Shapley Value and its Modified Solutions : Axiomatic and Non-cooperative Characterizations

Yukihiko Funaki

Waseda University, Tokyo, JAPAN

19 Oct. 2017, KIER seminar

Schedule of the Talk

- 1. Introduction
- 2. Several characterizations of the Shapley value
- 3. α-egalitarian Shapley value
- 4. δ-discounted Shapley value
- 5. Weak Surplus Monotonicity Axiom
- 6. **r**-Egalitarian Shapley value

1. Introduction : the Shapley value

- The Shapley value — One of the most famous solution concepts of cooperative games.
- Axiomatizations of the Shapley value :
  Original(Shapley[1953]),
  Strong Monotonicity(Young[1985]),
  Consistency(Sobolev[1973], Hart and Mas-Colell[1989]),
- Implementation(Pérez-Castrillo and Wettstein [2001])
- The Balanced Contribution property $\longrightarrow$ axiomatic characterizations of the Shapley value and related solutions (Myerson,1980).

Introduction : Modifications

- Convex combination of the Shapley value and the Equal Division value (Joosten[1996]) $\rightarrow$ α-egalitarian Shapley value (van den Brink, Funaki and Ju[2013], Casajus and Huettner[2014]))
- Shapley value of a discounted game (Joosten[1996]) $\rightarrow$ δ-discounted Shapley value
- Several monotonicity axioms $\rightarrow$ the above values, Consensus values(Ju et al.[2007]) and some modifications(Yokote and Funaki[2015])

Introduction : Modifications

- The Balanced Contribution property for Equal Contributors $\longrightarrow$ a class of solutions called **r-egalitarian Shapley values** (Yokote, Funaki and Kongo[2017]).
- This class contains the egalitarian Shapley values and the discounted Shapley values.

2. Several characterizations of the Shapley value

(N, v) : n-person TU game ($n = |N|$)
$N \subset \mathcal{N}$: the set of players
v : a characteristic function from 2^N to $\mathbb{R}$ with $v(\emptyset) = 0$
Γ : set of all games
Γ^N : set of all games with the player set N
$x = (x_i)_{i \in N} \in \mathbb{R}^N$: A payoff vector of a game (N, v).
A value function (one point solution) ψ on Γ:

$$(N, v) \in \Gamma \mapsto \psi(N, v) \in \mathbb{R}^N,$$

The Shapley value

The Shapley value

$$Sh_i(N,v)=\sum_{\substack{S\subseteq N\\ i\in S}}\frac{(n-s)!(s-1)!}{n!}(v(S)-v(S\setminus\{i\}))\quad \forall i\in N,$$

where $n=|N|, s=|S|$.

Axioms for the Shapley value

- **Axiom (EFFiciency)**: $\sum_{i\in N}\phi_i(N,v)=v(N)$
- **Axiom (NULL player)**: For a null player i ($\Leftrightarrow v(S\cup\{i\})=v(S)\ \forall S\subseteq N\setminus\{i\}$), $\phi_i(N,v)=0$.
- **Axiom (SYMmetry)**: If i and j are substitutes ($\Leftrightarrow v(S\cup\{i\})=v(S\cup\{j\})\ \forall S\subseteq N\setminus\{i,j\}$), then $\phi_i(N,v)=\phi_j(N.v)$.
- **Axiom (ADDitivity)**: For any games (N,v) and (N,w), $\phi_i(N.v+w)=\phi_i(N,v)+\phi_i(N,w)\ \ \forall i\in N$. Here, $(v+w)(S)=v(S)+w(S)\ \ \forall S\subseteq N$.

Theorem (Shapley[1953])

ϕ satisfies EFF, ADD, SYM and NULL $\iff \phi=Sh$.

Monotonicity

Axiom (Strong MONotonicity):
If $v(S\cup\{i\})-v(S)\geq w(S\cup\{i\})-w(S)$ for all $S\subseteq N\setminus\{i\}$, then $\phi_i(N,v)\geq\phi_i(N,w)$.

Theorem (Young[1985])

ϕ satisfies EFF, SYM and strong MON $\iff \phi=Sh$.

Axiom (Marginality):
If $v(S\cup\{i\})-v(S)=w(S\cup\{i\})-w(S)$ for all $S\subseteq N\setminus\{i\}$, then $\phi_i(N,v)=\phi_i(N,w)$.

Consistencies

Axiom (standardness for two-player games) For every $(N,v)\in\Gamma$ with $N=\{i,j\},\ i\neq j$, it holds that

$$\phi_i(N,v)=v(\{i\})+\frac{v(N)-v(\{i\})-v(\{j\})}{2}.$$

Definitions: Take $(N,v)\in\Gamma$ with $n\geq 2$, $j\in N$, $x\in\mathbb{R}^N$.

Complement reduced game w.r.t j **and** x is given by
$v^x(S)=v(S\cup\{j\})-x_j$ for all $\emptyset\neq S\subseteq N\setminus\{j\},\quad v^x(\emptyset)=0$.

Projection reduced game w.r.t j **and** x is given by
$v^x(S)=v(S)$ for all $S\subset N\setminus\{j\},\quad v(N\setminus\{j\})=v(N)-x_j$.

Convex consistency

Definition: **Convex Reduced Game**
For $(N,v)\in\Gamma$ with $n\geq 2$, $j\in N$, $x\in\mathbb{R}^N$, the **Convex reduced game w.r.t** j **and** x is the game $(N\setminus\{j\},v^x)$ given by

$$v^x(S)=\frac{|S|}{n-1}(v(S\cup\{j\})-x_j)+\frac{n-1-|S|}{n-1}v(S)\text{ for all }S\subseteq N\setminus\{j\}.$$

Definition: **Convex Consistency**
Let ϕ be a value on Γ. ϕ satisfies **Convex consistency** on Γ $\iff$ For every $(N,v)\in\Gamma$ with $n\geq 3$, $j\in N$, and $x=\phi(N,v)$, $\phi_i(N\setminus\{j\},v^x)=\phi_i(N,v)$ for $i\in N\setminus\{j\}$.

Theorem (Sobolev[1973])

ϕ satisfies Convex consistency on Γ and standardness for two-person games, $\iff \phi=Sh$.

HM Consistency

Definition: **Hart and Mas-Colell Reduced Game**
Given $(N,v)\in\Gamma$ with $n\geq 2$, $j\in N$, and a value ϕ, the **Hart and Mas-Colell reduced game w.r.t.** j is the game $(N\setminus\{j\},v^\phi)$ given by

$$v^\phi(S)=v(S\cup\{j\})-\phi_j(S\cup\{j\},v)\text{ for all }S\subseteq N\setminus\{j\}.$$

Definition: **Hart and Mas-Colell Consistency**
Let ϕ be a value on Γ. ϕ satisfies **Hart and Mas-Colell consistency** on Γ
$\iff$ For every $(N,v)\in\Gamma$ with $j\in N$,

$$\phi_i(N\setminus\{j\},v^\phi)=\phi_i(N,v)\quad\text{for }i\in N\setminus\{j\}.$$

Theorem (Hart and Mas-Colell[1989])

ϕ satisfies Hart and Mas-Colell consistency on Γ and standardness for two-person games, $\iff$ $\phi = Sh$.

Implementation

Pérez-Castrillo and Wettstein [2001] give an extensive form game called a bidding mechanism.

The bidding game for a set of players $N = \{1, ..., n\}$:

$t = 1$: Each player $i \in N$ makes bids $b^i = (b^i_j)_{j \neq i} \in \mathbb{R}^{n-1}$. For each $i \in N$, let $B^i = \sum_{j \neq i}(b^i_j - b^j_i)$, be the net bid of player i measuring its willingness to be the proposer. Let $h = \arg\max_i(B^i)$ where, in case there are multiple maximizers, h is randomly chosen among the maximizers. Once chosen, player h pays b^h_j to every player $j \neq h$.

$t = 2$: Player h makes a proposal, which specifies the offer y^h_j in $\mathbb{R}$ to every player $j \neq h$.

$t = 3$: The players other than h, sequentially, either accept or reject the offer.
If the offer is accepted by every player, each player $j \neq h$ receives y^h_j and player h obtains the worth of the grand coalition minus the payments $\sum_{j \neq h} y^h_j$. Then h gets $v(N) - \sum_{j \neq h} y^h_j - \sum_{j \neq h} b^h_j$ in total, and each $j (j \neq h)$ gets $y^h_j + b^h_j$.
If the offer is rejected by at least one player, then all players except for h proceed to play a sub bidding mechanism with player set $N \backslash \{h\}$ whereas player h obtains its stand-alone worth $v(\{h\})$, that is, $v(\{h\}) - \sum_{j \neq h} b^h_j$ in total.

Definition: A TU-game (N, v) is zero-monotonic if $v(N) \geq v(S) + \sum_{i \in N \backslash S} v(\{i\})$ for all $S \subset N$.

Theorem(Pérez-Castrillo and Wettstein[2001])

If the game (N, v) is zero-monotonic, then the outcome in any subgame perfect equilibrium of the bidding mechanism coincides with the payoff vector of the Shapley value.

3. α-egalitarian Shapley value

The Equal Devision value

$$ED_i(N, v) = \frac{v(N)}{n} \quad \forall i \in N.$$

α-egalitarian Shapley value ($\alpha \in [0, 1]$)

$$\varphi^\alpha(N, v) = \alpha Sh(N, v) + (1 - \alpha) ED(N, v)$$

Consistency

Axiom (α-standardness for two-player games) Let $\alpha \in [0, 1]$. Then for every $(N, v) \in \Gamma$ with $N = \{i, j\},\ i \neq j$, it holds that

$$\phi_i(N, v) = \alpha v(\{i\}) + \frac{v(N) - \alpha v(\{i\}) - \alpha v(\{j\})}{2}.$$

- $\alpha = 1$ yields *standardness for 2-person games*
- $\alpha = 0$ yields *egalitarian standardness for 2-person games*

Consistency

Theorem (van den Brink, Funaki and Ju[2013])

Take any $\alpha \in [0,1]$. ϕ satisfies Convex consistency on Γ and α-standardness for two-person games $\iff \phi = \varphi^\alpha$.

Monotonicity

Axiom (Weak MONotonicity) If $v(N) \geq w(N)$ and $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \setminus \{i\}$, then $\phi_i(N,v) \geq \phi_i(N,w)$.

Theorem (van den Brink, Funaki and Ju[2013]),Casajus and Huettner[2014])

Let $|N| \geq 3$. ϕ satisfies EFF, ADD, and weak MON $\iff \exists\ \alpha \in [0,1]\ \ s.t.\ \ \phi = \varphi^\alpha$.

Implementation

We adapt Pérez-Castrillo and Wettstein [2001] bidding mechanism to get the α-egalitarian Shapley values.
The bidding game for a set of players $N = \{1, ..., n\}$:

$t = 1, 2$: The same as the bidding mechanism.
$t = 3$: The players other than h, sequentially, either accept or reject the offer.
If the offer is accepted by every player, each player $j \neq h$ receives y_j^h and player h obtains the worth of the grand coalition minus the payments $\sum_{j \neq h} y_j^h$. Then h gets $v(N) - \sum_{j \neq h} y_j^h - \sum_{j \neq h} b_j^h$ in total, and each $j (j \neq h)$ gets $y_j^h + b_j^h$.

Implementation

If the offer is rejected by at least one player, then with probability $(1 - \alpha)$, where $\alpha \in [0,1]$, the game stops and all players including the proposer h get zero payoffs, (that is, $-\sum_{j \neq h} b_j^h$ in total,) while with probability α all players except for h proceed to play a sub bidding mechanism with player set $N \backslash \{h\}$ whereas player h obtains its stand-alone worth $v(\{h\})$, (that is, $v(\{h\}) - \sum_{j \neq h} b_j^h$ in total).
However from now on, in case of rejection, the remaining players other than player h keep playing the bidding mechanism, which is the same as the one in Pérez-Castrillo and Wettstein [2001].

Theorem

Theorem(van den Brink, Funaki and Ju[2013])

If the game (N, v) is zero-monotonic, then the outcome in any subgame perfect equilibrium of this bidding mechanism coincides with the payoff vector of the α-egalitarian Shapley value.

4. δ-discounted Shapley value

Modified Implementation
We consider a more consistent mechanism.
$t = 1, 2$: The same.
$t = 3$: The players other than h, sequentially, either accept or reject the offer.
If the offer is accepted by every player, each player $j \neq h$ receives y_j^h and player h obtains the worth of the grand coalition minus the payments $\sum_{j \neq h} y_j^h$. Then h gets $v(N) - \sum_{j \neq h} y_j^h - \sum_{j \neq h} b_j^h$ in total, and each $j (j \neq h)$ gets $y_j^h + b_j^h$.

If the offer is rejected by at least one player, then with probability $(1-\delta)$, where $\delta \in [0,1]$, the game stops and all players including the proposer h get zero payoffs, (that is, $-\sum_{j\neq h} b_j^h$ in total,) while with probability δ all players except for h proceed to play a sub bidding mechanism with player set $N\backslash\{h\}$ whereas player h obtains its stand-alone worth $v(\{h\})$, (that is, $v(\{h\})-\sum_{j\neq h} b_j^h$ in total).
In case of rejection, the remaining players other than player h play the bidding game which is the same as the case $t=1$.
(Go back to $t=1$.)

What is the value which is implemented by this mechanism?

δ-discounted Shapley value ($\delta \in [0,1]$)

$$\psi_i^\delta(N,v) = \sum_{\substack{S\subseteq N\setminus\{i\} \\ S\neq\emptyset}} \frac{|S|!(n-|S|-1)!}{n!} \cdot$$

$$\delta^{n-|S|-1}\big(v(S\cup\{i\}) - \delta\cdot v(S)\big) \quad \text{for all } i\in N.$$

Theorem(van den Brink and Funaki[2015])

Let $\delta\in[0,1]$ and and $v\in\Gamma$ be a zero monotonic game. Then the outcome in any subgame perfect equilibrium of the bidding mechanism coincides with the payoff vector of the δ-discounted Shapley value $\psi^\delta(N,v)$.

δ-discounted Shapley value ($\delta \in [0,1]$)

$$\psi^\delta(N,v) = Sh(N,w^\delta),$$
$$w^\delta(S) = \delta^{n-|S|}v(S) \text{ for all } S$$

Consistency

Theorem (Joosten[1996])

Take any $\delta\in[0,1]$. ϕ satisfies Hart and Mas-Colell consistency on Γ and δ-standardness for two-person games $\iff$ $\phi=\psi^\delta$.

Monotonicity

Axiom (δ-MONotonicity) $\phi_i(N,v)\geq\phi_i(N,w)$ for two games $(N,v),(N,w)$ and $i\in N$ such that $v(S\cup\{i\})-\delta v(S)\geq w(S\cup\{i\})-\delta w(S)$ for all $S\subseteq N\setminus\{i\}$.

Theorem(van den Brink and Funaki[2014])

Take any $\delta\in[0,1]$. ϕ satisfies EFF, SYM, δ-MON and δ-standardness for two-person games $\iff$ $\phi=\psi^\delta$.

5. Weak Surplus Monotonicity Axiom

- Strong Monotonicity (Mariginal contribution Monotonicity) $\longrightarrow$ the Shapley value (Young [1985])
- Weak Monotonicity (Marginal + Grand coalition) $\longrightarrow$ the Egalitarian Shapley value (van den Brink et al. [2013], Casajus and Huettner [2014])

The **consensus value** (Ju et al. [2007]): for $\alpha\in[0,1]$,

$$CV^\alpha(N,v) = \alpha Sh(N,v) + (1-\alpha)CIS(N,v).$$

Weak Surplus Monotonicity

Axiom: Weak Surplus Monotonicity (WSM)

Let $v, w \in \Gamma$ and $i \in N$. If

- $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$, $\forall\ S \subseteq N \backslash \{i\}$.
- $v(N) \geq w(N)$, and
- $v(N) - \sum_{j \in N} v(\{j\}) \geq w(N) - \sum_{j \in N} w(\{j\})$,

then $\psi_i(N, v) \geq \psi_i(N, w)$.

Theorem(Yokote and Funaki[2015])

Let $n \geq 6$. Then, a solution ψ satisfies EFF, SYM and WSM $\iff$ There exist $\alpha, \beta, \gamma \in [0, 1]$ s.t.

$$\psi(N, v) = \alpha ES^{\beta}(N, v) + (1 - \alpha) CV^{\gamma}(N, v).$$

Weak Surplus Monotonicity

Corollary

Let $n \geq 6$. Then, a solution ψ satisfies EFF, SYM and WSM $\iff$ There exist $\alpha_1, \alpha_2, \alpha_3 \in [0, 1]$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$, s.t.

$$\psi(N, v) = \alpha_1 Sh(N, v) + \alpha_2 ED(N, v) + \alpha_3 CIS(N, v).$$

An Example

Conisder (N, v) and (N, w), where $N = \{1, 2, 3\}$

- $v(1) = 0, v(2) = v(3) = 50,$
- $v(12) = v(13) = v(23) = 60, v(N) = 110,$
- $w(1) = w(2) = w(3) = 0,$
- $w(12) = w(13) = w(23) = 10, w(N) = 60,$

These satisfy:

- $v(1) = w(1),\ v(12) - v(2) = w(12) - w(2),$
  $v(13) - v(3) = w(13) - w(3),$
  $v(123) - v(23) = w(123) - w(23),$

and

- $v(N) = 110 > w(N) = 60,$

but

- $v(N) - \sum_{j \in N} v(j) = 10 < w(N) - \sum_{j \in N} w(j) = 60.$

Proof of Theorem

- Sketch of the proof.

For each $T \subseteq N,\ T \neq \emptyset$, we define u_T by

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

For each $T \subseteq N, |T| \geq 2$, we define $\bar{u}_T$ by

$$\bar{u}_T(S) = \begin{cases} 1 & \text{if } |S \cap T| = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Theorem

Define $u^1 = \sum_{i \in N} u_i$, $u^2 = \sum_{T \subseteq N : |T| = 2} u_T$. Then, the following set is a basis of game space Γ^N.

$$\{u^1\} \cup \{u_1 - u_i : i \in N, i \neq 1\} \cup \{u^2\}$$
$$\cup \{u_{12} - u_T : T \subseteq N, |T| = 2, T \neq \{1, 2\}\} \cup \{\bar{u}_T : |T| \geq 3\}$$

$$V^1 = \{u^1\} \cup \{u^2\} \cup \{\bar{u}_T : |T| \geq 3\}, \qquad \Gamma^1 = \text{Sp}(V^1),$$
$$V^2 = \{u_{12} - u_T : T \subseteq N, |T| = 2, T \neq \{1, 2\}\}, \quad \Gamma^2 = \text{Sp}(V^2),$$
$$V^3 = \{u_1 - u_i : i \in N, i \neq 1\}, \qquad \Gamma^3 = \text{Sp}(V^3).$$

Proof of Theorem

Firstly, we show that

$$\psi(N, v + w^3) = \psi(N, v) + \psi(N, w^3) \text{ for all } v \in \Gamma, w^3 \in \Gamma^3. \quad \text{(A)}$$

Next, we show that

$$\psi(N, v + w^2) = \psi(N, v) + \psi(N, w^2) \text{ for all } v \in \Gamma, w^2 \in \Gamma^2. \quad \text{(B)}$$

We also show that

$$\psi_i(N, w^1) = \frac{w^1(N)}{n} \text{ for all } w^1 \in \Gamma^1. \quad \text{(C)}$$

Proof of Theorem

Then for $v \in \Gamma$, we can express v by $v = v^1 + v^2 + v^3$, where $v^j \in \Gamma^j$, and

$$\begin{aligned}\psi_i(N, v) = \psi_i(N, v^1 + v^2 + v^3) &\overset{(A)}{=} \psi_i(N, v^1 + v^2) + \psi(N, v^3)\\ &\overset{(B)}{=} \psi_i(N, v^1) + \psi_i(N, v^2) + \psi_i(N, v^3)\\ &\overset{(C)}{=} \psi_i(N, v^2) + \psi_i(N, v^3) + \frac{v(N)}{n}.\end{aligned}$$

Cases for $n \leq 5$

- $n = 1$: EFF uniquely determines ψ.
- $n = 2$: There is another solution that satisfies EFF, SYM, WSM, but is not a convex combination of the solutions.
  - Casajus and Huettner [2014a].
- $n = 3$: We have another complicated solution that satisfies EFF, SYM, WSM, but is not a convex combination of the solutions.
- $n = 4, 5$: Open questions.

Surplus Monotonicity

Axiom:Surplus Monotonicity (SM)

Let $v, w \in \Gamma$ and $i \in N$. If

- $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$, $\forall\ S \subseteq N \backslash \{i\}$,
- $v(N) - \sum_{j \in N} v(\{j\}) \geq w(N) - \sum_{j \in N} w(\{j\})$,

then $\psi_i(N, v) \geq \psi_i(N, w)$.

Theorem(Yokote and Funaki[2015])

Let $n \geq 6$. Then, a solution $\psi(N, v)$ satisfies EFF, SYM and SM $\iff$ There exists $\alpha \in [0, 1]$ s.t. $\psi(N, v) = CV^{\alpha}(N, v)$.

Dual solutions

We consider a dual of WSM.

- $v^*(S) = v(N) - v(N \setminus S)\ \forall S \subseteq N$: dual game
- $\psi^*(N, v) = \psi(N, v^*)$: dual solution
- Axiom using v^* and ψ^*: dual axiom of v and ψ
- Dual axioms characterize a dual solution.

$$ENSC_i(N, v) = v(N) - v(N \backslash \{i\}) + \frac{v(N) - \sum_{j \in N}(v(N) - v(N \backslash \{j\})}{n}$$

$$ENSC(N, v) = CIS^*(N, v)$$

Axiomatization of Dual solution

Axiom: Dual Weak Surplus Monotonicity (DWSM)

Let $v, w \in \Gamma$ and $i \in N$. If

- $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$, $\forall\ S \subseteq N \backslash \{i\}$,
- $v(N) \geq w(N)$, and
- $v(N) - \sum_{j \in N}(v(N) - v(N \backslash \{j\})) \geq w(N) - \sum_{j \in N}(w(N) - w(N \backslash \{j\}))$,

then $\psi_i(N, v) \geq \psi_i(N, w)$.

Theorem (Yokote and Funaki [2015])

Let $n \geq 6$. Then, a solution ψ satisfies EFF, SYM and DWSM if and only if there exist $\alpha, \beta \in [0, 1]$ s.t. $\psi = \alpha ES^{\beta} + (1 - \alpha) ENSC$.

Summary Table $\quad \Delta_i(v) = v(S \cup \{i\}) - v(S)$

	Sufficient condition based on:				Solutions			
Axiom	$\Delta_i v$	$v(N)$	$v(N) - \sum_i v(i)$	dual	Sh	ED	CIS	ENSC
WSM	○	○	○		○	○	○	
D-WSM	○	○		○	○	○		○
WM	○	○			○	○		
SM	○		○		○		○	
D-SM	○			○	○			○
WGM		○	○			○	○	
D-WGM		○		○		○		○
STM	○				○			
SSM			○				○	
D-SSM				○				○
GM		○				○		

6. **r**-Egalitarian Shapley value

- For $N \subseteq \mathcal{N}$, $i \in N$ and $\lambda \in \mathbb{R}$, we define $(N, v_{\lambda,i}) \in \Gamma$ by

$$v_{\lambda,i}(S) = \begin{cases} v(S) + \lambda & \text{if } i \in S, \\ v(S) & \text{otherwise.} \end{cases}$$

Axiom: *Weak Strategic Invariance*, WSI.

For any $(N, v) \in \Gamma$, $i \in N$ and $\lambda \in \mathbb{R}$,

$$\psi_i(N, v_{\lambda,i}) = \psi_i(N, v) + \lambda.$$

Weak balanced contribution property

Axiom: *Balanced contribution property* (Myerson[1980]).

For any $(N, v) \in \Gamma$ and $i, j \in N$, $i \neq j$,

$$\psi_i(N, v) - \psi_i(N \backslash \{j\}, v) = \psi_j(N, v) - \psi_j(N \backslash \{i\}, v).$$

- "For any $i, j \in N$" seems to be a strong condition.

Axiom: *Balanced contribution property for equal contributors*, BCEC.

For any $(N, v) \in \Gamma$ and $i, j \in N$, $i \neq j$, $v(N \backslash \{i\}) = v(N \backslash \{j\})$,

$$\psi_i(N, v) - \psi_i(N \backslash \{j\}, v) = \psi_j(N, v) - \psi_j(N \backslash \{i\}, v).$$

r-Egalitarian Shapley value

- For any $(N, v) \in \Gamma$ and a sequence of real numbers $\mathbf{r} = \{r_k\}_{k=1}^n$, $n = |N|$, we define $(N, v^{\mathbf{r}}) \in \Gamma$ by

$$v^{\mathbf{r}}(S) = r_s v(S) \text{ for all } S \subseteq N,$$

  where $s := |S|$.
- We define the **r-egalitarian Shapley value** $ESh^{\mathbf{r}}$ by

$$ESh^{\mathbf{r}}(N, v) = (1 - r_n) \cdot \frac{v(N)}{n} + Sh(N, v^{\mathbf{r}}) \text{ for } (N, v) \in \Gamma.$$

Theorem(Yokote, Funaki and Kongo[2016])

A solution ψ on Γ^N satisfies EFF, WSI and BCEC $\iff$ There exists $\mathbf{r} = \{r_k\}_{k=1}^n$ such that $\psi(N, v) = ESh^{\mathbf{r}}(N, v)$.

Relationships with other solutions

- For $\alpha \in [0, 1]$, $r_k = \alpha \ \forall\ k = 1, \cdots, n$.
  $\Rightarrow$ α-egalitarian Sh.
- For $\delta \in [0, 1]$, $r_k = \delta^{n-k} \ \forall\ k = 1, \cdots, n$.
  $\Rightarrow$ δ-discounted Sh (**for** n**-person games**).
- For $\xi \in [0, 1]$, $r_1 = 1 - \xi$, $r_k = 1 - \dfrac{k \cdot \xi}{(k-1) \cdot \xi + 1} \ \forall\ k \neq 1$
  $\Rightarrow$ generalized solidarity value (Casajus and Huettner[2014b]).
- Many variants of the Shapley value satisfies the same axiom, BCEC,

Implementation

- $t = 1, 2$ are the same as the original mechanism.

$t = 1$ Each player $i \in N$ makes bids $b^i_j \in \mathbb{R}$ for every $j \neq i$. For each $i \in N$, let $B^i = \sum_{j \neq i}(b^i_j - b^j_i)$ be the net bid of player i. Let h be the player with the highest net bid. Player h pays every other player $j \in N \backslash h$, its offered bid b^h_j. Player h becomes the proposer in the next stage.

$t = 2$ Player h proposes an offer $y^h_j \in \mathbb{R}$ to every $j \in N \backslash h$.

$t = 3$ The players other than h, sequentially, either accept or reject the offer. If at least one player rejects it, then the offer is rejected. Otherwise, the offer is accepted.

Implementation

- If the offer is accepted, then each player $j \in N \backslash h$ receives y^h_j and player h obtains the remainder

$$v(N) - \sum_{j \neq h} y^h_j.$$

  If the offer is rejected then player h leaves the game and obtains $v(\{h\})$, while the players in $N \backslash h$ pay $\dfrac{1 - r_{n-1}}{n-1} v(N \backslash h)$ and proceed to the next round.

Theorem(Yokote, Funaki and Kongo[2016])

This mechanism implements $ESh^{\mathbf{r}}$ in any subgame perfect equilibrium.

References

- van den Brink, R., Y. Funaki and Y. Ju [2013]: "Reconciling marginalism with egalitarianism: consistency, monotonicity, and implementation of egalitarian Shapley values," Social Choice and Welfare, 40(3), 693-714.
- van den Brink, R. and Y. Funaki [2015]: "Implementation and Axiomatization of Discounted Shapley Values," Social Choice and Welfare, 45(2), 329-344.
- Casajus, A., and F. Huettner [2014a]: "Weakly monotonic solutions for cooperative games," Journal of Economic Theory, 154, 162-172.
- Casajus, A., and F. Huettner [2014b]: "On a class of solidarity values," European Journal of Operational Research, 236(2), 583-591.

References

- Driessen, T.S.H. and Y. Funaki [1991]: "Coincidence of and collinearity between game theoretic solutions," Operations Research Spektrum, 13(1), 15-30.
- Hart, S., and A. Mas-Colell [1996]: "Bargaining and value," Econometrica: Journal of the Econometric Society, 357-380.
- Joosten, R. [1996]: "Dynamics, equilibria and values," dissertation, Maastricht University.
- Ju, Y., P. Borm and P. Ruys [2007]: "The consensus value: a new solution concept for cooperative games," Social Choice and Welfare, 28(4), 685-703.
- Myerson, R. B. [1980]: "Conference structures and fair allocation rules," International Journal of Game Theory, 9(3), 169-182.

References

- Pérez-Castrillo, D., and D. Wettstein [2001]: "Bidding for the surplus: a non-cooperative approach to the Shapley value," Journal of Economic Theory, 100(2), 274-294.
- Shapley L.S. [1953]: "A value for n-person games," In: Roth AE (ed) The Shapley value. Cambridge University Press, Cambridge, 41-48
- Sobolev, A.I., [1973], "The functional equations that give the payoffs of the players in an n-person game (in Russian)". In: Advances in Game Theory (Ed. E. Vilkas). Izdat. "Mintis", Vilnius, 151-153.

References

- Young, H. P. [1985]: "Monotonic solutions of cooperative games," International Journal of Game Theory, 14(2), 65-72.
- Yokote, K. and Y. Funaki [2015]: *Weak Surplus Monotonicity characterizes convex combinations of egalitarian Shapley value and Consensus value,* WINPEC Working Paper Series No.E1504, Waseda University, Japan..
- Yokote, K., Y. Funaki and T.Kongo[2017]: "The Balanced Contribution Property for Equal Contributors," to appear in it Games and economic behavior.