lambda'}(\tau') = \frac{1}{1+r}$. Then it holds that $P = I - K_\lambda R$ and $P' = I - K_{\lambda'} R'$. Note that the relation between P and R and that between P' and R' are linear with the slopes of $-K_\lambda$ and $-K_{\lambda'}$, respectively. We show that if $\lambda > \lambda'$, then $K_\lambda > K_{\lambda'}$. Calculating $K_\lambda - K_{\lambda'}$, we have

$$\begin{aligned} &K_\lambda - K_{\lambda'} \\ =& \frac{1+\rho+(1+r)(\lambda-\delta)+\sqrt{(1+r)^2\delta^2+(1+\rho+(1+r)\lambda)^2+2\delta(1+r)(1+(1-r)\lambda+(1+2\lambda)\rho)}}{2\delta(1+r)(1+\rho)} \\ -& \frac{1+\rho+(1+r)(\lambda'-\delta)+\sqrt{(1+r)^2\delta^2+(1+\rho+(1+r)\lambda')^2+2\delta(1+r)(1+(1-r)\lambda'+(1+2\lambda')\rho)}}{2\delta(1+r)(1+\rho)} \\ =& \frac{(1+r)\lambda+\sqrt{(1+r)^2\delta^2+(1+\rho+(1+r)\lambda)^2+2\delta(1+r)(1+(1-r)\lambda+(1+2\lambda)\rho)}}{2\delta(1+r)(1+\rho)} \\ -& \frac{(1+r)\lambda'+\sqrt{(1+r)^2\delta^2+(1+\rho+(1+r)\lambda')^2+2\delta(1+r)(1+(1-r)\lambda'+(1+2\lambda')\rho)}}{2\delta(1+r)(1+\rho)}. \end{aligned}$$

Comparing inside the square roots, the former is larger than the latter. Thus we have $K_\lambda > K_{\lambda'}$. Though $\tau = \tau'$ holds only if $R = R' = 0$, $R > 0$ and $R' > 0$ by the lemma 3.4. Therefore $\tau \neq \tau'$. □

This proposition states that the point at which the indifference curve of each consumer is tangential to the feasibility frontier differs according to the type.

By the lemma 3.5, the optimal schedule satisfies that $\sigma_L(\tau_L) = \sigma_H(\tau_H) = \frac{1}{1+r}$ so we have

$$\begin{aligned} P_L &= I - K_L R_L \\ P_H &= I - K_H R_H. \end{aligned}$$

As we have mentioned, the feasibility condition is satisfied with equality for the optimal schedule. Hence, according to the feasibility condition, we have

$$\begin{aligned}
&(1+r)[n_L P_L + n_H P_H] = n_L R_L + n_H R_H \\
\Leftrightarrow &(1+r)[n_L(I - K_L R_L) + n_H(I - K_H R_H)] = n_L R_L + n_H R_H \\
\Leftrightarrow &(1+r)[I - n_L K_L R_L - n_H K_K R_H] = n_L R_L + n_H R_H \\
\Leftrightarrow &(1+(1+r)K_H)n_H R_H = (1+r)I - (1+(1+r)K_L)n_L R_L \\
\Leftrightarrow &R_H = \frac{(1+r)I}{(1+(1+r)K_H)n_H} - \frac{(1+(1+r)K_L)n_L}{(1+(1+r)K_H)n_H} R_L.
\end{aligned}$$

Thus the summarized problem is that

$$\begin{aligned}
\max_{\{\tau_\lambda\}_{\lambda=L,H}} \; & n_L \left[(1+\lambda_L)\log(I - P_L) + \delta \log R_L - \lambda_L \log\left(I - P_L + \frac{R_L}{1+r\rho}\right)\right] \\
& + n_H \left[(1+\lambda_H)\log(I - P_H) + \delta \log R_H - \lambda_H \log\left(I - P_H + \frac{R_H}{1+r\rho}\right)\right] \\
\text{s.t. } & P_L = I - K_L R_L, \; P_H = I - K_H R_H \\
& R_H = \frac{(1+r)I}{(1+(1+r)K_H)n_H} - \frac{(1+(1+r)K_L)n_L}{(1+(1+r)K_H)n_H} R_L.
\end{aligned}$$

Substituting P_L, P_H, and R_H, this problem can be seen as simply one variable problem. The first-order condition is

$$\begin{aligned}
\frac{\partial W}{\partial R_L} =& \frac{n_L[(n_H + n_L)(1+(1+r)K_L)R_L - (1+r)I](1+\delta)}{R_L(n_L(1+(1+r)K_L)R_L - (1+r)I)} = 0 \\
\Leftrightarrow & R_L = \frac{(1+r)I}{1+(1+r)K_L}.
\end{aligned}$$

Then we have

$$\begin{aligned}
P_L &= I - \frac{K_L(1+r)I}{1+(1+r)K_L} = \frac{I}{1+(1+r)K_L} \\
R_H &= \frac{(1+r)I - (1+(1+r)K_L)n_L \frac{(1+r)I}{(1+(1+r)K_L)}}{(1+(1+r)K_H)n_H} = \frac{(1+r)I}{1+(1+r)K_H}. \\
P_H &= I - K_H \frac{(1+r)I}{1+(1+r)K_H} = \frac{I}{1+(1+r)K_H}.
\end{aligned}$$

Seeing this result, we find that the optimal schedule satisfies $R_\lambda = (1+r)P_\lambda$ for $\lambda \in \Lambda$. In this situation, the incentive compatibility condition is strictly satisfied. Therefore, we have the following theorem.

Theorem 3.3. *Assume $u(c) = \log c$ and $|\Lambda| = 2$. The optimal pension schedule has the following form: for any $\lambda \in \Lambda$,*

$$\begin{aligned}
P_\lambda &= \frac{I}{1+(1+r)K_\lambda}, \\
R_\lambda &= (1+r)P_\lambda.
\end{aligned}$$

We can see some things from the result. First, there is no monetary transfer among types since the optimal plans are balanced for each type. Second, similar to the identical-type case, there are no private saving and borrowing at the optimal schedule. This implies that the monetary market is balanced. Third, and importantly, the optimal schedule does not depend on the distribution of types.Indeed, it does not contain n_L and n_H. Furthermore, consider the following mechanism. The government shows a pension schedule such that

$$\left\{(R, P) \in T \colon P = \frac{R}{1+r}\right\}$$

and let each consumer announce the maximal amount of pension she foresees wanting in old age. Then the consumer is enrolled in the pension plan that is in accord with her decision. Though formal proof is omitted, this mechanism implements the optimal schedule with a weakly dominant strategy. This follows by the convexity of individual welfare and not by the existing externality between types. Note that this mechanism does not need any information about types, such as the distribution and even what types there are.

4 Generalization

In this section, we generalize our model. One generalization is about the number of types. First, we consider the case of many finite types. Then we consider the case of continuous types. As in the previous section, we specialize the normative utility function as a logarithm function.

4.1 Finite types

Consider the set of types $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$, where $m < \infty$. Denote the population of type λ_i by n_i, where it is assumed that $\sum_{i=1}^{m} n_i = 1$ and $n_i > 0$ for all $\lambda_i \in \Lambda$. The problem is that

$$\begin{aligned} &\max_{S} \sum_{i=1}^{m} n_i \left\{(1+\lambda_i)\log(I - P_i) + \delta \log R_i - \lambda_i \log\left(I - P_i + \frac{R_i}{1+\rho}\right)\right\} \\ &\text{s.t. } (1+r)\sum_{i=1}^{m} n_i P_i \geqslant \sum_{i=1}^{m} n_i R_i. \end{aligned}$$

We can apply the lemma 3.5 also in this model. Thus, by substituting $P_i = I - K_i R_i$ in the problem, we have

$$\begin{aligned} &\max_{S} \sum_{i=1}^{m} n_i \left\{(1+\lambda_i)\log K_i R_i + \delta \log R_i - \lambda_i \log\left(\left(K_i + \frac{1}{1+\rho}\right) R_i\right)\right\} \\ &\text{s.t. } (1+r)\sum_{i=1}^{m} n_i (I - K_i R_i) \geqslant \sum_{i=1}^{m} n_i R_i. \end{aligned}$$

The associated Lagrangian is

$$\begin{aligned} \mathscr{L} \equiv &\mu_0 \left[\sum_{i=1}^{m} n_i \left\{(1+\lambda_i)\log K_i R_i + \delta \log R_i - \lambda_i \log\left(\left(K_i + \frac{1}{1+\rho}\right) R_i\right)\right\}\right] \\ &+ \mu_1 \left[(1+r)\sum_{i=1}^{m} n_i (I - K_i R_i) - \sum_{i=1}^{m} n_i R_i\right]. \end{aligned}$$

The necessary condition is

$$\frac{\partial \mathscr{L}}{\partial R_j} = \mu_0 \left[n_j \left\{ \frac{1+\lambda_i}{R_j} + \frac{\delta}{R_j} - \frac{\lambda_j}{R_j} \right\} \right] + \mu_1[-(1+r)n_jK_j - n_j] = 0,\ \forall \lambda \in \Lambda$$
$$(\mu_0, \mu_1) \geqslant 0$$
$$(\mu_0, \mu_1) \neq 0$$
$$\mu_1 \left[(1+r)\sum_{i=1}^{m} n_i(I - K_iR_i) - \sum_{i=1}^{m} n_iR_i \right] = 0.$$

Note that $[-(1+r)n_jK_j - n_j] < 0$. Then if $\mu_0 = 0$, the first order condition is satisfied only if $\mu_1 = 0$. This contradicts the non-zero condition of Lagrange multipliers. Thus $\mu_0 > 1$ and we can standardize this as $\mu_0 = 1$. Then the first-order conditions are rearranged as

$$n_j \frac{1+\delta}{R_j} - \mu_1[(1+r)n_jK_j - n_j] = 0,\ \forall \lambda \in \Lambda. \tag{12}$$

μ_1 is not zero since the first term is strictly positive. So the feasibility condition is satisfied with equality. By (12), we have

$$\mu_1 = \frac{1}{(1+r)K_j + 1} \left(\frac{1+\delta}{R_j} \right).$$

Thus for any $\lambda_j \in \Lambda$ and $\lambda_k \in \Lambda$, it follows that

$$\frac{1}{(1+r)K_j + 1} \left(\frac{1+\delta}{R_j} \right) = \frac{1}{(1+r)K_k + 1} \left(\frac{1+\delta}{R_k} \right)$$
$$\Leftrightarrow [(1+r)K_j + 1]R_j = [(1+r)K_k + 1]R_k. \tag{13}$$

By the feasibility condition with equality, we have

$$(1+r)\sum_{i=1}^{m} n_i(I - K_iR_i) - \sum_{i=1}^{m} n_iR_i = 0$$
$$\Leftrightarrow \sum_{i=1}^{m} n_i((1+r)K_i + 1)R_i = (1+r)I \sum_{i=1}^{m} n_i = (1+r)I.$$

Fixing arbitrarily $\lambda_j \in \Lambda$ and using (13), it follows that

$$(1+r)I = \sum_{i=1}^{m} n_i((1+r)K_i + 1)R_i = ((1+r)K_j + 1)R_j \sum_{i=1}^{m} n_i$$
$$= ((1+r)K_j + 1)R_j$$
$$\Leftrightarrow R_j = \frac{(1+r)I}{(1+r)K_j + 1}.$$

Substituting this to $P_j = I - K_jR_j$, we have

$$P_j = I - K_j \frac{(1+r)I}{(1+r)K_j + 1} = \frac{((1+r)K_j + 1)I - (1+r)K_jI}{(1+r)K_j + 1} = \frac{I}{(1+r)K_j + 1}.$$

Thus in the solution $R_j = (1+r)P_j$ is satisfied for any type $\lambda_j \in \Lambda$. This is the same result as in the two types case.

4.2 Continuous types

Let $\Lambda = [\underline{\lambda}, \overline{\lambda}]$, where $\underline{\lambda} \geqslant 0$. $\lambda \in \Lambda$ are distributed according to the distribution function F. $(R(\lambda), P(\lambda))$ denotes the pension plan for the consumer whose type is $\lambda \in \Lambda$. We assume that $R(\lambda)$ and $P(\lambda)$ are continuous and differentiable.

Similarly, considering the problem with complete information, it follows that $P(\lambda) = I - K(\lambda)R(\lambda)$, $c_1^\lambda = I - P(\lambda)$ and $c_2^\lambda = R(\lambda)$ for each λ, where $K(\lambda)$ is the constant that has the same form as K_λ in the previous section. The problem is that

$$\begin{aligned}
\max_S & \int_{\underline{\lambda}}^{\overline{\lambda}} \left[(1+\lambda)\log(I - P(\lambda)) + \delta \log(R(\lambda)) - \lambda \log\left(I - P(\lambda) + \frac{R(\lambda)}{1+\rho}\right)\right] dF(\lambda) \\
\text{s.t.} & \int_{\underline{\lambda}}^{\overline{\lambda}} [(1+r)P(\lambda) - R(\lambda)]\, dF(\lambda) = 0 \\
& P(\lambda) = I - K(\lambda)R(\lambda),
\end{aligned}$$

where we can assume that the solution satisfies the feasibility condition with equality for the same reason as in the previous section. Since the feasibility constraint includes a integration, we rewrite it. We define a new state variable h as

$$h(\lambda) \equiv \int_{\underline{\lambda}}^{\lambda} [(1+r)P(x) - R(x)]\, dF(x).$$

Then, h satisfies the following conditions, conversely implying the feasibility condition.

$$\begin{aligned}
h'(\lambda) &= (1+r)P(\lambda) - R(\lambda), \\
h(\underline{\lambda}) &= 0 \text{ and } h(\overline{\lambda}) = 0.
\end{aligned}$$

Thus, we have the rewritten problem:

$$\begin{aligned}
\max_S & \int_{\underline{\lambda}}^{\overline{\lambda}} \left[(1+\lambda)\log(I - P(\lambda)) + \delta \log(R(\lambda)) - \lambda \log\left(I - P(\lambda) + \frac{R(\lambda)}{1+\rho}\right)\right] dF(\lambda) \\
\text{s.t. } & h'(\lambda) = (1+r)P(\lambda) - R(\lambda), \\
& h(\underline{\lambda}) = 0 \text{ and } h(\overline{\lambda}) = 0 \\
& P(\lambda) = I - K(\lambda)R(\lambda).
\end{aligned}$$

The associated Hamiltonian is,

$$\begin{aligned}
\mathscr{H} \equiv & (1+\lambda)\log(I - P(\lambda)) + \delta \log(R(\lambda)) - \lambda \log\left(I - P(\lambda) + \frac{R(\lambda)}{1+\rho}\right) + \mu(\lambda) h'(\lambda) \\
= & (1+\lambda)\log(I - P(\lambda)) + \delta \log(R(\lambda)) - \lambda \log\left(I - P(\lambda) + \frac{R(\lambda)}{1+\rho}\right) \\
& + \mu(\lambda) \left[(1+r)P(\lambda) - R(\lambda)\right],
\end{aligned}$$

where μ is the co-state variable. Substituting $P(\lambda) = I - K(\lambda)R(\lambda)$, we have,

$$\begin{aligned}
\mathscr{H} = & (1+\lambda)\log(K(\lambda)R(\lambda)) + \delta \log(R(\lambda)) - \lambda \log\left(\left(K(\lambda) + \frac{1}{1+\rho}\right) R(\lambda)\right) \\
& + \mu(\lambda) \left[(1+r)I - ((1+r)K(\lambda) + 1)R(\lambda)\right].
\end{aligned}$$

By Pontryagin's principle,

$$\begin{aligned}\frac{\partial \mathscr{H}}{\partial R(\lambda)} &= \frac{1+\lambda}{R(\lambda)} + \frac{\delta}{R(\lambda)} - \frac{\lambda}{R(\lambda)} - \mu(\lambda)((1+r)K(\lambda)+1)R(\lambda) \\ &= \frac{1+\delta}{R(\lambda)} - \mu(\lambda)((1+r)K(\lambda)+1 = 0 \qquad (14) \\ \mu'(\lambda) &= -\frac{\partial \mathscr{H}}{\partial h(\lambda)}.\end{aligned}$$

Note that $-\partial \mathscr{H}/\partial h(\lambda) = 0$, so $u'(\lambda) = 0$. This implies that $u(\lambda)$ does not depend on λ. Hence we can simply wreeite $\mu(\lambda) = \mu$. (14) can be rearranged as,

$$R(\lambda) = \frac{1+\delta}{((1+r)K(\lambda)+1)\mu}.$$

By the feasibility condition,

$$\begin{aligned}&\int_{\underline{\lambda}}^{\overline{\lambda}} (1+r)(I - K(\lambda)R(\lambda)) - R(\lambda)dF(\lambda) \\ &= \int_{\underline{\lambda}}^{\overline{\lambda}} (1+r)I - ((1+r)K(\lambda)+1)\, R(\lambda)dF(\lambda) \\ &= \int_{\underline{\lambda}}^{\overline{\lambda}} (1+r)I - ((1+r)K(\lambda)+1)\, \frac{1+\delta}{((1+r)K(\lambda)+1)\,\mu} dF(\lambda) \\ &= \left((1+r)I - \frac{1+\delta}{\mu}\right) \int_{\underline{\lambda}}^{\overline{\lambda}} dF(\lambda) = (1+r)I - \frac{1+\delta}{\mu} = 0 \\ \Leftrightarrow \mu &= \frac{1+\delta}{(1+r)I}\end{aligned}$$

Therefore we have,

$$\begin{aligned}R(\lambda) &= \frac{1+\delta}{((1+r)K(\lambda)+1)\mu} = \frac{1+\delta}{((1+r)K(\lambda)+1)\frac{1+\delta}{(1+r)I}} = \frac{(1+r)I}{(1+r)K(\lambda)+1}, \\ P(\lambda) &= I - K(\lambda)R(\lambda) = I - \frac{(1+r)K(\lambda)I}{(1+r)K(\lambda)+1} = \frac{I}{(1+r)K(\lambda)+1}.\end{aligned}$$

Again we obtained the relation $R(\lambda) = (1+r)P(\lambda)$. For the same reason as in the previous section, if we construct the pension schedule of such plans, the IC condition is strictly satisfied. The following theorem summarizes this section.

Theorem 4.1. *Assume $u(c) = \log c$ and $\Lambda = [\underline{\lambda}, \overline{\lambda}]$. The optimal pension schedule has the following form: for any $\lambda \in \Lambda$,*

$$\begin{aligned}P(\lambda) &= \frac{I}{(1+r)K(\lambda)+1} \\ R(\lambda) &= (1+r)P(\lambda).\end{aligned}$$

5 Discussion

5.1 The effect of a borrowing constraint

So far, we have assumed that consumers are allowed to borrow money in the first period. In this section, we consider the special case in which consumers can borrow no money;

that is, we restrict $s \geqslant 0$[3]. In the context of self-control preference, this assumption has an important meaning. The impossibility of borrowing after paying a premium strengthens the funded pension scheme as a commitment device.

With the borrowing constraint in place, the budget constraint is simply

$$B(\tau) = \{(c_1, c_2) \in \mathbb{R}^2_+ : c_1 + s \leqslant I - P, c_2 \leqslant (1+r)(I - c_1) + R\}.$$

For simplicity, we specialize a normative utility function as $u(c) = \log(c)$. Then the consumption in period 1 is

$$c_1(\tau) = \begin{cases} \frac{(1+\lambda)[(1+r)(I-P)+R]}{(1+r)(1+\delta+\lambda)} & \text{if } 0 \leqslant P < I - \frac{(1+\lambda)R}{\delta(1+r)} \\ I - P & \text{if } I - \frac{(1+\lambda)R}{\delta(1+r)} \leqslant P \leqslant I. \end{cases}$$

Similar to the case in section 3.1, we assume the identical type. Then $R = (1+r)P$ follows.

Calculating an optimal pension plan, we have

$$P_\lambda = \begin{cases} \text{any number } P \in [0, \frac{\delta I}{1+\delta}] & \text{if } \lambda = 0 \\ \frac{\delta I}{1+\delta} & \text{if } \lambda > 0 \end{cases}$$
$$R_\lambda = (1+r)P_\lambda, \ \forall \lambda \geqslant 0.$$

Note that P does not depend on the type if $\lambda > 0$. This optimal P is equal to optimal saving when there is no temptation and no pension policy. Intuitively, by the consumption decision above, a consumer who paid $\frac{\delta I}{1+\delta}$ consumes all of remaining money in period 1. Since $\frac{\delta I}{1+\delta}$ is the optimal saving, the optimal consumption includes all of the remaining money. Thus, naturally, welfare is maximized without the harm of temptation. The interest rate for the pension is the same as that for private saving, so there is no difference in the amount of payout between the pension and saving. However, the pension, which make her pay a premium in advance has a role as a commitment device. Saving does not have the role since a consumer decide how much to save after she faces the temptation. Furthermore, importantly, now the consumer is not allowed to borrow, so the effect of an increase in the premium further strengthens the budget set: there is no temptation to borrow. This result is very different from that in sections 3 and 4.

5.2 Income diversity

In this section, we study the case when the income can be differ for each consumer. The income is one of the elements in $\mathcal{I} = \{I_1, I_2, \ldots, I_m\}$ and the degree of temptation is drawn from $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, where $0 < I_1 < I_2 < \cdots < I_m < \infty$ and $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \infty$. Consumers are characterized by the pair $(I_s, \lambda_t) \in \Theta \subseteq \mathcal{I} \times \Lambda$. n_{st} denotes the proportion of the consumers having $(I_s, \lambda_t) \in \Theta$. A pension plan for consumer with (I_s, λ_t) is the pair (R_{st}, P_{st}). We assume that R_{st} and P_{st} are weakly positive and that consumers have to be able to make their payment for the pension, that is, $P_{st} \leqslant I_s$.

Differently from the degree of temptation, it is natural that the government can observe the income of consumers. In fact, the government uses the information for an income tax imposition. Thus we assume that the government can observe the income of consumers but does not know the degree of temptation. However, as in the previous section, we consider the problem with complete information at first. Then we can use lemma 2.3.5 again because the result does not depend on income, so we can focus on the pension plans

[3] There may be various strengths of the constraint, but here we consider only the strongest borrowing constraint.

which satisfies $P_{st} = I_s - K_t R_{st}$. Using this, the constraint that $0 \leqslant P_{st} \leqslant I_s$ is rewritten as $0 \leqslant R_{st} \leqslant I_s/K_t$. The feasibility condition can also be rearranged as

$$(1+r) \sum_{(s,t)\in\Theta} n_{st} P_{st} \geqslant \sum_{(s,t)\in\Theta} n_{st} R_{st} \iff (1+r) \sum_{(s,t)\in\Theta} n_{st}(I_s - K_t R_{st}) \geqslant \sum_{(s,t)\in\Theta} n_{st} R_{st}$$

$$\iff \sum_{(s,t)\in\Theta} n_{st} \{(1+r)I_s - (1+(1+r)K_t)R_{st}\} \geqslant 0.$$

The solution satisfies the following property.

Lemma 5.1. *If a pension schedule is optimal, it holds that $R_{st} > 0$ for all s and t.*

Proof. At first, we show that there is at least one type whose pension return is strictly positive. Suppose that there exists consumer with (I_s, λ_t) whose pension plan is $R_{st} = 0$. Then private welfare of the consumer is

$$(1+\lambda_t)\log(I_s - P_{st}) + \delta \log R_{st} - \lambda_t \log\left(I_s - P_{st} + \frac{R_{st}}{1+\rho}\right)$$
$$= (1+\lambda_t)\log(I_s - P_{st}) + \delta \log 0 - \lambda_t \log(I_s - P_{st}) = \log(I_s - P_{st}) + \delta \log 0 = -\infty.$$

Note that private welfare is bounded above by the constraints. Then the social welfare in this case is $-\infty$. On the other hand, let $(R'_{st}, P'_{st}) = (\varepsilon, \varepsilon/(1+r))$ for all s and t with $\varepsilon > 0$ that satisfies $\varepsilon/(1+r) < I_1$. A pension schedule composed of this plan is feasible. In fact, it holds that

$$\sum_{(s,t)\in\Theta} n_{st} \left\{(1+r)P'_{st} - R_{st}\right\} = \sum_{(s,t)\in\Theta} n_{st} \{\varepsilon - \varepsilon\} = 0.$$

The social welfare of this schedule is

$$\sum_{(s,t)\in\Theta} n_{st} \left\{(1+\lambda_t)\log\left(I_s - \frac{\varepsilon}{1+r}\right) + \delta \log \varepsilon - \lambda_t \log\left(I_s - \frac{\varepsilon}{1+r} + \frac{\varepsilon}{1+\rho}\right)\right\}$$

This is strictly larger than $-\infty$ since the antilogarithm of the first and the second term is greater than 0 and the third term is bounded above by $\varepsilon/(1+r) < I_1 < \infty$ and $\lambda_k < \infty$. Therefore, the schedule that assign $R_{st} = 0$ for all consumers is not a solution.

Next suppose that $R_{st} = 0$ for some s, t in the optimal shcedule. Z denote the set of these (s,t). By the discussion above, there is at least one $\tilde{s}$, $\tilde{t}$ whose plan satisfies $R_{\tilde{s}\tilde{t}} > 0$. As we saw, social welfare when there exist some consumers whose pension return is 0 is $-\infty$. Let $R'_{\tilde{s}\tilde{t}}$ be $\varepsilon \in (0, R_{st})$ and the return for consumers whose initial return is 0 be $R'_{\tilde{s}\tilde{t}}$ that satisfies

$$\sum_{(s,t)\in Z} n_{st} R'_{st} = n_{\tilde{s}\tilde{t}}(R_{\tilde{s}\tilde{t}} - \varepsilon).$$

For other consumers that does not included in $Z \cup \{\tilde{s}, \tilde{t}\}$, their plans are not be changed from initial plans. This assures that the feasibility condition holds. Actually,

$$\begin{aligned}
&\sum_{(s,t)\in\Theta} n_{st}\{(1+r)P'_{st} - R'_{st}\} \\
&= \sum_{(s,t)\in Z} n_{st}\{(1+r)P_{st} - R'_{st}\} + \sum_{(s,t)\notin Z\cup\{(\tilde{s},\tilde{t})\}} n_{st}\{(1+r)P_{st} - R_{st}\} + n_{\tilde{s}\tilde{t}}\{(1+r)P_{\tilde{s}\tilde{t}} - \varepsilon\} \\
&= \sum_{(s,t)\in Z} n_{st}(1+r)P_{st} + \sum_{(s,t)\notin Z\cup\{(\tilde{s},\tilde{t})\}} n_{st}\{(1+r)P_{st} - R_{st}\} + n_{\tilde{s}\tilde{t}}\{(1+r)P_{\tilde{s}\tilde{t}} - R_{\tilde{s}\tilde{t}}\} \\
&= \sum_{(s,t)\in\Theta} \{(1+r)P_{st} - R_{st}\} \geqslant 0
\end{aligned}$$

since the last line is feasibility condition for initial schedule. By the same reason as the discussion above, the social welfare of the revised schedule is strictly larger than $-\infty$. Therefore, the schedule that assign zero return to more than one consumer is not optimal. □

We use this result in the analysis below. The government's problem is as follows:

$$\begin{aligned}
\max_{S} \quad & \sum_{(s,t)\in\Theta} n_{st}\left\{(1+\lambda_t)\log K_t R_{st} + \delta \log R_{st} - \lambda_t \log\left(K_t + \frac{1}{1+\rho}\right) R_{st}\right\} \\
\text{s.t} \quad & \sum_{(s,t)\in\Theta} n_{st}\left\{(1+r)I_s - (1+(1+r)K_t)R_{st}\right\} \geqslant 0 \\
& 0 \leqslant R_{st} \leqslant I_s/K_t, \ \forall (s,t)\in\Theta.
\end{aligned}$$

The associated Lagrangian is

$$\begin{aligned}
\mathcal{L} &= \mu_0 \sum_{(s,t)\in\Theta} n_{st}\left[(1+\lambda_t)\log K_t R_{st} + \delta \log R_{st} - \lambda_t \log\left(K_t + \frac{1}{1+\rho}\right) R_{st}\right] \\
&+ \mu_1 \sum_{(s,t)\in\Theta} n_{st}\left[(1+r)I_s - (1+(1+r)K_t)R_{st}\right] + \sum_{(s,t)\in\Theta} \mu_{2st} R_{st} + \sum_{(s,t)\in\Theta} \mu_{3st}\left(\frac{I_s}{K_t} - R_{st}\right),
\end{aligned}$$

where μ_0, μ_1, μ_{2st} and μ_{3st} are Lagrange multipliers. Note that $\mu_{2st} = 0$ for all (s,t) in the solution because of the complementary slackness conditions for the constraint $R_{st} \geqslant 0$ and lemma 5.1. Then the first order conditions for R_{st} are

$$\frac{\partial \mathcal{L}}{\partial R_{st}} = \mu_0 n_{st} \frac{1+\delta}{R_{st}} - \mu_1 n_{st}(1+(1+r)K_t) - \mu_{3st} = 0. \tag{15}$$

We can see that μ_0 is strictly positive in the solution. Suppose that $\mu_0 = 0$. Then $\mu_1 = 0$ and $\mu_{3st} = 0$ must hold for all (s,t) by (15). However, this contradicts non-zero condition of Lagrange multipliers. Hence we can standardize μ_0 as 1. Then (15) is rewritten as

$$\mu_1 = \left(\frac{1+\delta}{R_{st}} - \frac{\mu_{3st}}{n_{st}}\right)\frac{1}{1+(1+r)K_t}. \tag{16}$$

The left hand side is independent of (s,t), the right hand side is equivalent for all (s,t).

We can exclude a schedule with $R_{st} = I_s/K_t$ for all (s,t) from a candidate of solution since this schedule is infeasible. In addition, here we show the following lemma.

Lemma 5.2. *For all $(s,t)\in\Theta$, $R_{st} < I_s/K_t$.*

Proof. To show this, we assume that $R_{st} = I_s/K_t$ for some $(s,t) \in \Theta$ and derive optimal plans for other $(\tilde{s},\tilde{t}) \neq (s,t)$. We propose a contradiction between a condition that $R_{\tilde{s}\tilde{t}} > 0$ and a condition that $\mu_{3st} \geqslant 0$. Define C to be $C \equiv \{(s,t) \in \Theta\colon R_{st} = I_s/K_t\}$. By the definition and the complementary slackness, $\mu_{3s't'} = 0$ for $(s',t') \notin C$, which implies two things. First, $\mu_1 > 0$ by (16), that is, the feasibility condition is satisfied with equality. Second, by (16) again, it holds that

$$\begin{aligned}&\frac{1+\delta}{R_{st}}\frac{1}{1+(1+r)K_t} = \frac{1+\delta}{R_{s't'}}\frac{1}{1+(1+r)K_{t'}}\\ \Longleftrightarrow\ &(1+(1+r)K_t)R_{st} = (1+(1+r)K_{t'})R_{s't'}\end{aligned} \tag{17}$$

for any (s,t), $(s',t') \notin C$. Using this relation, we have

$$\begin{aligned}&\sum_{(s,t)\in\Theta} n_{st}\left[(1+r)I_s - (1+(1+r)K_t)R_{st}\right]\\ &= \sum_{(s,t)\notin C} n_{st}\left[(1+r)I_s - (1+(1+r)K_t)R_{st}\right]\\ &\quad + \sum_{(s,t)\in C} n_{st}\left[\frac{(1+r)K_tI_s - (1+(1+r)K_t)I_s}{K_t}\right] = 0\\ \Longleftrightarrow\ &(1+(1+r)K_{\tilde{t}})R_{\tilde{s}\tilde{t}}\sum_{(x,t)\notin C} n_{st} = (1+r)\sum_{(s,t)\notin C} n_{st}I_s - \sum_{(s,t)\in C}\frac{n_{st}I_s}{K_t}\\ \Longleftrightarrow\ &R_{\tilde{s}\tilde{t}} = \frac{1}{(1+(1+r)K_{\tilde{t}})\sum_{(s,t)\notin C} n_{st}}\left[(1+r)\sum_{(s,t)\notin C} n_{st}I_s - \sum_{(s,t)\in C}\frac{n_{st}I_s}{K_t}\right]\end{aligned}$$

for any $(\tilde{s},\tilde{t}) \notin C$. This is positive if and only if

$$(1+r)\sum_{(s,t)\notin C} n_{st}I_s > \sum_{(s,t)\in C}\frac{n_{st}I_s}{K_t}. \tag{18}$$

On the other hand, it must hold that $\mu_{3st} \geqslant 0$ for $(s,t) \in C$. By (16), for any $(s,t) \in C$ and $(s',t') \notin C$, we have the following:

$$\begin{aligned}&\frac{1+\delta}{R_{s't'}}\frac{1}{1+(1+r)K_{t'}} = \left(\frac{1+\delta}{R_{st}} - \frac{\mu_{3st}}{n_{st}}\right)\frac{1}{1+(1+r)K_t}\\ \Rightarrow\ &\mu_{3st} = n_{st}(1+\delta)\left(\frac{K_t}{I_s} - \frac{(1+(1+r)K_t)\sum_{(s,t)\notin C} n_t^s}{(1+r)\sum_{(s,t)\notin C} n_t^s I_s - \sum_{(s,t)\in C}(n_{st}I_s/K_t)}\right).\end{aligned}$$

A sufficient and necessary condition for $\mu_{3st} \geqslant 0$ is

$$\begin{aligned}&\frac{K_t}{I_s} \geqslant \frac{(1+(1+r)K_t)\sum_{(s,t)\notin C} n_t^s}{(1+r)\sum_{(s,t)\in C} n_t^s I_s - \sum_{(s,t)\in C}(n_{st}I_s/K_t)}\\ \Longleftrightarrow\ &\frac{I_s}{K_t}\frac{(1+(1+r)K_t)\sum_{(s,t)\notin C} n_t^s}{(1+r)\sum_{(s,t)\in C} n_t^s I_s - \sum_{(s,t)\in C}(n_{st}I_s/K_t)} \leqslant 0\\ \Longleftrightarrow\ &(1+r)\sum_{(s,t)\notin C} n_t^s I_s < \sum_{(s,t)\in C}\frac{n_{st}I_s}{K_t}\end{aligned} \tag{19}$$

Thus, (19) contradicts (18). □

By lemma, we can focus on an inner solution. Then by the complementary slackness conditions, $\mu_{3st} = 0$ for all $(s,t) \in \Theta$ and we can derive the relation (17) for any pair (s,t), $(s',t') \in \Theta$. Moreover, the feasibility condition is satisfied with equality by (16) and $\mu_{3st} = 0$. Substituting (17) to the feasibility condition, we have

$$R_{st} = \frac{(1+r)\sum_{(\tilde{s},\tilde{t})\in\Theta} n_{\tilde{s}\tilde{t}} I_{\tilde{s}}}{1+(1+r)K_t}$$

for all $(s,t) \in \Theta$. To derive P_{st}, substitute R_{st} to $P_{st} = I_s - K_t R_{st}$. Then

$$\begin{aligned} P_{st} &= I_s - K_t \frac{(1+r)\sum_{(\tilde{s},\tilde{t})\in\Theta} n_{\tilde{s}\tilde{t}} I_{\tilde{s}}}{1+(1+r)K_t} \\ &= \frac{[1+(1+r)K_t]I_s - K_t(1+r)n_{st}I_s - K_t(1+r)\sum_{(s',t')\neq(s,t)} n_{s't'}I_{s'}}{1+(1+r)K_t} \\ &= \frac{I_s + (1+r)K_t\sum_{(s',t')\neq(s,t)} n_{s't'}I_s - (1+r)K_t\sum_{(s',t')\neq(s,t)} n_{s't'}I_{s'}}{1+(1+r)K_t} \\ &= \frac{I_s + (1+r)K_t\sum_{(s',t')\neq(s,t)} n_{s't'}(I_s - I_{s'})}{1+(1+r)K_t}. \end{aligned}$$

We need to check whether consumers have incentive to untruthfully tell their degree of temptation. Since their optimal plans that we have derived is determined on the locus of $\sigma_{st}(\tau_{st}) = 1/(1+r)$, one sufficient condition for no-deviation is that plans for arbitrary two consumers with the same income are on the same line with the gradient of $1/(1+r)$. Actually, this is satisfied.

Proposition 5.1. *For any (s,t) and (s,t'), it holds that*

$$\frac{P_{st} - P_{st'}}{R_{st} - R_{st'}} = \frac{1}{1+r}.$$

Proof. We calculate $R_{st} - R_{st'}$ and $P_{st} - P_{st'}$ in practice.

$$R_{st} - R_{st'} = \frac{(1+r)\sum_{(\tilde{s},\tilde{t})\in\Theta} n_{\tilde{s}\tilde{t}} I_{\tilde{s}}}{1+(1+r)K_t} - \frac{(1+r)\sum_{(\tilde{s},\tilde{t})\in\Theta} n_{\tilde{s}\tilde{t}} I_{\tilde{s}}}{1+(1+r)K_{t'}} = \frac{(1+r)(K_{t'} - K_t)\sum_{(\tilde{s},\tilde{t})\in\Theta} n_{\tilde{s}\tilde{t}} I_{\tilde{s}}}{[1+(1+r)K_t][1+(1+r)K_{t'}]},$$

$$\begin{aligned} P_{st} - P_{st'} &= (I_s - K_t R_{st}) - (I_s - K_t' R_{st'}) \\ &= \frac{K_{t'}((1+r)\sum_{(\tilde{s},\tilde{t})\in\Theta} n_{\tilde{s}\tilde{t}} I_{\tilde{s}})}{1+(1+r)K_t} - \frac{K_t((1+r)\sum_{(\tilde{s},\tilde{t})\in\Theta} n_{\tilde{s}\tilde{t}} I_{\tilde{s}})}{1+(1+r)K_{t'}} \\ &= \frac{K_{t'}[1+(1+r)K_{t'}] - K_t[1+(1+r)K_t]}{[1+(1+r)K_t][1+(1+r)K_{t'}]} \sum_{(\tilde{s},\tilde{t})\in\Theta} n_{\tilde{s}\tilde{t}} I_{\tilde{s}} \\ &= \frac{(K_{t'} - K_t)\sum_{(\tilde{s},\tilde{t})\in\Theta} n_{\tilde{s}\tilde{t}} I_{\tilde{s}}}{[1+(1+r)K_t][1+(1+r)K_{t'}]}. \end{aligned}$$

Hence, we have the intended result. □

Therefore, we can conclude that the solution we have derived is not only for the problem with complete information but also for the problem with incomplete information.

Theorem 5.1. *In the unique optimal schedule, the plan of consumer with (I_s, λ_t) is*

$$\begin{aligned} R_{st} &= \frac{(1+r)\sum_{(\tilde{s},\tilde{t})\in\Theta} n_{\tilde{s}\tilde{t}} I_{\tilde{s}}}{1+(1+r)K_t} \\ P_{st} &= \frac{I_s + (1+r)K_t\sum_{(s',t')\neq(s,t)} n_{s't'}(I_s - I_{s'})}{1+(1+r)K_t}. \end{aligned}$$

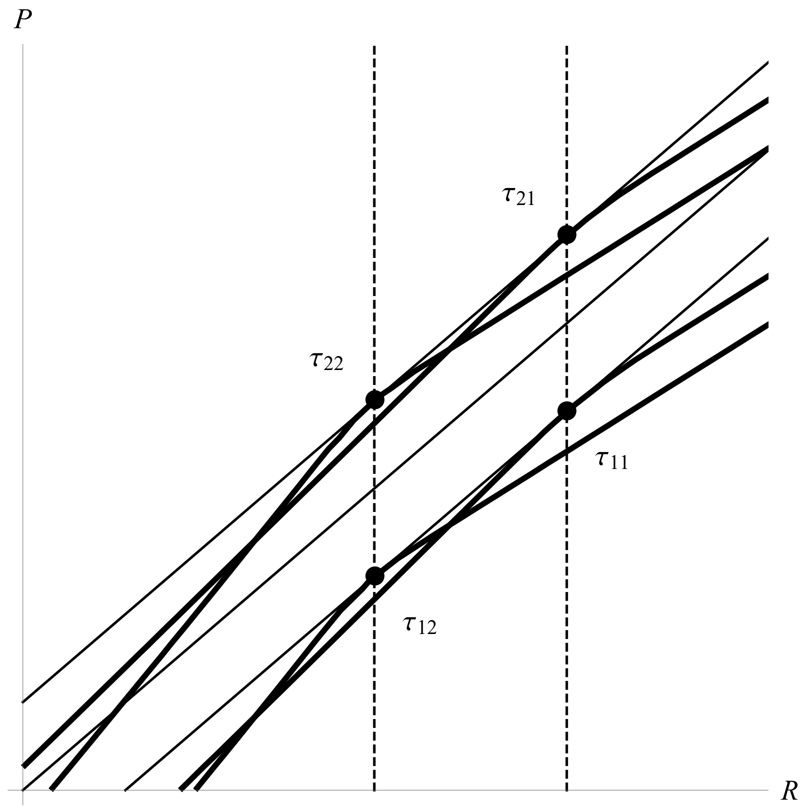

Figure 6: An optimal schedule

Figure 6 shows an example of the optimal schedule when $\mathcal{I} = \{I_1, I_2\}$ and $\Lambda = \{\lambda_1, \lambda_2\}$. Four points are the elements of optimal schedule. Note that the consumers can lie about only their degree of temptation. Thus the deviation that we have to think is between (I_1, λ_1) and (I_1, λ_2) and between (I_2, λ_1) and (I_2, λ_2). We can see that the deviation makes loss for any types.

The optimal schedule above has some characteristic property. First, clearly, this schedule is the generalization of the result in the previous section with identical income. You can see this by assuming $I_1 = I_2 = \cdots = I_m$ in the plan for (s, t). Second, the pension return is equivalent for two consumers if and only if their degree of temptation are equivalent. That is, the determination of the amount of return depends only on the degree of temptation and not on income. On the other hand, third, the payment for the pension fund depends both of income and the degree of temptation. Especially, consumers with relatively higher income pay more amount, since the summation in the numerator of P_{st} become larger when I_s is high. This implies that there is a monetary transfer from high-income consumers to low-income consumers through the differences of payments.

6 Conclusion

We have considered optimally funded pensions for consumers who face the temptation to overconsume and as well as for those who have enough self-control to withstand their temptation. Since funded pensions tighten consumers' budgets , they can serve as commitment devices to avoid overconsumption. We applied the pension to an economy in which consumers have heterogeneous self-control. We showed that funded pensions can improve social welfare even if the interest rates they draw are the same as those for private saving. In addition, consumers do not save individually when they choose the pension plan that is optimal for them. Furthermore, interestingly, lower pension premium and lower pension payout are applied for a higher temptation economy. This result is related to borrowing constraints. In an identical-type economy, an increase in the premium leads to increased pension income, and this augments possibility of debt. Since consumers are tempted to overconsume, this works stronger for consumers who have strong temptations. This effect is greater than the benefit of strengthening the budget set. As a main result, we have considered an optimal pension schedule when there exist two or more types in the economy. In that situation, we show the necessary conditions for the optimal schedule. If the normative utility function is a logarithm, it is characterized in the same way as that for an economy with identical-type. We show that monetary transfer among types will not occur for the optimal schedule. An important result is that the optimal schedule does not depend on the distribution of types, that is, what the government has to know is only what types are in the economy. This make the operation of the pension policy easier.

We have showed the construction of optimal pension scheme in which income redistribution occurs. In the future research, we will extend the model to the situation of overlapping generations model. We study the relation between relieving of self-control cost and intergenerational redistribution. Then we can this make scheme fitted to the more realistic pension policy.

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