Multiproduct-Firm Oligopoly: An Aggregative Games Approach*

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Abstract

This paper introduces a new class of demand systems to study oligopolistic pricing games with multiproduct firms. We focus on the class of demand systems that can be derived from discrete/continuous choice with iid type 1 extreme-value taste shocks. We also show that these demand systems are integrable with quasi-linear preferences. The pricing game is aggregative and payoff functions are uni-modal, although not necessarily quasi-concave. Firms' fitting-in and best-response functions can be entirely summarized by a uni-dimensional sufficient statistic, called the iota-markup. This allows us to show that, under fairly weak conditions, the pricing game has a Nash equilibrium. Under stronger conditions, this equilibrium is unique. We also provide an algorithm which exploits the aggregative nature of the game to compute the pricing equilibrium with multiproduct firms and CES demands. The algorithm always converges. As an application, we derive a number of results on the dynamic optimality of myopic merger policy under differentiated Bertrand competition.

1 Introduction

We introduce a new class of demand systems that nests the standard multinomial logit and CES demand systems. Using this demand system, we analyze an oligopoly pricing model with multiproduct firms. Exploiting the aggregative games structure of the model, we prove

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existence of equilibrium under fairly general conditions and provide sufficient conditions for uniqueness of equilibrium. We apply the model to static and dynamic merger analysis.

Analyzing the behavior of multiproduct firms in oligopolistic markets appears to be of a first-order importance. Multiproduct firms are endemic and play an important role in the economy. Even when defining products quite broadly at the NAICS 5-digit level, multiproduct firms account for 91% of total output and 41% of the total number of firms (Bernard, Redding, and Schott (2010)). Similarly, many markets are characterized by oligopolistic competition: Even at the 5-digit industry level, concentration ratios are fairly high: for instance, in U.S. manufacturing, the average NAICS 5-digit industry has a four-firm concentration ratio of 35% (Source: Census of U.S. Manufacturing, 2002).

While there has been a lot of interest in multiproduct firms in the industrial organization and international trade literatures, researchers have generally shied away from dealing with the theoretical difficulties arising in oligopolistic models with multiproduct firms. The first source of difficulties is the high dimensionality of firms' strategy sets. The second source is that even with "well behaved" demand systems such as the multinomial logit demand system, firms' payoff functions are typically not quasi-concave when firms offer multiple products (Spady (1984), Hanson and Martin (1996)). The third is that action sets are not bounded, and that it is often difficult to find natural upper bounds on prices.[1] The fourth is that payoff functions typically fail to be supermodular or log-supermodular.

In light of these technical difficulties, it is perhaps not surprising that the burgeoning literature on multiproduct firms in international trade has focused almost exclusively on models of monopolistic competition (Bernard, Redding, and Schott (2011), Dhingra (2013), Mayer, Melitz, and Ottaviano (2014), Nocke and Yeaple (2014)).[2] In industrial organization, multiproducts firms are at the heart of the literature on bundling but the existing models are highly stylized.[3] Multiproduct firms feature very prominently in the empirical industrial organization literature on demand estimation where marginal costs are backed out under the *assumption* that the pricing equilibrium exists and that first-order conditions are sufficient (Berry (1994), Berry, Levinsohn, and Pakes (1995), Nevo (2001)).[4]

In the first part of this paper, we introduce a new class of quasi-linear demand systems

[1]For example, in the case of multinomial logit demand without an outside option, even a single-product firm's best-response price goes to infinity when the prices of rivals' offerings become large.

[2]An exception is Eckel and Neary (2010) who study (identical) multiproduct firms in a Cournot model with linear demand.

[3]Much of the bundling literature focuses on monopoly and/or two goods only (Schmalensee (1984), McAfee, McMillan, and Whinston (1989), Armstrong and Vickers (2010), Chen and Riordan (2013)).

[4]A theoretical foundation of these assumptions is missing so far, and the assumptions are likely to be violated in applications.

which nests standard CES and multinomial logit demand systems with heterogeneous quality and price-sensitivity parameters. We provide necessary and sufficient conditions for the demand system to be integrable, i.e., to be derivable from a representative consumer choice problem. We show that, under the same conditions, the demand system is also derivable from a discrete-continuous choice problem with random utility.[5] As the demand system satisfies the Independence of Irrelevant Alternatives (IIA) axiom, the representative consumer's indirect (sub-)utility function can be written as as a function of a single-dimensional aggregator, which is given by the sum (over all products) of transforms of product prices. This property implies that the pricing game between multiproduct firms is aggregative; that is, each firm's profit can be written as a function of its own prices and the single-dimensional aggregator.

In the second part of the paper, we study a pricing game between multiproduct firms with arbitrary product portfolios and product-level heterogeneity in marginal costs and qualities (price-sensitivity parameters). The dimensionality of the problem is reduced, first, because the pricing game is aggregative, and, second, by showing that a firm's multidimensional pricing strategy can be fully summarized by a unidimensional sufficient statistic. This allows us to establish equilibrium existence under mild conditions, and equilibrium uniqueness under more stringent conditions. In case equilibrium is not unique, equilibria can be Pareto-ranked for the players (firms), with firms' ranking being the inverse of consumers' ranking of equilibria. The reduction in the dimensionality of relevant strategy sets not only helps proving existence and uniqueness but also computing equilibria efficiently, as we show. In the special cases of CES and multinomial logit demands, an additional aggregation property obtains: A firm's product portfolio with associated qualities and marginal costs can be fully summarized by a unidimensional sufficient statistic.

In the third part of the paper, we apply the pricing game to static and dynamic merger analysis. For the special cases of CES and multinomial logit demands, we extend Nocke and Whinston (2010)'s result on the dynamic optimality of myopic merger approval policy to mergers between arbitrary multiproduct firms.

2 Discrete/Continuous Consumer Choice

We consider a demand model in which consumers make discrete/continuous choices: a consumer first decides which variety to patronize, and then, how much of this variety to consume. We formalize discrete/continuous choice as follows:

[5] See Novshek and Sonnenschein (1979), Hanemann (1984), Dubin and McFadden (1984), Smith (2004) and Chan (2006) for references on discrete-continuous choice.

Definition 1. *A discrete/continuous choice model of consumer demand is a collection* $(h_j)_{j\in\mathcal{N}}$*, where* $\mathcal{N}$ *is a finite and non-empty set, and, for every* $j \in \mathcal{N}$*,* h_j *is a* $\mathcal{C}^3$ *function from* $\mathbb{R}_{++}$ *to* $\mathbb{R}_{++}$ *such that* $h_j' < 0$ *and* $\log(h_j)'' \geq 0$*.*

For every j, $v_j \equiv \log(h_j)$ is an indirect subutility function in a hypothetical quasi-linear economy in which only variety j and an outside good are available. Conditions $v_j' \leq 0$ and $v_j'' \geq 0$ are necessary and sufficient for v_j to be an indirect subutility function (see Nocke and Schutz, 2015). By Roy's identity, the demand for product j in this hypothetical economy is given by $q_j \equiv -v_j'$. Therefore, assumption $h_j' < 0$ means that the demand for variety j never vanishes. Let $\mathcal{H}$ be the set of $\mathcal{C}^3$, strictly decreasing, and log-convex functions from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$.

Let y be the consumer's income. We normalize the price of the outside good to 1. The consumer makes discrete-continuous choices as follows. He first observes all varieties' prices $(p_j)_{j\in\mathcal{N}}$, and a vector of taste shocks $(\varepsilon_j)_{j\in\mathcal{N}}$. If he chooses variety $k \in \mathcal{N}$, then he consumes $q_k(p_k)$ units of product k, uses the rest of his income to consume the outside good, and receives utility $y + v_k(p_k) + \varepsilon_k$. Therefore, the consumer chooses variety k only if

$$\forall j \in \mathcal{N}, \quad y + v_k(p_k) + \varepsilon_k \geq y + v_j(p_j) + \varepsilon_j.$$

We assume that the components of vector $(\varepsilon_j)_{j\in\mathcal{N}}$ are identically and independently drawn from a type-1 extreme value distribution. Therefore, by Holman and Marley's theorem, variety k is chosen with probability

$$\mathbb{P}_k(p) = \Pr\left(v_k(p_k) + \varepsilon_k = \max_{j\in\mathcal{N}} (v_j(p_j) + \varepsilon_j)\right) = \frac{\mathrm{e}^{v_k(p_k)}}{\sum_{j\in\mathcal{N}} \mathrm{e}^{v_j(p_j)}} = \frac{h_k(p_k)}{\sum_{j\in\mathcal{N}} h_j(p_j)}.$$

It follows that the expected demand for product k is given by

$$\mathbb{P}_k(p)q_k(p_k) = \frac{\mathrm{e}^{v_k(p_k)}}{\sum_{j\in\mathcal{N}} \mathrm{e}^{v_j(p_j)}} \left(-v_k'(p_k)\right) = \frac{-h_k'(p_k)}{\sum_{j\in\mathcal{N}} h_j(p_j)}.$$

This motivates the following definition:

Definition 2. *The demand system generated by discrete/continuous choice model* $(h_j)_{j\in\mathcal{N}}$ *is:*

$$D_k\left((p_j)_{j\in\mathcal{N}}\right) = \frac{-h_k'(p_k)}{\sum_{j\in\mathcal{N}} h_j(p_j)}, \quad \forall k \in \mathcal{N}, \quad \forall (p_j)_{j\in\mathcal{N}} \in \mathbb{R}_{++}^{\mathcal{N}}. \tag{1}$$

Our class of demand systems nests standard multinomial logit (if $h_j(p_j) = \mathrm{e}^{\frac{a_j - p_j}{\lambda}}$ for all $j \in \mathcal{N}$, where $a_j \in \mathbb{R}$ and $\lambda > 0$ are parameters) and CES demands (if $h_j(p_j) = a_j p_j^{1-\sigma}$, where

$a_j > 0$ and $\sigma > 1$ are parameters) as special cases. The fact that CES demands can be derived from discrete/continuous choice was already pointed out by Anderson, De Palma, and Thisse (1987) in a slightly different framework without an outside good. As seen in equation (1), the class of demand systems that can be derived from discrete/continuous choice is much wider than CES and multinomial logit demands. Notice also that, by virtue of the iid type-1 extreme value distribution assumption, all these demand systems have the Independence of Irrelevant Alternatives (IIA) property. The IIA property will allow us to greatly simplify the multiproduct-firm pricing problem in Section 4.

The consumer's expected utility can be computed using standard formulas (see, e.g., Anderson, de Palma, and Thisse (1992), Section 2.10.4):

$$\mathbb{E}\left(y + \max_{j \in \mathcal{N}} v_j(p_j)\right) = y + \log\left(\sum_{j \in \mathcal{N}} \mathrm{e}^{v_j(p_j)}\right) = y + \log\left(\sum_{j \in \mathcal{N}} h_j(p_j)\right). \tag{2}$$

Therefore, consumer surplus is aggregative, in that it only depends on the value of aggregator $H \equiv \sum_{j \in \mathcal{N}} h_j(p_j)$.

Integrability. While much of the empirical industrial organization scholars have adopted discrete choice models as a way of deriving consumer demands, other strands of literature, such as the international trade literature, mainly use a representative consumer approach. In the following, we investigate whether demand system (1) can be obtained from the maximization of the utility function of a representative consumer with quasi-linear preferences. We recall the following definition from Nocke and Schutz (2015):

Definition 3. *Let $n \geq 1$ and $D : \mathbb{R}^n_{++} \longrightarrow \mathbb{R}^n_+$. We say that D is quasi-linearly integrable if there exists a function $u : \mathbb{R}^n_+ \to \mathbb{R} \cup \{-\infty\}$ such that for every $(p, y) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$ such that $p \cdot D(p) \leq y$, vector $(y - p \cdot D(p), D(p))$ is the unique solution of*

$$\max_{(q_0, q)} \{q_0 + u(q)\} \ \text{ s.t. } \ q_0 + p \cdot q \leq y, \ \ q_0 \geq 0 \ \text{ and } \ q \geq 0.$$

When this is the case, we say that u (resp. $v : p \in \mathbb{R}^n_{++} \mapsto u(D(p))$) is a direct (resp. indirect) subutility function for demand system D.

We prove the following proposition:[6]

[6]In Appendix A.1, we prove a more general result. We derive necessary and sufficient conditions for

Proposition 1. *Let D be the demand system generated by discrete/continuous choice model $(h_j)_{j \in \mathcal{N}}$. D is quasi-linearly integrable. Moreover, v is an indirect subutility function for D if and only if there exists a constant $\alpha \in \mathbb{R}$ such that*

$$v\left((p_j)_{j \in \mathcal{N}}\right) = \alpha + \log\left(\sum_{j \in \mathcal{N}} h_j(p_j)\right).$$

Proof. See Appendix A.1. □

Therefore, any demand system that can be derived from discrete/continuous choice can also be derived from quasi-linear utility maximization. The second part of the proposition says that the expected utility of a consumer making discrete/continuous choice and the indirect utility of the associated representative consumer coincide (up to an additive constant, which we can safely ignore). Therefore, the results we will derive on consumer welfare do not depend on the way the demand system has been generated. Whether we use discrete/continuous or a representative consumer approach, all that matters is the value of aggregator H.

Remark: Consumer heterogeneity. One limitation of our approach is that the ε taste shocks are the only source of consumer heterogeneity. This concern could be addressed as follows: Let $t \in \mathbb{R}^{\mathcal{N}}$, the consumer's type, be a random vector drawn from some probability distribution P. Assume that h_j depends not only on p_j, but also on the j-th component of the consumer's type: $h_j(p_j, t_j)$. If the consumer observes his type before choosing which variety to patronize, then the expected demand for variety k is given by:

$$D_k\left((p_j)_{j \in \mathcal{N}}\right) = \int_{\mathbb{R}^{\mathcal{N}}} \frac{-\frac{\partial h_k}{\partial p_k}(p_k, t_k)}{\sum_{j \in \mathcal{N}} h_j(p_j, t_j)} dP(t),$$

i.e., the associated demand system is a mixture of the demand systems introduced in Definition 2. Notice that random coefficients logit demands are a special case. Unfortunately, our aggregative games approach does not allow us to handle such demand systems.

The case where the consumer observes his type after having chosen which product to buy is easier to accommodate. The consumer's expected utility from choosing variety j is given

demand system

$$D_k\left((p_j)_{j \in \mathcal{N}}\right) = \frac{g_k(p_k)}{\sum_{j \in \mathcal{N}} h_j(p_j)}, \quad \forall k \in \mathcal{N}, \quad \forall (p_j)_{j \in \mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$$

to be quasi-linearly integrable.

by:

$$\int_{\mathbb{R}^{\mathcal{N}}} \log\left(h_j\left(p_j, t_j\right)\right) dP(t) \equiv \log\left(h_j(p_j)\right).$$

Under some technical conditions (which allow us to differentiate under the integral sign), the consumer's expected demand for product j conditional on having picked product j is:

$$\int_{\mathbb{R}^{\mathcal{N}}} -\frac{\partial}{\partial p_j}\left(\log\left(h_j(p_j, t_j)\right)\right) dP(t) = -\frac{\partial}{\partial p_j}\int_{\mathbb{R}^{\mathcal{N}}} \log\left(h_j(p_j, t_j)\right) dP(t) = -\frac{d}{dp_j}\left(\log\left(h_j(p_j)\right)\right).$$

Differentiating once more under the integral sign, we also see that $h_j(\cdot)$ is log-convex if $h_j(\cdot, t_j)$ is log-convex for every t_j. Therefore, $h_j(\cdot)$ is the logarithm of an indirect subutility function. Moreover, the expected demand for product k is given by:

$$D_k\left(\left(p_j\right)_{j\in\mathcal{N}}\right) = \frac{-h_k'(p_k)}{\sum_{j\in\mathcal{N}} h_j(p_j)}, \quad \forall k \in \mathcal{N}, \quad \forall \left(p_j\right)_{j\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++},$$

which is the same expression as in Equation (1). Another way of phrasing this discussion is that our aggregative games tools allow us to handle consumer heterogeneity in *ex post* demand, but not in choice probabilities.

3 Multiproduct Monopoly Pricing with an Outside Option

Fix a discrete/continuous choice model $\left(h_j\right)_{j\in\mathcal{N}}$. In this section, we partition set of products $\mathcal{N}$ into two non-empty subsets: $\mathcal{N}^m$ and $\mathcal{N}^0$. A monopolist is the sole owner of all the products in set $\mathcal{N}^m$. The constant unit cost of product $k \in \mathcal{N}^m$ is denoted $c_k > 0$. Products in set $\mathcal{N}^0$ are exogenously priced at $\left(p_j^0\right)_{j\in\mathcal{N}^0}$. The monopolist therefore chooses its products' prices $\left(p_j\right)_{j\in\mathcal{N}^m}$ so as to maximize

$$\sum_{k\in\mathcal{N}^m} (p_k - c_k)\frac{-h_k'(p_k)}{\sum_{j\in\mathcal{N}^m} h_j(p_j) + \sum_{j\in\mathcal{N}^0} h_j\left(p_j^0\right)} = \sum_{k\in\mathcal{N}^m} (p_k - c_k)\frac{-h_k'(p_k)}{\sum_{j\in\mathcal{N}^m} h_j(p_j) + H^0},$$

where $H^0 = \sum_{j\in\mathcal{N}^0} h_j\left(p_j^0\right)$. H^0 is called the (value of) the outside option.

The goal of this section is to derive conditions under which the above profit function and maximization problem are well-behaved. We first study whether first-order conditions are sufficient for local and/or global optimality. This is an important question, because the aggregative games approach we will use in the next section relies on first-order conditions.

Next, we investigate whether the profit maximization problem has a solution, and whether the solution is unique.

3.1 Sufficiency of First-Order Conditions

3.1.1 Definitions and Statement of the Theorem

We first define a multiproduct firm as a collection of products, along with a constant unit cost for each product:

Definition 4. *A multiproduct firm is a pair* $\left((h_j)_{j\in\mathcal{N}}, (c_j)_{j\in\mathcal{N}}\right)$, *where* $\mathcal{N} = \{1,\dots,n\}$ *is a finite and non-empty set, and for every* $j \in \mathcal{N}$, $h_j \in \mathcal{H}$, *and* $c_j > 0$. *The profit function associated with multi-product firm* M *is:*

$$\Pi(M)\left(p, H^0\right) = \sum_{k\in\mathcal{N}} (p_k - c_k)\frac{-h_k'(p_k)}{\sum_{j\in\mathcal{N}} h_j(p_j) + H^0}, \quad \forall p \in \mathbb{R}_{++}^{\mathcal{N}},\ \forall H^0 > 0.$$

In the following, it will be useful to study multiproduct firms that can be constructed from a set of products (i.e., a set of indirect subutility functions) smaller than $\mathcal{H}$:

Definition 5. *The set of multiproduct firms that can be constructed from set* $\mathcal{H}' \subseteq \mathcal{H}$ *is:*

$$\mathcal{M}(\mathcal{H}') = \bigcup_{n\in\mathbb{N}_{++}} \left(\mathcal{H}'^n \times \mathbb{R}_{++}^n\right).$$

We can now define well-behaved multiproduct firms and well-behaved sets of products:

Definition 6. *We say that multiproduct firm* $M \in \mathcal{M}(\mathcal{H})$ *is well-behaved if for every* $(p, H^0) \in \mathbb{R}_{++}^{n+1}$, $\nabla_p \Pi(M)(p, H^0) = 0$ *implies that* p *is a local maximizer of* $\Pi(M)(., H^0)$. *We say that product set* $\mathcal{H}' \subseteq \mathcal{H}$ *is well-behaved if every* $M \in \mathcal{M}(\mathcal{H}')$ *is well-behaved.*

Put differently, a set of products is well-behaved if for every multiproduct firm that can be constructed from this set, for every value the outside option H^0 can take, first-order conditions are sufficient for local optimality. In the following, we look for the "largest" well-behaved set of products, where the meaning of "large" will be made more precise shortly.

We define the following differential operators: for every $h \in \mathcal{H}$, $\gamma(h) \equiv \frac{(h')^2}{h''}$, $\rho(h) = \frac{h}{\gamma(h)}$, $\iota(h)(x) \equiv x\frac{h''(x)}{-h'(x)}$, for all $x > 0$, and $\nu(h)(p,c) \equiv \frac{p-c}{p}\iota(h)(p)$ for every $p > 0$ and $c > 0$. The condition that h is strictly decreasing and log-convex is equivalent to $h' < 0$, $h'' > 0$ and $\rho(h) \geq 1$.

$\iota(h)$ has a straightforward economic interpretation: Consider a hypothetical single-product firm selling product h. Suppose that this firm behaves in a monopolistically competitive way, in the sense that it does not internalize the impact of its price on aggregator H. This firm therefore faces demand $-h'(p)/H$ and, since it takes H as given, it believes that the price elasticity of demand for its product is equal to the elasticity of $-h'(p)$, which is exactly $\iota(h)(p)$. It will be useful to define the following set of products:

$$\mathcal{H}^{\iota} = \left\{ h \in \mathcal{H} : \quad \forall p > 0, \quad \iota(h)(p) > 1 \Longrightarrow (\iota(h))'(p) \geq 0 \right\}.$$

Condition $\iota(h)(p) > 1 \Longrightarrow (\iota(h))'(p) \geq 0$ essentially says that the price elasticity of the monopolistic competition demand for product h is non-decreasing when the price is high enough. This condition is sometimes called Marshall's second law of demand.

We also define the set of CES and logit products as follows:

$$\begin{aligned} \mathcal{H}^{CES} &= \left\{ h \in \mathcal{H} : \quad \exists (a, \sigma) \in \mathbb{R}_{++} \times (1, \infty) \text{ s.t. } \forall p > 0,\ h(p) = a p^{1-\sigma} \right\}, \\ \mathcal{H}^{logit} &= \left\{ h \in \mathcal{H} : \quad \exists (a, \lambda) \in \mathbb{R} \times \mathbb{R}_{++} \text{ s.t. } \forall p > 0,\ h(p) = \mathrm{e}^{\frac{a-p}{\lambda}} \right\}. \end{aligned}$$

Clearly, if $h \in \mathcal{H}^{CES}$, then $\iota(h) = \sigma$ for some constant $\sigma > 1$, and if $h \in \mathcal{H}^{logit}$, then $\iota(h)(p) = p/\lambda$ for every $p > 0$, and for some constant $\lambda > 0$. Therefore, $\mathcal{H}^{CES} \subseteq \mathcal{H}^{\iota}$ and $\mathcal{H}^{logit} \subseteq \mathcal{H}^{\iota}$.

We are now in a position to state our theorem:

Theorem 1. *$\mathcal{H}^{\iota}$ is the largest (in the sense of set inclusion) set $\mathcal{H}' \subseteq \mathcal{H}$ such that $\mathcal{H}^{CES} \subseteq \mathcal{H}'$ and $\mathcal{H}'$ is well-behaved.*

In words, $\mathcal{H}^{\iota}$ is the largest set of products that contains CES products and that is well-behaved.

3.1.2 Proof of Theorem 1

We first introduce new notation. For every $k \in \mathcal{N}$, define $\gamma_k \equiv \gamma(h_k)$, $\rho_k \equiv \rho(h_k)$ and $\iota_k \equiv \iota(h_k)$. Define also $\nu_k(p_k, c_k) \equiv \frac{p_k - c_k}{p_k} \iota_k(p_k)$. Note that

$$\frac{\partial \nu_k}{\partial p_k} = \frac{c_k}{p_k^2} \iota_k(p_k) + \frac{p_k - c_k}{p_k} \iota_k'(p_k). \tag{3}$$

In addition, since $\iota_k(p_k) = p_k \frac{-h_k'(p_k)}{\gamma_k(p_k)}$, we also have that

$$\frac{\partial \nu_k}{\partial p_k} = \frac{(\nu_k(p_k, c_k) - 1)\, h_k'(p_k) - \nu_k(p_k, c_k)\gamma_k'(p_k)}{\gamma_k(p_k)}. \tag{4}$$

Differentiating the monopolist's profit with respect to p_k, we obtain:

$$\begin{aligned} \frac{\partial \Pi\,(M)}{\partial p_k} &= \frac{-h_k'(p_k)}{H}\left(1 - \frac{p_k - c_k}{p_k} p_k \frac{-h_k''(p_k)}{-h_k'(p_k)} + \sum_{j\in\mathcal{N}} (p_j - c_j)\frac{-h_j'(p_j)}{H}\right), \\ &= \frac{-h_k'(p_k)}{H}\left(1 - \nu_k(p_k, c_k) + \sum_{j\in\mathcal{N}} \nu_j(p_j, c_j)\frac{\gamma_j(p_j)}{H}\right), \end{aligned} \tag{5}$$

where $H = \sum_{j\in\mathcal{N}} h_j(p_j) + H^0$. Therefore, if the first-order conditions hold at price vector p, then, for every k in $\mathcal{N}$,

$$\nu_k(p_k, c_k) = 1 + \sum_{j\in\mathcal{N}} \nu_j(p_j, c_j)\frac{\gamma_j(p_j)}{H}. \tag{6}$$

Since the right-hand side of the above equation does not depend on the identity of product k, it follows that

$$\nu(p_i, c_i) = \nu(p_j, c_j), \quad \forall i, j \in \mathcal{N}.$$

We say that price vector p satisfies the *common ι-markup property.* This is an important property, which we will extend and discuss in greater detail in Section 3.2. For now, we note that the constant ι-markup property allows us to rewrite the first-order condition for product k as follows:

$$\nu_k(p_k, c_k)\left(1 - \sum_{j\in\mathcal{N}} \frac{\gamma_j(p_j)}{H}\right) = 1. \tag{7}$$

Since we are interested in the sufficiency of first-order conditions for local optimality, we need to calculate the Hessian of the monopolist's profit function. This is done in the following lemma:

Lemma 1. *Let $M \in \mathcal{M}\,(\mathcal{H})$, $p >> 0$ and $H^0 > 0$. If $\nabla_p \Pi(M)\,(p, H^0) = 0$, then the Hessian of $\Pi(M)\,(., H^0)$, evaluated at price vector p, is diagonal, with typical diagonal element*

$$\frac{h_k'(p_k)}{H^0 + \sum_{j\in\mathcal{N}} h_j(p_j)} \frac{\partial \nu_k}{\partial p_k}(p_k, c_k).$$

Proof. See Appendix B.1. □

The following lemma is an immediate consequence of Lemma 1 and equation (3):

Lemma 2. *Set $\mathcal{H}^{\iota}$ is well-behaved.*

Proof. See Appendix B.2. $\square$

The next step is to rule out products that are not in $\mathcal{H}^{\iota}$. This is done in the following lemma:

Lemma 3. *Let $h \in \mathcal{H} \backslash \mathcal{H}^{\iota}$. Then, $\mathcal{H}^{CES} \cup \{h\}$ is not well-behaved.*

Proof. See Appendix B.3. $\square$

The idea behind the proof is as follows. Since $h_1 \equiv h \notin \mathcal{H}^{\iota}$, there exists a price $\hat{p} > 0$ such that $\iota(\hat{p}) > 1$ and $\iota'(\hat{p}) < 0$. By equation (3), $\partial \nu_1 (\hat{p}, c_1) < 0$ for $c_1 > 0$ low enough. We pair product h_1 with a well-chosen CES product $h_2 \in \mathcal{H}^{CES}$, and find a $p_2 > 0$, a c_2 and an $H^0 > 0$ that ensure that the first-order condition for product 1 (given price $\hat{p}$ and marginal cost c_1) holds, while the first-order condition for the other product holds as well. Since $\partial \nu_1 (\hat{p}, c_1) < 0$, we can then use Lemma 1 to show that the multiproduct firm we have constructed is not well-behaved.

Combining Lemma 2 and 3 proves Theorem 1. In the following, we focus on multiproduct firms that are constructed from set $\mathcal{H}^{\iota}$, since any bigger set would necessarily imply that, for some multiproduct firms, first-order conditions would not be sufficient for optimality, thereby invalidating our aggregative games approach.

We close this section by noting that multiproduct-firms are special, in the sense that, compared to single-product firms, they require strictly stronger restrictions on the set of admissible products to be well-behaved. This statement is formalized in the following proposition:

Proposition 2. *Let $h \in \mathcal{H}$, $c > 0$ and $M = (h, c)$. The following assertions are equivalent:*

(i) Firm M is well-behaved.

(ii) For every $p > 0$ such that $\iota(h)(p) > 1$, $(\iota(h))'(p) \geq 0$ or $(\rho(h))'(p) \geq 0$.

Proof. See Appendix B.4. $\square$

3.2 Existence and Uniqueness of a Solution, and Sufficiency of First-Order Conditions for Global Optimality

We start by proving the following technical lemma:

Lemma 4. *Let $h \in \mathcal{H}^\iota$. Then:*

(i) There exists a unique scalar $\underline{p}(h) \geq 0$ such that for every $p > 0$, $\iota(h)(p) > 1$ if and only if $p > \underline{p}(h)$. Moreover, $(\iota(h))'(p) \geq 0$ for all $p > \underline{p}(h)$.

(ii) $\bar{\mu}(h) \equiv \lim_{p\to\infty} \iota(h)(p) > 1$.

(iii) For every $p > \underline{p}(h)$, $(\gamma(h))'(p) < 0$.

(iv) $\lim_{p\to\infty} \gamma(h)(p) = 0$.

(v) $\lim_{p\to\infty} ph'(p) = 0$.

(vi) If $\lim_\infty h = 0$ and $\bar{\mu}(h) < \infty$, then $\lim_{p\to\infty} \rho(h)(p) = \frac{\bar{\mu}(h)}{\bar{\mu}(h)-1}$.

Proof. See Appendix B.5. □

Fix a multiproduct firm $M = \left((h_j)_{j\in\mathcal{N}}, (c_j)_{j\in\mathcal{N}}\right) \in (\mathcal{H}^\iota)^{\mathcal{N}} \times \mathbb{R}^{\mathcal{N}}_{++}$. It is useful to allow firms to set infinite prices, to ensure that the profit maximization problem has a solution:

Definition 7. *The profit function associated with multi-product firm $M = \left((h_j)_{j\in\mathcal{N}}, (c_j)_{j\in\mathcal{N}}\right)$ is:*[7]

$$\Pi(M)\left(p, H^0\right) = \sum_{\substack{k\in\mathcal{N} \\ p_k<\infty}} (p_k - c_k) \frac{-h_k'(p_k)}{\sum_{\substack{j\in\mathcal{N} \\ p_j<\infty}} h_j(p_j) + \sum_{\substack{j\in\mathcal{N} \\ p_j=\infty}} \lim_\infty h_j + H^0}, \ \forall p \in (0,\infty]^{\mathcal{N}}, \ \forall H^0 > 0. \quad (8)$$

The assumption is that, if $p_k = \infty$, then the firm simply does not supply product k, and therefore does not earn any profit on this product. In the discrete/continuous choice model, the consumer still receives a type-1 extreme value draw ε_k for product k, so he might still end up "choosing" product k, but he will not consume a positive amount of it.[8] This explains the

[7] Throughout the paper, we adopt the convention that the sum of an empty collection of real numbers is equal to zero.

[8] Suppose that $\lim_\infty -h'/h = l > 0$ (the limit exists, since h is log-convex). There exists $x_0 > 0$ such that $-h'(x)/h(x) > l/2$ for all $x \geq x_0$. Integrating this inequality, we see that

$$-\log\left(\frac{h(x)}{h(x_0)}\right) > \frac{l}{2}(x - x_0), \quad \forall x > x_0.$$

Taking exponentials on both side, and letting x go to infinity, we obtain that $\lim_\infty h = 0$. Therefore, if the consumer's conditional demand for the product given that he chooses this product is strictly positive when the price is infinite, then his probability of choosing this product is zero. Conversely, if the probability that the consumer chooses this product when the price is infinite is strictly positive ($\lim_\infty h > 0$), then the consumer's conditional demand at infinite price must be equal to zero. These two cases (as well as the case in which the conditional demand *and* the choice probability are identically zero) are consistent with the interpretation that the product is simply not available.

presence of $\sum_{\substack{j\in\mathcal{N}\\ p_j=\infty}} \lim_\infty h_j$ in the denominator of $\Pi(M)$. In the following, we write $h_j(\infty)$ instead of $\lim_\infty h_j$.

Our goal is to study the following maximization problem for every $H^0 > 0$:

$$\max_{p\in(0,\infty]^{\mathcal{N}}} \Pi(M)\left(p, H^0\right), \tag{9}$$

Lemma 5. *If $h_j \in \mathcal{H}^\iota$ for every $j \in \mathcal{N}$, then maximization problem* (9) *has a solution for every $H^0 > 0$. Moreover, if p solves maximization problem* (9)*, then $p_j \geq c_j$ for all $j \in \mathcal{N}$.*

Proof. See Appendix B.6. □

The lemma is proven by exploiting the fact that $\prod_{j\in\mathcal{N}}[c_j,\infty]$ can essentially be treated like a compact set (since there is a continuous bijection between this set and $[0,1]^{\mathcal{N}}$), and that, if $(h_j)_{j\in\mathcal{N}} \in (\mathcal{H}^\iota)^{\mathcal{N}}$, then $\Pi(M)\,(.,H^0)$ is "continuous" on $\prod_{j\in\mathcal{N}}[c_j,\infty]$ (in the sense that, for every price vector p with finite and/or infinite components, $\lim_{\tilde{p}\to p} \Pi(M)\,(\tilde{p},H^0) = \Pi(M)\,(p,H^0)$). Prices below marginal cost are easy to rule out.

The next step is to solve the firm's maximization problem using first-order conditions. The problem is that the profit function is not necessarily differentiable at infinite prices, so we will need to modify the definition of first-order conditions to account for that. Note first that, if all products in $\mathcal{N}' \subsetneq \mathcal{N}$ are priced at infinity, then profit function $\Pi\,(.,H^0)$ is still $\mathcal{C}^2$ in $(p_j)_{j\in\mathcal{N}\setminus\mathcal{N}'} \in \mathbb{R}_{++}^{\mathcal{N}\setminus\mathcal{N}'}$, as can be seen by inspecting profit function (8). Next, we slightly abuse notation, by denoting $\left(p_k, (p_j)_{j\in\mathcal{N}\setminus\{k\}}\right)$ the price vector with k-th component p_k, and with other components given by $(p_j)_{j\in\mathcal{N}\setminus\{k\}}$. We generalize first-order conditions as follows:

Definition 8. *We say that the generalized first-order conditions of maximization problem* (9) *hold a price vector $\tilde{p} \in (0,\infty]^{\mathcal{N}}$ if for every $k \in \mathcal{N}$,*

(a) $\frac{\partial \Pi(M)}{\partial p_k}\,(\tilde{p},H^0) = 0$ *whenever $\tilde{p}_k < \infty$, and*

(b) $\Pi(M)\,(\tilde{p},H^0) \geq \Pi(M)\left(\left(p_k, (\tilde{p}_j)_{j\in\mathcal{N}\setminus\{k\}}\right), H^0\right)$ *for every $p_k \in \mathbb{R}_{++}$ whenever $\tilde{p}_k = \infty$.*

It is obvious that generalized first-order conditions are necessary for optimality:

Lemma 6. *If $p \in (0,\infty]^{\mathcal{N}}$ solves maximization problem* (9)*, then the generalized first-order conditions are satisfied at price profile p.*

Next, we want to show that, if the generalized first-order conditions hold at a price vector, then this price vector satisfies a generalized version of the common ι-markup property introduced in Section 3.1.2. To define this generalized common ι-markup property, we first need to establish a few facts about functions $\nu_j(.,.)$:

Lemma 7. *For every $h \in \mathcal{H}^\iota$ and $c > 0$, function $\nu(h)(.,c)$ is a strictly increasing $\mathcal{C}^1$-diffeomorphism from (c,∞) to $(0,\bar{\mu}(h))$. Denote its inverse function by $r(h)(.,c)$. Then, for all $\mu \in (0,\bar{\mu}(h))$,*

$$\frac{\partial r(h)}{\partial \mu} = \frac{\gamma(h)\left(r(h)(\mu,c)\right)}{\mu\left(-\left(\gamma(h)\right)'\left(r(h)(\mu,c)\right)\right) - (\mu-1)\left(-h'\left(r(h)(\mu,c)\right)\right)} > 0. \tag{10}$$

In addition, $r(h)$ is strictly increasing in μ and c, and $r(h)(\mu,c) > \underline{p}(h)$ whenever $\mu \geq 1$.

Proof. See Appendix B.7. □

In the following, it will be convenient to extend functions $\nu(h)$ and $r(h)$ by continuity as follows:

$$\begin{aligned} \nu(h)(\infty,c) &= \bar{\mu}(h),\ \forall c > 0, \\ r(h)(\mu,c) &= \infty,\ \forall \mu \geq \bar{\mu}(h),\ \forall c > 0. \end{aligned}$$

We can now generalize the common ι-markup property to price vectors with infinite components (for every k, put $r_k \equiv r(h_k)$):

Definition 9. *We say that price vector $p \in (0,\infty]^{\mathcal{N}}$ satisfies the common ι-markup property if there exists a number $\mu \geq 0$, called the ι-markup, such that*

$$\forall k \in \mathcal{N}, \quad p_k = r_k(\mu, c_k).$$

For every $k \in \mathcal{N}$, extend γ_k by continuity at infinity: $\gamma_k(\infty) = 0$. The following lemma essentially rewrites condition (6) (replace ν_j by μ and p_j by r_j for every $j \in \mathcal{N}$) when infinite prices are allowed. It allows us to simplify first-order conditions considerably:

Lemma 8. *Suppose that the generalized first-order conditions for maximization problem* (9) *hold at price vector $p \in (0,\infty]^{\mathcal{N}}$. Then, p satisfies the common ι-markup property. The corresponding ι-markup, μ, solves the following equation on interval $(1,\infty)$:*

$$\mu = 1 + \mu \frac{\sum_{j\in\mathcal{N}} \gamma_j\left(r_j(\mu,c_j)\right)}{\sum_{j\in\mathcal{N}} h_j\left(r_j(\mu,c_j)\right) + H^0}. \tag{11}$$

In addition, $\Pi(M)\left(p,H^0\right) = \mu - 1$.

Proof. See Appendix B.8. □

All we need to do now is study equation (11):

Lemma 9. *Equation* (7) *has a unique solution on interval* $(1, \infty)$.

Proof. See Appendix B.9. □

Combining Lemmas 5–9 allows us to conclude our study of maximization problem (9). The profit maximization problem has a solution, and generalized first-order conditions are necessary for optimality. Moreover, there exists at most one price vector that satisfies the first-order condition. Therefore, the profit maximization problem has a solution, and first-order conditions are sufficient for global optimality.

Theorem 2. *Maximization problem* (9) *has a unique solution. The generalized first-order conditions associated with this maximization problem are necessary and sufficient for global optimality. The optimal price vector satisfies the common* ι*-markup property, and the corresponding* μ *is the unique solution of equation* (11)*. The maximized value of the objective function is* $\mu - 1$.

Proof. See Appendix B.10. □

3.3 Comments and Comparative Statics

Although the firm's profit function is not necessarily strictly quasi-concave, we find that the profit maximization problem has a unique solution, and that generalized first-order conditions are necessary and sufficient for optimality. The optimal price vector satisfies the common ι-markup property: there exists a firm-level scalar $\mu > 1$ such that for every product $k \in \mathcal{N}$ that is sold at the monopolist's optimum, $\nu_k(p_k, c_k) = \mu$. To gain some intuition, it is useful to rewrite this condition as follows:

$$\forall k \in \mathcal{N} \text{ s.t. } p_k < \infty, \quad \frac{p_k - c_k}{p_k} = \frac{\mu}{\iota_k(p_k)}.$$

This formula resembles the inverse elasticity rule, in that the Lerner index on product k is equal to the inverse of the monopolistic competition price elasticity of demand for product k, multiplied by a number, μ, which summarizes the impact of an increase in p_k on the aggregator $H = H^0 + \sum_{j \in \mathcal{N}} h_j(p_j)$, as well as cannibalization effects. What is remarkable is that this μ is independent of the identity the product under consideration.

The profit maximization problem then boils down to finding the optimal value of μ. From a computational point of view, this result is very useful, as it allows the modeler to find the firm's optimal prices by solving a single equation in one unknown (equation (11)), instead of resorting to complex, and potentially time-consuming, multivariate optimization methods.

We will also use this result to simplify multiproduct oligopoly pricing games in the next section.

Let $H^*\left((c_j)_{j\in\mathcal{N}}, H^0\right)$, $\mu^*\left((c_j)_{j\in\mathcal{N}}, H^0\right)$, $p_k^*\left((c_j)_{j\in\mathcal{N}}, H^0\right)$ ($k \in \mathcal{N}$) be the equilibrium values of H, μ and p_k, respectively. Define also

$$\mathcal{N}^*\left((c_j)_{j\in\mathcal{N}}, H^0\right) = \left\{k \in \mathcal{N} : \quad p_k^*\left((c_j)_{j\in\mathcal{N}}, H^0\right) < \infty\right\}$$

as the set of products that are sold in equilibrium.

We obtain the following comparative statics:

Proposition 3. *As H^0 increases, μ^* and p_k^* ($k \in \mathcal{N}$) decrease, H^* increase, and $\mathcal{N}^*$ expands. As c_j increases, μ^* and p_k^* ($k \neq j$) decrease, p_j^* increases, and $\mathcal{N}^*$ expands. The impact on H^* is ambiguous.*

Proof. See Appendix B.11. □

The first part of the proposition says that, as the outside option becomes more attractive, the monopolist's market power weakens, prices fall down and consumer surplus improves. In addition, the model predicts that a firm that operates in a more competitive environment (higher H^0) tends to sell more products (larger $\mathcal{N}^*$). To understand the intuition, consider a simple environment in which the monopolist only has two CES products, $h_1(p_1) = p_1^{1-\sigma_1}$ and $h_2(p_2) = p_2^{1-\sigma_2}$, with $1 < \sigma_1 < \sigma_2$. Applying Roy's identity, we obtain the conditional demand for product $i \in \{1, 2\}$ given that the consumer purchases this product: $q_i(p_i) = \frac{\sigma_i - 1}{p_i}$. Therefore, the monopolist makes a conditional profit of $(\sigma_i - 1)\frac{p_i - c_i}{p_i}$ on product i.

We first focus on the extreme case in which the outside option is worthless: $H^0 = 0$. In this case, consumers will necessarily buy one of the monopolist's product. Note that the conditional profit on product 1 is bounded above by $\sigma_1 - 1$, whereas the conditional profit on product 2 can be made arbitrarily close to $\sigma_2 - 1 > \sigma_1 - 1$. The monopolist therefore wants to retire product 1 to prevent consumers from choosing this product, and ensure that they will go for product 2, which yields a higher conditional profit.[9]

On the other hand, if the outside option is sufficiently attractive ($H^0 >> 0$), then the monopolist starts worrying about consumers switching to it. The monopolist therefore has

[9] This idea is a bit hard to formalize, since, when $H^0 = 0$, the monopolist's profit function,

$$(p_1, p_2) \mapsto \sum_{i=1}^{2} (\sigma_i - 1) \frac{p_i - c_i}{p_i} \frac{p_i^{1-\sigma_i}}{p_1^{1-\sigma_1} + p_2^{1-\sigma_2}},$$

does not have a limit at (∞, ∞). However, the supremum of this function, $\sigma_2 - 1$, can be approached by letting p_1 go to infinity at a faster rate than p_2.

incentives to lower its prices and to make product 1 available.

The second part of Proposition 3 says that, as the marginal cost of product j increases, the monopolist passes this cost increase on to consumers (p_j^* rises), and lowers the prices of its other products. The latter effect is due to the fact that the monopolist has less incentives to divert sales towards products j, since this product is now less profitable. The proposition also says that, everything else equal, a less productive firm tends to sell more products. This comes from the fact that an inefficient monopolist makes relatively low profit on each of the product it sells, and therefore has less incentives to withdraw a product to divert sales towards other products. Finally, since the marginal cost increase has asymmetric effects on prices, the impact on consumer surplus is ambiguous.

4 The Pricing Game

In this section, we study a price competition game between multiproduct firms with arbitrary product heterogeneity. The pricing game is defined as follows:

Definition 10. *A pricing game is a triple* $\left((h_j)_{j\in\mathcal{N}}, \mathcal{F}, (c_j)_{j\in\mathcal{N}}\right)$*, where:*

- $(h_j)_{j\in\mathcal{N}} \in (\mathcal{H}^\iota)^{\mathcal{N}}$ *is a discrete/continuous choice model of consumer demand,*
- $\mathcal{F}$*, the set of firms, is a partition of* $\mathcal{N}$ *such that* $|\mathcal{F}| \geq 2$*,*
- $(c_j)_{j\in\mathcal{N}} \in \mathbb{R}_{++}^{\mathcal{N}}$ *is a profile of marginal costs.*

Since $\bigcup_{f\in\mathcal{F}} f = \mathcal{N}$, this definition does not seem to allow for an exogenously priced outside option. However, such an outside option is easy to incorporate. Let $(h_j)_{j\in\mathcal{N}}$ be a discrete/continuous choice model of consumer demand. Partition $\mathcal{N}$ into two sets: $\widetilde{\mathcal{N}}$, the set of products sold by oligopoly players, and $\mathcal{N}^0$, the set of products sold at exogenous prices $(p_j)_{j\in\mathcal{N}^0}$. Let $H^0 = \sum_{j\in\mathcal{N}^0} h_j(p_j) \geq 0$ be the value of the outside option. We can now define another discrete/continuous choice model, $\left(\widetilde{h}_j\right)_{j\in\widetilde{\mathcal{N}}}$ as follows: for every $j \in \widetilde{\mathcal{N}}$, $\widetilde{h}_j = h_j + \frac{H^0}{|\widetilde{\mathcal{N}}|}$. Note that this transformation affects neither consumer surplus nor expected demands. Therefore, it is equivalent to study price competition game with discrete/continuous choice model $(h_j)_{j\in\mathcal{N}}$ and exogenous prices $(p_j)_{j\in\mathcal{N}^0}$, and price competition game with discrete/continuous choice model $\left(\widetilde{h}_j\right)_{j\in\widetilde{\mathcal{N}}}$ and no outside option.

We now define our equilibrium concept (note that we continue to allow infinite prices):

Definition 11. *The normal-form game associated with pricing game* $\left((h_j)_{j\in\mathcal{N}}, \mathcal{F}, (c_j)_{j\in\mathcal{N}}\right)$ *is triple* $\left(\mathcal{F}, \left((0,\infty]^f\right)_{f\in\mathcal{F}}, \left(\Pi^f\right)_{f\in\mathcal{F}}\right)$*, where for every* $f \in \mathcal{F}$*,* $p^f \in (0,\infty]^f$ *and* $p^{-f} \in (0,\infty]^{\mathcal{N}\setminus f}$*,*

$$\Pi^f(p^f, p^{-f}) = \sum_{\substack{k\in f \\ p_k<\infty}} (p_k^f - c_k) \frac{-h_k'(p_k^f)}{\sum_{j\in f} h_j(p_j^f) + \sum_{j\in\mathcal{N}\setminus f} h_j(p_j^{-f})}.$$

A pricing equilibrium of pricing game $\left((h_j)_{j\in\mathcal{N}}, \mathcal{F}, (c_j)_{j\in\mathcal{N}}\right)$ *is a Nash equilibrium of the associated normal-form game.*

4.1 An Aggregative Games Approach to Equilibrium Existence

There are three main difficulties associated with the equilibrium existence problem. First, nothing guarantees that payoff functions are quasi-concave. In fact, it is well known that, with standard multinomial logit demands, profit functions can fail to be quasi-concave (Spady (1984), Hanson and Martin (1996)). Second, firms' action sets are unbounded, and therefore not compact. In addition, in many cases, best-response functions are not bounded above, and it is therefore not possible to impose an upper bound on the set of feasible prices. For instance, using the same notation as in Section 3, the optimal price of a monopolist owning a unique CES product tends to infinity as H^0 goes to infinity. Third, payoff functions are not necessarily supermodular. With single-product firms, it is possible to get around this issue by showing that profit functions are log-supermodular (see Milgrom and Roberts (1990), Vives (2000)).[10] We are not aware of a similar trick for multi-product firms.

The first two difficulties imply that standard existence theorems for compact games (such as Nash or Glicksberg's theorems) based on Kakutani's fixed-point theorem cannot be applied. The last two difficulties imply that existence theorems based on supermodularity theory and Tarski's fixed-point theorem (see Milgrom and Roberts (1990), Topkis (1998)) have no bite.

The idea behind our existence proof is to reduce the dimensionality of the problem in two ways. First, we know from Theorem 2 that a firm's optimal price can be fully summarized by a uni-dimensional sufficient statistics, the firm's ι-markup, which is pinned down by a single equation in one unknown. Second, the pricing game is aggregative (see Selten (1970)),

[10] If $f = \{k\}$ is a single-product firm, then, for every $j \in \mathcal{N}$ such that $j \neq k$,

$$\frac{\partial^2 \log\left(\pi^f\right)}{\partial p_k \partial p_j} = \frac{h_j'(p_j) h_k'(p_k)}{\sum_{i\in\mathcal{N}} h_i(p_i)} > 0.$$

in that the profit of a firm only depends on its own profile of prices and on uni-dimensional sufficient statistic $H = \sum_{j\in\mathcal{N}} h_j(p_j)$.

Price vector $p \in (0,\infty]^{\mathcal{N}}$ is a Nash equilibrium if and only if, for every $f \in \mathcal{F}$, $(p_j)_{j\in f}$ maximizes $\Pi^f\left(\cdot,(p_j)_{j\in\mathcal{N}\setminus f}\right)$. Note that, in any Nash equilibrium, each firm sets at least one finite price, since a firm setting only infinite prices makes zero profit. Let $f \in \mathcal{F}$, and fix a price vector $(p_j)_{j\in\mathcal{N}\setminus f}$ with at least one finite component. Define $H^{-f} = \sum_{j\in\mathcal{N}\setminus f} h_j(p_j)$. Then, firm f chooses $(p_j)_{j\in f}$ so as to maximize

$$\sum_{\substack{k\in f\\ p_k<\infty}} (p_k^f - c_k)\frac{-h_k'(p_k^f)}{\sum_{j\in f} h_j(p_j^f) + H^{-f}} = \Pi\left(\left((h_j)_{j\in\mathcal{N}},(c_j)_{j\in\mathcal{N}}\right)\right)\left((p_j)_{j\in\mathcal{N}}, H^{-f}\right).$$

We know from Theorem 2 that this maximization problem boils down to finding the unique solution of the following equation on interval $(1,\infty)$:

$$\mu^f = 1 + \mu^f \frac{\sum_{j\in f}\gamma_j\left(r_j(\mu^f,c_j)\right)}{\sum_{j\in f} h_j\left(r_j(\mu^f,c_j)\right) + H^{-f}}.$$

The equilibrium existence problem therefore reduces to finding a profile of ι-markups $\left(\mu^f\right)_{f\in\mathcal{F}} \in (1,\infty)^{\mathcal{F}}$ such that

$$\mu^f = 1 + \mu^f \frac{\sum_{j\in f}\gamma_j\left(r_j(\mu^f,c_j)\right)}{\sum_{g\in\mathcal{F}}\sum_{j\in g} h_j\left(r_j(\mu^g,c_j)\right)}, \quad \forall f \in \mathcal{F}.$$

This is, in turn, equivalent to finding an aggregator level $H > 0$ and a profile of ι-markups $\left(\mu^f\right)_{f\in\mathcal{F}} \in (1,\infty)^{\mathcal{F}}$ such that $H = \sum_{g\in\mathcal{F}}\sum_{j\in g} h_j\left(r_j(\mu^g,c_j)\right)$ and for all $f \in \mathcal{F}$,

$$\mu^f = 1 + \mu^f \frac{\sum_{j\in f}\gamma_j\left(r_j(\mu^f,c_j)\right)}{H}. \tag{12}$$

Our approach to equilibrium existence consists in showing that this nested fixed point problem has a solution. We start by studying the inner fixed point problem:

Lemma 10. *For every $f \in \mathcal{F}$, for every $H > 0$, equation (12) has a unique solution in μ^f on interval $(1,\infty)$. Denote this solution by $m^f(H)$.*
Function $m^f(.)$ is continuous, strictly decreasing, and satisfies $\lim_\infty m^f = 1$ and $\lim_{0^+} m^f = \bar{\mu}^f$, where $\bar{\mu}^f = \max_{j\in f}\bar{\mu}_j$.

In addition, if H is such that $m^f(H) \notin \{\bar{\mu}_j\}_{j\in f}$, then m^f is $\mathcal{C}^1$ in a neighborhood of H and

$$m^{f\prime}(H) = -\frac{1}{H}\frac{m^f(H)(m^f(H)-1)}{1+m^f(H)(m^f(H)-1)\frac{\sum_{k\in f,\ m^f(H)<\bar{\mu}_k} r_k'\left(m^f(H)\right)\left(-\gamma_k'\left(r_k\left(m^f(H)\right)\right)\right)}{\sum_{k\in f,\ m^f(H)<\bar{\mu}_k}\gamma_k\left(r_k\left(m^f(H)\right)\right)}} < 0. \quad (13)$$

Proof. See Appendix C.1. □

We can now take care of the outer fixed point problem, which, by Lemma 10 consists in finding an $H > 0$ such that $\Omega(H) = 1$, where

$$\Omega(H) \equiv \frac{\sum_{f\in\mathcal{F}}\sum_{k\in f} h_k\left(r_k\left(m^f(H)\right)\right)}{H}.$$

The following lemma guarantees that the outer fixed-point problem has a solution:

Lemma 11. *There exists $H^* > 0$ such that $\Omega(H^*) = 1$.*

Proof. See Appendix C.2. □

We can conclude:

Theorem 3. *Pricing game $\left((h_j)_{j\in\mathcal{N}}, \mathcal{F}, (c_j)_{j\in\mathcal{N}}\right)$ has an equilibrium. In any equilibrium, the common ι-markup property holds for every firm. Moreover, a firm's equilibrium profit is equal to its ι-markup minus 1.*

4.2 Comments and Comparative Statics

Comparing equilibria. If we know that H^* is an equilibrium aggregator level, then we can compute consumer surplus ($\log(H^*)$), the profit of firm $f \in \mathcal{F}$ ($m^f(H^*) - 1$) and the price of product $k \in f$ ($r_k\left(m^f(H^*), c_k\right)$). In addition, the following proposition states that, if there are multiple equilibria, then these equilibria can be Pareto-ranked among players (firms), with this ranking being the inverse of consumers' ranking of equilibria:

Proposition 4. *Suppose that there are two pricing equilibria with aggregators H_1^* and $H_2^* > H_1^*$, respectively. Then, each firm $f \in \mathcal{F}$ makes a strictly larger profit in the first equilibrium (with aggregator H_1^*), whereas consumers' indirect utility is higher in the second equilibrium (with aggregator H_2^*).*
In addition, there is an equilibrium with the largest value of H and an equilibrium with the lowest value of H.

Proof. See Appendix C.3. □

Markups. Our class of demand systems can generate rich patterns of equilibrium markups within a firm's product portfolio. To see this, let us first consider the extreme case of CES products with common σ's ($h_j(p_j) = a_j p_j^{1-\sigma}$ for all $j \in \mathcal{N}$). In this case, $\iota_j = \sigma$ for all j, and the common ι-markup property states that, in equilibrium, $\frac{p_j - c_j}{p_j} = \frac{\mu^f}{\sigma}$ for all $j \in f$. Therefore, firm f charges the same Lerner index for all the products in its product portfolio, and firm f charges higher absolute markups on products that it produces less efficiently (since $p_j - c_j = \frac{\mu^f}{\sigma - \mu^f} c_j$ increases with c_j).

These markup patterns are not robust to small changes in the demand system. Suppose for instance that all products are still CES products, but with potentially heterogeneous σ's ($h_j(p_j) = a_j p_j^{1-\sigma_j}$ for all $j \in \mathcal{N}$). Then, in equilibrium, $\frac{p_j - c_j}{p_j} = \frac{\mu^f}{\sigma_j}$, and firm f no longer has incentives to charge the same Lerner index over all its products (unless all these products share the same σ). Similarly, it does not necessarily charge higher absolute markups on high marginal cost products.

The same point could be made about the other extreme case in which all products are logit with common λ's ($h_j(p_j) = \exp\left(\frac{a_j - p_j}{\lambda}\right)$ for all $j \in \mathcal{N}$). In this case, in equilibrium, a multiproduct firm charges the same absolute markup over all its products (since $\iota_j(p_j) = p_j/\lambda$ for all j), and sets a lower Lerner index on high marginal cost products. Again, this can be overturned by allowing the λ's to differ across products.

In general, the pattern of markups within a firm's product portfolio depends on supply-side considerations ($(c_j)_{j \in f}$) and on demand-side conditions, as captured by functions $(\iota_j)_{j \in f}$.

Comparative statics. We now reintroduce an outside option H^0, and ask how an increase in H^0 affects the set of equilibria. We prove the following proposition:

Proposition 5. *Suppose that H^0 increases. Then, in both the equilibrium with the smallest and largest value of the aggregator H, this induces (i) a decrease in the profit of all firms, (ii) a decrease in the prices of all goods, (iii) an increase in consumer surplus, and (iv) an expansion of the set of products sold at a finite price.*

Proof. See Appendix C.4. □

Therefore, an increase in the value of the outside option has the same impact as in the monopoly case: prices, ι-markups and firms' profits fall down, the set of products sold expands, and consumers are better off. The result is proven by showing that an increase in H^0 implies an upward shift in function $\Omega(\cdot)$.

Similar techniques can be used to analyze the impact of entry. Suppose that firm $f^0 \in \mathcal{F}$ is initially inactive, i.e., $p_j = \infty$ for every $j \in f^0$. Solving pricing game $\left((h_j)_{j \in \mathcal{N} \setminus f^0}, \mathcal{F} \setminus \{f^0\}, (c_j)_{j \in \mathcal{N} \setminus f^0}\right)$

with outside option $H^0 = \sum_{j \in f^0} h_j(\infty)$ gives us the set of pre-entry equilibrium aggregator levels. The set of post-entry equilibrium aggregator levels can be obtained by solving pricing game $\left((h_j)_{j\in\mathcal{N}}, \mathcal{F}, (c_j)_{j\in\mathcal{N}}\right)$ with outside option 0. We prove the following proposition:

Proposition 6. *Suppose that a new firm enters. Then, in both the equilibrium with the smallest and largest value of the aggregator H, this induces (i) a decrease in the profit of all incumbent firms, (ii) a decrease in the prices of all goods, (iii) an increase in consumer surplus, and (iv) an expansion of the set of products sold at a finite price.*

Proof. The result is proven in the same way as Proposition 5. After entry, function $\Omega(\cdot)$ shifts upward. □

An increase in a product (product k)'s marginal cost has ambiguous effects, for reasons that were already discussed in the monopoly case. The firm owning this product, call it firm f, has less incentives so shift sales towards this product, and therefore lowers its ι-markup μ^f. What is unclear is whether this decrease in μ^f raises or lowers firm f's contribution to the aggregator $\sum_{j\in f} h_j(r_j(m^f(H), c_j))$. On the one hand, h_k decreases due to the direct effect of the increase in c_k. On the other hand, h_j increases for every j in f such that $j \neq k$, due to the fact that μ^f decreases. Therefore, function Ω may shift up or down, and the impact on the equilibrium aggregator level is unclear. In the end, consumers and rival firms ($g \neq f$) may end up benefiting or suffering from an increase in c_k.[11]

Examples. Consider the following family of functions: for every $\lambda > 0$, $\phi \in [0,1]$ and $x > 0$,

$$h^{\phi,\lambda}(x) = \begin{cases} \exp\left(-\lambda \frac{x^\phi - 1 + \phi^2}{\phi}\right) & \text{if } \phi > 0, \\ x^{-\lambda} & \text{if } \phi = 0. \end{cases}$$

Note that for every $x, \lambda > 0$, $\lim_{\phi\to 0} h^{\phi,\lambda}(x) = h^{0,\lambda}(x)$, i.e., $h^{\phi,\lambda}$ converges pointwise to $h^{0,\lambda}$ when ϕ goes to zero.[12] Note that $h^{\phi,\lambda}$ is CES when $\phi = 0$ (with $\sigma = \lambda + 1$) and logit

[11] We do obtain clear-cut comparative statics with logit and CES demands. See Propositions 11 and 13 in Section 5.

[12] To see this, note that

$$\begin{aligned}
\lim_{\phi\to 0} \frac{x^\phi - 1 + \phi^2}{\phi} &= \lim_{\phi\to 0} \frac{\exp(\phi \log(x)) - 1 + \phi^2}{\phi}, \\
&= \lim_{\phi\to 0} \frac{\log(x)\exp(\phi\log(x)) + 2\phi}{1}, \\
&= \log(x),
\end{aligned}$$

where the second line follows by L'Hospital's rule (note that $\exp(\phi\log(x)) - 1 \underset{\phi\to 0}{\longrightarrow} 0$).

when $\phi = 1$. Therefore, our family of functions $h^{\phi,\lambda}$ bridges the gap between CES and logit demands.

We still need to check that $h^{\phi,\lambda} \in \mathcal{H}^\iota$ for every $\phi \in [0,1]$ and $\lambda > 0$. Clearly, $h^{\phi,\lambda} > 0$. Taking minus the logarithmic derivative of $h^{\phi,\lambda}$ gives us the conditional demand for the product:

$$-\frac{h^{\phi,\lambda\prime}(x)}{h^{\phi,\lambda}(x)} = \lambda x^{\phi-1}, \quad \forall x > 0.$$

This function is indeed positive and decreasing. When $\phi = 0$, conditional demand is λ/x, which is indeed the conditional demand for a CES product. When $\phi = 1$, conditional demand is constant, as in the logit case. Cases where $\phi \in (0,1)$ allow us to capture situations where conditional demand decreases faster than in the logit case, but slower than in the CES case.

Finally, we compute $\iota^{\phi,\lambda}$:

$$\iota^{\phi,\lambda}(x) = -\frac{d\log(-h'(x))}{d\log x} = 1 - \phi + \lambda x^{\phi}, \quad \forall x > 0,$$

which is indeed non-decreasing for all $\phi \in [0,1]$ and $\lambda > 0$. Therefore, $h^{\phi,\lambda} \in \mathcal{H}^\iota$, and all the comparative statics and existence results proven so far apply.

An almost complete characterization of set $\mathcal{H}^\iota$. One way of finding elements of $\mathcal{H}^\iota$ is to start with a function h that is positive, decreasing and log-convex, and to check that the associated ι function is non-decreasing whenever it is strictly greater than 1. This is tedious, because nothing guarantees that ι will have the right monotonicity property. Another possibility is to start with a function ι that is non-decreasing, to integrate a second-order differential equation to obtain a function h, and to adjust constants of integration to ensure that h is positive, decreasing and log-convex. The following proposition states that there exist constants of integration such that h does belong to $\mathcal{H}^\iota$:

Proposition 7. *Let $\widetilde{\iota} : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ be a $\mathcal{C}^1$ function such that $\widetilde{\iota}$ is non-decreasing, $\lim_{0^+} \widetilde{\iota} > 0$, and $\widetilde{\iota}(x) > 1$ for some $x > 0$. For every $(\alpha, \beta) \in \mathbb{R}^2_{++}$, let*

$$h^{\alpha,\beta}(x) = \alpha\left(\beta - \int_1^x \exp\left(-\int_1^t \frac{\widetilde{\iota}(u)}{u} du\right) dt\right).$$

Then, there exists $\underline{\beta} > 0$ such that, for every $h \in \mathcal{H}^\iota$, $\iota(h) = \widetilde{\iota}$ if and only if $h = h^{\alpha,\beta}$ for some $\alpha > 0$ and $\beta \geq \underline{\beta}$.

Proof. See Appendix C.5. □

Non-linear pricing. We now assume that firms can charge two-part tariffs: for every $j \in \mathcal{N}$, p_j (resp. F_j) denotes the variable (resp. fixed) part of the two-part tariff contract for product j. In equilibrium, firms find it optimal to set all variable parts equal to marginal cost, and compete on the fixed parts. Intuitively, if firm f sets $p_j \neq c_j$ for some $j \in f$, then it is profitable for this firm to deviate to $p_j = c_j$, and to adjust the fixed part in such a way that product j is chosen with the same probability as before. Since this deviation raises the joint surplus of the consumer and the firm (the consumer consumes the efficient quantity), and since the consumer receives the same expected surplus as before (otherwise, the choice probabilities would not be the same as before), this deviation is indeed profitable.

When all firms set variable parts equal to marginal costs, the consumer's indirect utility when choosing product j (net of the taste shock) is $\log h_j(c_j) - F_j$, and his conditional demand for product j is $-h'_j(c_j)/h_j(c_j)$. Therefore, the profit of firm f is given by:

$$\Pi^f = \sum_{k \in f} \left(F_k - c_k \frac{-h'_k(c_k)}{h_k(c_k)} \right) \frac{h_k(c_k)\, \mathrm{e}^{-F_k}}{\sum_{j \in \mathcal{N}} h_j(c_j)\, \mathrm{e}^{-F_j}}.$$

These are the payoff functions of pricing game $\left(\left(\widetilde{h}_j \right)_{j \in \mathcal{N}}, \mathcal{F}, (\widetilde{c}_j)_{j \in \mathcal{N}} \right)$ with linear tariffs, where, for every $j \in \mathcal{N}$, $\widetilde{h}_j(F_j) = \exp\left(\log h_j(c_j) - F_j\right)$ (i.e., firms only have logit products), and $\widetilde{c}_j = c_j \frac{-h'_j(c_j)}{h_j(c_j)}$. We know that this pricing game has an equilibrium, and in fact, we will soon show that this equilibrium is unique. The next step should be to compare the equilibria of the original pricing game with linear pricing and the unique equilibrium of $\left(\left(\widetilde{h}_j \right)_{j \in \mathcal{N}}, \mathcal{F}, (\widetilde{c}_j)_{j \in \mathcal{N}} \right)$. One possible avenue (which we have not yet had time to explore) would be to investigate whether social welfare and consumer surplus are higher under non-linear pricing.

It is interesting to notice that, under non-linear pricing, all products are always being sold, whereas, as discussed before, this is not necessarily the case under linear pricing. This comes from the fact that the non-linear pricing game is equivalent to a linear pricing game with logit demands, and logit products are such that $\bar{\mu}_j = \infty$ (since $\widetilde{\iota}_j(F_j) = F_j$). Intuitively, a firm is better able to extract additive taste shock ε_j under non-linear pricing than under linear pricing.

4.3 Equilibrium Uniqueness

We now turn our attention to the question of equilibrium uniqueness. The idea is to derive conditions under which $\Omega'(H) < 0$ whenever $\Omega(H) = 1$,[13] which ensures that there is exactly one value of H such that $\Omega(H) = 1$. To avoid non-differentiability issues (recall from Lemma 10 that m^f is differentiable everywhere but at points H such that $m^f(H) = \bar{\mu}_k$ for some $k \in f$), we assume that, for every $f \in \mathcal{F}$ and $j \in f$, $\bar{\mu}^f = \bar{\mu}_j$. This ensures that all products are always sold at finite prices (firm f would make zero profit if it were to set $\mu^f \geq \bar{\mu}^f$) and that $\Omega(\cdot)$ is $\mathcal{C}^1$ on $\mathbb{R}_{++}$. We also introduce the following notation: for all $j \in \mathcal{N}$, $\theta_j \equiv h'_j/\gamma'_j$.

We can now state our uniqueness theorem:

Theorem 4. *Let $\left((h_j)_{j\in\mathcal{N}}, \mathcal{F}, (c_j)_{j\in\mathcal{N}}\right)$ be a pricing game. Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Suppose that, for every firm $f \in \mathcal{F}$, at least one of the following conditions holds:*

(i) $\min_{j\in f} \inf_{p_j > \underline{p}_j} \rho_j(p_j) \geq \max_{j\in f} \sup_{p_j > \underline{p}_j} \theta_j(p_j)$.

(ii) $\bar{\mu}^f \leq \mu^*(\simeq 2.78)$, *and for every* $j \in f$, $\lim_\infty h_j = 0$[14] *and* ρ_j *is non-decreasing on* $\left(\underline{p}_j, \infty\right)$.

(iii) There exists a function $h^f \in \mathcal{H}^\iota$ and $c^f > 0$ such that $h_j = h^f$ and $c_j = c^f$ for all $j \in f$. In addition ρ^f is non-decreasing on $\left(\underline{p}^f, \infty\right)$.

Then, the pricing game has a unique equilibrium.

Proof. The result follows from Lemmas D, F, G, H and K, stated and proven in Appendices D.1–D.4. □

Comments. The theorem is proven in two steps. We first show that uniqueness condition $\Omega'(H) < 0$ can be rewritten as $|\mathcal{F}|$ independent firm-level conditions (Lemmas D and F). In the second step, we show that each of conditions (i), (ii) and (iii) is sufficient for the firm-level condition (Lemmas G, H and K).

We first discuss the condition that ρ_j is non-decreasing on $\left(\underline{p}_j, \infty\right)$ for every $j \in f$. Consider a discrete / continuous choice model of demand in which only product j and outside option $H^0 > 0$ are available. Then, the expected demand for product j is given by:

[13] Another possibility would be to follow an index approach and compute the sign of the determinant of the Jacobian of the first-order conditions map. In Appendix D.8, we show that this approach is equivalent to ours.

[14] Condition $\lim_\infty h_j = 0$ can be weakened. See Propositions A and B, and Corollaries A and B in Appendix D.5.

$D_j(p_j, H^0) = -h_j'(p_j)/\left(h_j(p_j) + H^0\right)$. It is easy to show that function $p_j \in \left(\underline{p}_j, \infty\right) \mapsto 1/D_j(p_j, H^0)$ is convex if and only if ρ_j is non-decreasing on $\left(\underline{p}_j, \infty\right)$.[15] Caplin and Nalebuff (1991) argue that this convexity condition is "just about as weak as possible" (see the paragraph after Proposition 3 page 38). They show that, under this condition, single-product firms' profit functions are quasi-concave in own prices. In their framework, equilibrium existence then follows from Kakutani's fixed-point theorem.

We find that, although this convexity condition is not needed to obtain equilibrium existence, it guarantees equilibrium uniqueness, provided that some additional restrictions, contained in conditions (i), (ii) and (iii), are satisfied. Note that condition (i) is indeed a stronger version of the assumption that ρ_j is non-decreasing. This is because ρ_j is non-decreasing on $\left(\underline{p}_j, \infty\right)$ if and only if $\rho_j \geq \theta_j$ on the same interval. Condition (i) imposes that the highest possible value of θ_j ($j \in f$) be smaller than the lowest possible value of ρ_j ($j \in f$), which is indeed stronger.

Finally, note that if $f = \{j\}$ and ρ_j is non-decreasing on $\left(\underline{p}_j, \infty\right)$, then condition (iii) trivially holds. It is therefore easier to ensure equilibrium uniqueness for single-product firms than for multiproduct firms.

Examples. In the following, we provide examples of demand systems that satisfy (or do not satisfy) our uniqueness conditions. A priori, condition (i) seems tedious to check if the firm under consideration has heterogeneous products. The following proposition shows that a certain type of heterogeneity can be easily handled:

Proposition 8. *Let $h \in \mathcal{H}^\iota$ such that $\sup_{x > \underline{p}(h)} \theta(h)(x) \leq \inf_{x > \underline{p}(h)} \rho(h)(x)$. Let f be a finite*

[15] To see this, note that, for every $p_j > \underline{p}_j$ and $H^0 > 0$,

$$\begin{aligned}
\frac{\partial^2}{\partial p_j^2} \frac{1}{D_j(p_j, H^0)} &= -\left(\frac{h_j + H^0}{h_j'}\right)'', \\
&= -\left(\frac{\left(h_j'\right)^2 - h_j''\left(h_j + H^0\right)}{\left(h_j'\right)^2}\right)', \\
&= \left(\rho_j + \frac{H^0}{\gamma_j}\right)', \\
&= \rho_j' - \frac{\gamma_j' H^0}{\gamma_j^2},
\end{aligned}$$

which, by Lemma 4-(iii), is non-negative for every $p_j > \underline{p}_j$ and H^0 if and only if $\rho_j'(p_j) \geq 0$ for all $p_j > \underline{p}_j$.

and non-empty set, and, for every $j \in f$, $(\alpha_j, \beta_j, \delta_j, \epsilon_j) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_+$. For every $j \in f$, define

$$h_j(p_j) = \alpha_j h(\beta_j p_j + \delta_j) + \epsilon_j, \quad \forall p_j > 0.$$

Then, $h_j \in \mathcal{H}^\iota$ and $\bar{\mu}_j = \bar{\mu}^f$ for all $j \in f$, and $\max_{j \in f} \sup_{p_j > \underline{p}_j} \theta_j(p_j) \leq \min_{j \in f} \inf_{p_j > \underline{p}_j} \rho_j(p_j)$.

Proof. See Appendix D.6. $\square$

Proposition 8 can be applied as follows. Let $h(p) = \mathrm{e}^{-p}$ for all $p > 0$. We already know that $h \in \mathcal{H}^\iota$. In addition, $\rho(p) = \theta(p) = 1$ for all $p > 0$. By Proposition 8, if firm f is such that for all $j \in f$, there exist $\lambda_j > 0$ and $a_j \in \mathbb{R}$ such that $h_j(p_j) = \mathrm{e}^{\frac{a_j - p_j}{\lambda_j}}$ for all $p_j > 0$ (i.e., firm f only has logit products), then condition (i) in Theorem 4 holds for firm f.

Similarly, let $h(p) = p^{1-\sigma}$ for all $p > 0$ ($\sigma > 1$). Again, we already know that $h \in \mathcal{H}^\iota$. In addition, $\rho(p) = \theta(p) = 1$. Therefore, if firm f is such that for all $j \in f$, there exist $a_j, b_j, d_j > 0$ such that $h_j(p_j) = a_j (b_j p_j + d_j)^{1-\sigma}$ for all $p_j > 0$ (i.e., if firm f only has (generalized) CES products with a common σ), then condition (i) in Theorem 4 holds for firm f. Other candidates for the base h include $h(x) = \exp(\mathrm{e}^{-x})$, $h(x) = 1 + \frac{1}{1+\mathrm{e}^{1+x}}$, $h(x) = 1 + \frac{1}{\cosh(2+x)}$, etc.

Some functions satisfy condition (ii), but not condition (i). Consider the following function: $h(x) = \frac{1}{\log(1+\mathrm{e}^x)}$. It is easy to show that $h \in \mathcal{H}^\iota$, $\lim_\infty h_j = 0$, ρ is non-decreasing and $\bar{\mu} = 2 (< 2.78)$. Therefore, condition (ii) holds. However, condition $\sup \theta(x) \leq \inf \rho(x)$ is not satisfied.

It is easy to construct a multi-product firm that satisfies none of our uniqueness conditions. For instance, let $f = \{1, 2\}$, $h_1(p_1) = p_1^{1-\sigma_1}$ and $h_2(p_2) = p_2^{1-\sigma_2}$, where $\sigma_1 \neq \sigma_2$. Then, $\bar{\mu}_1 \neq \bar{\mu}_2$, and Theorem 4 does not apply. It is also possible to find single-product firms for which Theorem 4 has no bite. Consider for instance the family of function $h^{\phi,\lambda} \in \mathcal{H}^\iota$ introduced in Section 4.2. It is easy to show that $\rho\left(h^{\phi,\lambda}\right)(\cdot)$ is strictly decreasing whenever $\phi \in (0, 1)$. Therefore, none of our uniqueness conditions hold.

Equilibrium uniqueness when marginal costs are high or the outside option is attractive enough. As discussed above, Theorem 4 is not powerful enough to guarantee equilibrium uniqueness for every pricing game. In the following, we show that, for a given discrete / continuous choice model of consumer demand $(h_j)_{j \in \mathcal{N}}$ and a given partition of the set of products $\mathcal{F}$, pricing game $\left((h_j)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}}\right)$ has a unique equilibrium, provided that firms are sufficiently inefficient and that consumers have access to an outside option. From now on, we no longer assume that $\bar{\mu}_j = \bar{\mu}^f$ for every $f \in \mathcal{F}$ and $j \in f$.

Proposition 9. *Let* $(h_j)_{j\in\mathcal{N}} \in (\mathcal{H}^\iota)^{\mathcal{N}}$, *and let* $\mathcal{F}$ *be a partition of* $\mathcal{N}$ *containing at least two elements. Then,*

- *For every* $\underline{H}^0 > 0$, *there exists* $\underline{c} > 0$ *such that pricing game* $\left((h_j)_{j\in\mathcal{N}}, \mathcal{F}, (c_j)_{j\in\mathcal{N}}\right)$ *with outside option* H^0 *has a unique equilibrium whenever* $(c_j)_{j\in\mathcal{N}} \in [\underline{c}, \infty)^{\mathcal{N}}$ *and* $H^0 \geq \underline{H}^0$.
- *For every* $\underline{c} > 0$, *there exists* $\underline{H}^0 \geq 0$ *such that pricing game* $\left((h_j)_{j\in\mathcal{N}}, \mathcal{F}, (c_j)_{j\in\mathcal{N}}\right)$ *with outside option* H^0 *has a unique equilibrium whenever* $(c_j)_{j\in\mathcal{N}} \in [\underline{c}, \infty)^{\mathcal{N}}$ *and* $H^0 \geq \underline{H}^0$.

Proof. See Appendix D.7. □

5 Type Aggregation with Multinomial Logit and CES Demands

5.1 The CES case

In this section, we study a multiproduct-firm pricing game with CES demands and heterogeneous qualities and productivities. Let $\mathcal{N}$ be a finite set containing at least two elements. For every $k \in \mathcal{N}$, for every $x > 0$, let $h_k(x) = a_k x^{1-\sigma}$, where $a_k > 0$ is the quality of product k, and $\sigma > 1$ is the elasticity of substitution. We have already shown in Section 4.3 that any pricing game based on $(h_j)_{j\in\mathcal{N}}$ has a unique equilibrium.

Firm f's fitting-in function is pinned down by equation (12), which involves functions γ_k and r_k. With CES demands, $\gamma_k(x) = \frac{\sigma-1}{\sigma} a_k x^{1-\sigma}$. Recall that r_k is the inverse function of $\nu_k : p_k \mapsto \frac{p_k - c_k}{p_k}\iota_k(p_k)$. With CES demands, $\iota_k = \sigma$, so that ν_k is just σ times the Lerner index. Therefore, for every $\mu^f \in [1, \sigma)$, $r_k(\mu^f) = \frac{c_k}{1-\frac{\mu^f}{\sigma}}$. From now on, we redefine μ^f as $\frac{\mu^f}{\sigma}$, so that μ^f is firm f's Lerner index, and takes values between $1/\sigma$ and 1. Equation (12) can then be rewritten as follows:

$$\sigma\mu^f \left(1 - \frac{\sum_{k\in f} \frac{\sigma-1}{\sigma} a_k \left(\frac{c_k}{1-\mu^f}\right)^{1-\sigma}}{H}\right) = 1. \tag{14}$$

Put $T^f = \sum_{k\in f} a_k c_k^{1-\sigma}$. Simplifying and rearranging terms in (14), we get:

$$\mu^f = \frac{1}{\sigma - (\sigma-1)\frac{T^f}{H}(1-\mu^f)^{\sigma-1}}. \tag{15}$$

It follows from Lemma 10 that equation (15) has a unique solution. This implicitly defines a function $m\left(\frac{T^f}{H}\right)$. Firm f's fitting-in function is $H \mapsto m\left(\frac{T^f}{H}\right)$. An immediate implication is that firms f and g have the same type ($T^f = T^g$) if and only if they share the same fitting-in function.

Next, we claim, that if firms f and g have the same type, then their contributions to the aggregator are the same. To see this, we introduce the following notation: for a given level of aggregator H, $s_k = \frac{a_k p_k^{1-\sigma}}{H}$ is the market share of product $k \in \mathcal{N}$, and $s^f = \sum_{k \in f} s_k$ is the market share of firm $f \in \mathcal{F}$. Then, for every $f \in \mathcal{F}$ and $k \in f$,

$$s_k = \frac{a_k \left(r_k\left(\mu^f\right)\right)^{1-\sigma}}{H} = \frac{\left(1-\mu^f\right)^{\sigma-1} a_k c_k^{1-\sigma}}{H}.$$

Therefore,

$$s^f = \sum_{k \in f} \frac{\left(1-\mu^f\right)^{\sigma-1} a_k c_k^{1-\sigma}}{H} = \frac{T^f}{H}\left(1 - m\left(\frac{T^f}{H}\right)^{\sigma-1}\right) \equiv S\left(\frac{T^f}{H}\right).$$

Firm f's market share function is $H \mapsto S\left(\frac{T^f}{H}\right)$. Therefore, firms f and g share the same market share function if and only if $T^f = T^g$. Put differently, firm f and g's contributions to the aggregator are identical if and only if they have the same type.

Recall that H is an equilibrium aggregator level if and only if $\Omega(H) = 1$, where

$$\begin{aligned}
\Omega(H) &= \frac{1}{H} \sum_{f \in \mathcal{F}} \sum_{k \in f} h_k\left(r_k\left(m^f(H)\right)\right), \\
&= \sum_{f \in \mathcal{F}} \sum_{k \in f} s_k, \\
&= \sum_{f \in \mathcal{F}} S\left(\frac{T^f}{H}\right).
\end{aligned}$$

In words, H is an equilibrium aggregator level if and only if firms' market shares add up to 1.

Last, we claim that, if firms f and g have the same type, then they earn the same profit: for every $f \in \mathcal{F}$,

$$\pi^f = \sum_{k \in f} (p_k - c_k) \frac{a_k p_k^{-\sigma}}{H},$$

$$= \sum_{k \in f} \frac{p_k - c_k}{p_k} \frac{a_k p_k^{1-\sigma}}{H},$$
$$= m\left(\frac{T^f}{H}\right) S\left(\frac{T^f}{H}\right),$$
$$\equiv \pi\left(\frac{T^f}{H}\right).$$

We summarize these findings in the following proposition:

Proposition 10. *Let f and g be two CES multiproduct firms. Put $T^f = \sum_{k \in f} a_k c_k^{1-\sigma}$ and $T^g = \sum_{k \in g} a_k c_k^{1-\sigma}$. The following assertions are equivalent:*

(i) $T^f = T^g$.

(ii) Firms f and g have the same markup fitting-in function.

(iii) Firms f and g have the same market share fitting-in function.

(iv) Firms f and g have the same profit fitting-in function.

Under CES demands, firms' types are aggregative as well. Firms f and g may differ widely in terms of product portfolios, productivity and product qualities, but if their types are the same, i.e., if $T^f = T^g$, then they share the same fitting-in functions. This implies that, no matter what the competitive environment is, these two firms will always behave in the exact same way. If we replace firm f by firm g, then the equilibrium aggregator level will not change, the behavior of firm f's rivals will not be affected, and firm g will end up charging the same markup, having the same market share, and earning the same profit as firm f. Interestingly, given a multiproduct firm f, there always exists an equivalent single-product firm. To see this, define firm $\hat{f}$ as a firm selling only one product with quality $\hat{a} = \sum_{k \in f} a_k c_k^{1-\sigma}$ and marginal cost $\hat{c} = 1$. Then, $T^f = T^{\hat{f}}$, and firms f and $\hat{f}$ are therefore equivalent.

We also obtain the following comparative statics results:

Proposition 11. *In a multiproduct-firm pricing game with CES demands,*

(i) $m', S', \pi' > 0$.

(ii) For every $f \in \mathcal{F}$, $\frac{dH^*}{dT^f}, \frac{d\mu^{f*}}{dT^f}, \frac{ds^{f*}}{dT^f}, \frac{d\pi^{f*}}{dT^f} > 0$, where superscript $*$ indicates equilibrium values, and d/dT^f is the total derivative with respect to T^f (taking into account the impact of T^f on the equilibrium aggregator level).

(iii) For every $f, g \in \mathcal{F}$, $f \neq g$, $\frac{d\mu^{g*}}{dT^f}$, $\frac{ds^{g*}}{dT^f}$, $\frac{d\pi^{g*}}{dT^f} < 0$.

Proof. See Appendix E.1. □

Point (i) says that a firm charges a high markup, has a high market share, and makes high profits if it has many products, if it is highly productive, if it sells high-quality products (high T^f), or if it operates in a less competitive environment (low H). Points (ii) and (iii) say that if firm f's type increases, then consumers benefit, firm f's markup, market share and profit increase, to the detriment of its rivals.

5.2 The Logit Case

In this section, we study a multiproduct-firm pricing game with logit demands and heterogeneous qualities and productivities. Let $\mathcal{N}$ be a finite set containing at least two elements. For every $k \in \mathcal{N}$, for every $x > 0$, let $h_k(x) = \mathrm{e}^{\frac{a_k - x}{\lambda}}$, where $a_k \in \mathbb{R}$ is the quality of product k, and $\lambda > 0$ is a substitutability parameter. We have already shown that any pricing game based on $(h_j)_{j \in \mathcal{N}}$ has a unique equilibrium.

As in the previous section, we want to reexpress firm f's fitting-in function. With logit demands, $\gamma_k(x) = h_k(x)$, and $\nu_k(p_k) = \frac{p_k - c_k}{p_k} \iota_k(p_k) = \frac{p_k - c_k}{\lambda}$. Therefore, for every $\mu^f \in [1, \infty)$, $r_k(\mu^f) = \lambda \mu^f + c_k$. Equation (12) can then be rewritten as follows:

$$\mu^f \left(1 - \frac{1}{H} \sum_{k \in f} \exp\left(\frac{a_k - c_k - \lambda \mu^f}{\lambda} \right) \right) = 1. \tag{16}$$

Put $T^f = \sum_{k \in f} \exp\left(\frac{a_k - c_k}{\lambda}\right)$. Simplifying and rearranging terms in (16), we get:

$$\mu^f \left(1 - \frac{T^f}{H} \mathrm{e}^{-\mu^f} \right) = 1. \tag{17}$$

This uniquely pins down a function $m(.)$ such that $\mu^f = m\left(\frac{T^f}{H}\right)$. As before, define $s_k = \mathrm{e}^{\frac{a_k - p_k}{\lambda}} / H$ and $s^f = \sum_{k \in f} s_k$. Then,

$$s^f = \sum_{k \in f} \frac{\mathrm{e}^{\frac{a_k - c_k - \lambda \mu^f}{\lambda}}}{H} = \frac{T^f}{H} \exp\left(-m\left(\frac{T^f}{H} \right) \right) \equiv S\left(\frac{T^f}{H} \right).$$

We can then rewrite equilibrium condition $\Omega(H) = 1$ as $\sum_{f \in \mathcal{F}} S\left(\frac{T^f}{H}\right) = 1$. In addition, it is straightforward to check that

$$\pi^f = \mu^f s^f = m\left(\frac{T^f}{H}\right) S\left(\frac{T^f}{H}\right) \equiv \pi\left(\frac{T^f}{H}\right).$$

Therefore, firms' types are still aggregative under logit demands:

Proposition 12. *Let f and g be two logit multiproduct firms. Put $T^f = \sum_{k\in f} \exp\left(\frac{a_k - c_k}{\lambda}\right)$ and $T^g = \sum_{k\in g} \exp\left(\frac{a_k - c_k}{\lambda}\right)$. The following assertions are equivalent:*

(i) $T^f = T^g$.

(ii) Firms f and g have the same markup fitting-in function.

(iii) Firms f and g have the same market share fitting-in function.

(iv) Firms f and g have the same profit fitting-in function.

We also derive the following comparative statics:

Proposition 13. *In a multiproduct-firm pricing game with logit demands,*

(i) $m',\ S',\ \pi' > 0$.

(ii) For every $f \in \mathcal{F}$, $\frac{dH^*}{dT^f},\ \frac{d\mu^{f*}}{dT^f},\ \frac{ds^{f*}}{dT^f},\ \frac{d\pi^{f*}}{dT^f} > 0$.

(iii) For every $f, g \in \mathcal{F}$, $f \neq g$, $\frac{d\mu^{g*}}{dT^f},\ \frac{ds^{g*}}{dT^f},\ \frac{d\pi^{g*}}{dT^f} < 0$.

Proof. See Appendix E.2. □

To summarize, we obtain type aggregation both under CES and logit demands. With CES (resp. logit) demands, the relevant ι-markup is the Lerner index (resp. the unnormalized markup), and the relevant s^f is firm f's market share in value (resp. in volume).

6 An Algorithm for CES Demands

Numerically solving for the equilibrium of a multiproduct-firm pricing game in an industry with many firms and products can be a daunting task with standard methods, due to the high dimensionality of the problem. Exploiting the aggregative structure of the pricing game allows us to reduce this dimensionality tremendously: instead of solving a system of $|\mathcal{N}|$ non-linear equations in $|\mathcal{N}|$, we only need to look for and $H > 0$ such that $\Omega(H) = 1$. Of course, there usually will not be a closed-form expression for $\Omega(.)$, so we still need to approximate this function numerically. But $\Omega(H)$ is simple to compute as well, since all we need to do is solve for $|\mathcal{F}|$ separate equations, each with one unknown. Below, we describe how this general approach can be implemented to solve a multiproduct-firm pricing game with CES demands.

The algorithm use two nested loops. The inner loop computes $\Omega(H)$ for a given H. The outer loop iterates on H. We start by describing the inner loop. Fix some $H > 0$. As argued in Section 5.1, we need to compute

$$s^f = \frac{T^f}{H}\left(1-\mu^f\right)^{\sigma-1},$$

where μ^f is the unique solution of

$$\mu^f = \frac{1}{\sigma - (\sigma-1)\left(\frac{T^f}{H}\right)\left(1-\mu^f\right)^{(\sigma-1)}},$$

or, equivalently,

$$\underbrace{\mu_f\left(\sigma - (\sigma-1)\frac{T^f}{H}(1-\mu^f)^{\sigma-1}\right) - 1}_{\equiv \phi^f(\mu^f)} = 0. \tag{18}$$

To do so, we solve equation (12) numerically using the Newton-Raphson method. The derivative of ϕ^f can be computed analytically:

$$\phi^{f\prime}(\mu^f) = \sigma - (\sigma-1)\frac{T^f}{H}(1-\mu^f)^{\sigma-1} + \mu^f(\sigma-1)^2\frac{T^f}{H}(1-\mu^f)^{\sigma-2},$$

so we do not need to take finite differences to compute the Newton step. The usual problem with the Newton-Raphson method is that it may fail to converge if starting values are not good enough. This is a potentially major issue, because the value of $\Omega(H)$ used by the outer loop would then be inaccurate. Fortunately, the following starting value guarantees convergence:

$$\mu_0^f = \max\left(\frac{1}{\sigma},\ 1-\left(\frac{H}{T^f}\right)^{\frac{1}{\sigma-1}}\right).$$

In fact, the Newton method converges extremely fast (usually fewer than 5 steps). Notice, in addition, that this method can be easily vectorized by stacking up the μ^f's in a vector.

The outer loop iterates on H to solve equation $\Omega(\Gamma) - 1 = 0$. This can be done by using standard derivative-based methods (we currently use Matlab's implementation of the trust-region dogleg algorithm). The Jacobian can be computed analytically:

$$\Omega'(H) = -\sum_{g\in\mathcal{F}} \frac{T^g}{H^2} S'\left(\frac{T^g}{H}\right),$$
$$= \frac{-1}{H} \sum_{g\in\mathcal{F}} \frac{\frac{T^g}{H}(1-\mu^g)^{\sigma-1}}{1+(\sigma-1)^2 \frac{T^g}{H}(1-\mu^g)^{\sigma-2}(\mu^g)^2}.$$

We use the value of H that would prevail under monopolistic competition as starting value ($H_0 = \sum_{f\in\mathcal{F}} T^f \left(1-\frac{1}{\sigma}\right)^{\sigma-1}$), and we always get convergence (usually in about 20 steps).[16]

7 Application to Merger Policy

Throughout this section, we assume that demand is either of the CES or multinomial logit forms. As shown in Section 5, in this case an additional aggregation property obtains: a firm's product portolio (with heterogeneous qualities and marginal costs) can be fully summarized by its one-dimensional type T^f, where $T^f = \sum_{k\in f} a_k c_k^{1-\sigma}$ in the case of CES demand and $T^f = \sum_{k\in f} \exp\left(\frac{a_k - c_k}{\lambda}\right)$. For the following merger analysis, this allows us to expense with any restriction on merger-specific synergies: some of the merged firms' marginal costs may go up; other marginal costs may go down; some of the products' qualities may improve or degrade; the merged entity may end up developing new products, or instead withdrawing some of its products. All relevant information can be summarized in the merged firm's post-merger type.

7.1 Static Merger Policy

We consider a merger between firms f and g. Let H^* (resp., $\hat{H}^*$) denote the equilibrium value of the aggregator before (resp., after) the merger. As consumer surplus is increasing in the value of that aggregator, we say that the merger is *CS-increasing* (resp., *CS-decreasing*) if $\hat{H}^* > H^*$ (resp., $\hat{H}^* < H^*$); it is *CS-neutral* if $\hat{H}^* = H^*$. Assume that the merger partners' pre-merger types are T^f and T^g, respectively. Let T^M denote the merged firm's post-merger type.

We have:

Proposition 14. *There exists a cutoff $\hat{T}^M > T^f + T^g$ such that merger M is CS-neutral if $T^M = \hat{T}^M$, CS-increasing if $T^M > \hat{T}^M$, and CS-decreasing if $T^M < \hat{T}^M$. Moreover, if*

[16] In Breinlich, Nocke, and Schutz (2015), we use this algorithm to calibrate an international trade model with two countries, 160 manufacturing industries, CES demands and oligopolistic competition.

merger M is CS-nondecreasing (i.e., either CS-neutral or CS-increasing), then it is (strictly) profitable.

Proof. See Appendix F.1. □

Inequality $\hat{T}^M > T^f + T^g$ means that, for a merger to be CS-increasing, the merger has to involve synergies, as in Williamson (1968) and Farrell and Shapiro (1990). This implies that a CS-neutral merger must be profitable, since it does not affect the intensity of competition, but it allows the merging parties to benefit from synergies.

7.2 Dynamic Merger Review with CES/Logit Demand

We now turn to studying the interaction between mergers. Consider two mergers, M_1 and M_2, and assume that these mergers are disjoint, i.e., no firm takes part in more than one merger. In the context of a homogeneous goods Cournot model, Nocke and Whinston (2010) have established that that there is a sign-preserving complementarity in the consumer surplus effect of (disjoint) mergers that share the same sign in terms of their consumer surplus effect. The following proposition shows that this result carries over to mergers between arbitrary multiproduct firms, provided demand takes the CES or multinomial logit forms.

Proposition 15. *There is a sign-preserving complementarity in the consumer surplus effect of disjoint mergers that share the same sign in terms of their consumer surplus effect.. If merger M_i is CS-nondecreasing (and hence profitable) in isolation, it remains CS-nondecreasing (and hence profitable) if another merger M_j, $j \neq$, that is CS-nondecreasing in isolation takes place. If merger M_i is CS-decreasing in isolation, it remains CS-decreasing if another merger M_j, $j \neq$, that is CS-decreasing in isolation takes place.*

Proof. See Appendix F.2. □

Nocke and Whinston (2010)'s result on the interaction of CS-increasing and CS-decreasing mergers also carries over to our setting with multiproduct firms:

Proposition 16. *Suppose that mergers M_1 and M_2 are CS-nondecreasing and CS-decreasing, respectively, in isolation. Then, merger M_1 is CS-increasing (and hence profitable), conditional on merger M_2 taking place. Moreover, the joint profit of the firms involved in M_1 is strictly larger if both mergers take place than if neither does.*

Proof. The proof is identical to that of Proposition 2 in Nocke and Whinston (2010). It involves inverting the order of the two mergers: at the first step, merger M_2 and, at the second step, merger M_1. As consumer surplus must, by assumption, be (weakly) higher after both mergers have taken place than before, and because consumer surplus (strictly) falls at step 1 (again, by assumption), consumer surplus must (strictly) increase at step 2. That is, M_1 is CS-increasing, conditional on M_2 taking place. By Proposition 14, this implies that the joint profit of the firms in M_1 must go up at step 2. Finally, we assert that the joint profit of the firms in M_1 must go up at step 1 as well. This follows from an argument identical to that used in the proof of Proposition 4, as the CS-decreasing merger at step 1 induces a reduction in the equilibrium value of the aggregator H^*, which benefits all outsiders to that merger, including the firms involved in M_1. □

We now embed our pricing game in a dynamic model with endogenous mergers and merger policy, as in Nocke and Whinston (2010). There are T periods, and a set $\{M_1, M_2, ..., M_K\}$ of disjoint potential mergers. Merger M_k becomes feasible at the beginning of period t with probability $p_{kt} \in [0, 1]$, where $\sum_t p_{kt} \leq 1$. Conditional on becoming feasible, the post-merger type of the merged firm M_k is drawn from some distribution C_{kt}.[17] The feasibility of a particular merger (including its efficiency) is publicly observed by all firms. In each period, the firms involved in a feasible and not-yet-approved merger decide whether or not to propose their merger to the antitrust authority. We assume that bargaining is efficient so that the merger partners propose the merger if and only if it is in their joint interest to do so. Given a set of proposed mergers, the antitrust authority then decides which mergers to approve (if any). An approved merger is consummated immediately. Finally, at the end of each period, the firms play the pricing game, given current market structure. All firms as well as the antitrust authority discount payoffs with factor $\delta \leq 1$.

Following Nocke and Whinston (2010), we define a *myopically CS-maximizing merger policy* as an approval policy, where in each period, given the set of proposed mergers and current market structure, the antitrust authority approves a set of mergers that maximizes consumer surplus in the current period. The *most lenient myopically CS-maximizing merger policy* is a *myopically CS-maximizing merger policy* that approves the largest such set (i.e., including CS-neutral mergers). (As shown in Nocke and Whinston (2010) such a policy is well-defined.)

The following proposition shows that Nocke and Whinston (2010)'s result on the dynamic optimality of a myopic merger approval policy carries over to our multiproduct firm setting:

Proposition 17. *Suppose the antitrust authority adopts the most lenient myopically CS-*

[17] The main result would not change if we assumed instead an arbitrary stochastic process governing (i) when mergers become feasible, and (ii) the merged firms' post-merger type.

maximizing merger policy. Then, all feasible mergers being proposed in each period after any history is a subgame-perfect Nash equilibrium for the firms. The equilibrium outcome maximizes discounted consumer surplus (indirect utility) for any realized sequence of feasible mergers. Moreover, for each such sequence, every subgame-perfect Nash equilibrium results in the same optimal sequence of period-by-period consumer surpluses.

Proof. The result follows from Propositions 14, 15 and 16, which are the analogues of Corollary 1 and Proposition 1 and 2 in Nocke and Whinston (2010). See Nocke and Whinston (2010) for details. □

8 Concluding Remarks

TBW

A Proofs for Section 2

A.1 Proof of Proposition 1

The aim of this section is to prove the following result:

Theorem A. *Let $\mathcal{N}$ be a finite and non-empty set. For every $k \in \mathcal{N}$, let h_k (resp. g_k) be a $\mathcal{C}^2$ (resp. $\mathcal{C}^1$) function from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$. Suppose that $h'_k > 0$ for every k. Define demand system D as follows:*

$$D_k\left((p_j)_{j\in\mathcal{N}}\right) = \frac{g_k(p_k)}{\sum_{j\in\mathcal{N}} h_j(p_j)}, \quad \forall k \in \mathcal{N}, \quad \forall (p_j)_{j\in\mathcal{N}} \in \mathbb{R}_{++}^{\mathcal{N}}$$

The following assertions are equivalent:

(i) D is quasi-linearly integrable.

(ii) There exists a strictly positive scalar α such that, for every $k \in \mathcal{N}$, $g_k = -\alpha h'_k$. Moreover, $h''_k > 0$ for every $k \in \mathcal{N}$, and $\sum_{k\in\mathcal{N}} \gamma_k \leq \sum_{k\in\mathcal{N}} h_k$.

When this is the case, function $v(.)$ is an indirect subutility function for the associated demand system if and only if there exists $\beta \in \mathbb{R}$ such that $v(p) = \alpha \log\left(\sum_{j\in\mathcal{N}} h_j(p_j)\right) + \beta$ for every $p >> 0$.

We first state and prove two preliminary technical lemmas, which will be useful to prove Theorem A:

Lemma A. *For every $n \geq 1$, for every $(\alpha_i)_{1\leq i\leq n} \in \mathbb{R}^n$, define*

$$\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right) = \begin{pmatrix} 1-\alpha_1 & 1 & \cdots & 1 \\ 1 & 1-\alpha_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\alpha_n \end{pmatrix}$$

Then,[18]

$$\det\left(\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)\right) = (-1)^n \left(\left(\prod_{k=1}^n \alpha_k\right) - \sum_{j=1}^n \left(\prod_{\substack{1\leq k\leq n \\ k\neq j}} \alpha_k\right)\right)$$

[18]We adopt the convention that the product of an empty collection of real numbers is equal to 1.

Moreover, matrix $\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)$ is negative semi-definite if and only if $\alpha_i \geq 1$ for all $1 \leq i \leq n$ and

$$\sum_{i=1}^{n} \frac{1}{\alpha_i} \leq 1.$$

Proof. We prove the first part of the lemma by induction on $n \geq 1$. Start with $n = 1$. Then,

$$\det\left(\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)\right) = 1 - \alpha_1 = (-1)^1(\alpha_1 - 1),$$

so the property is true for $n = 1$.

Next, let $n \geq 2$, and assume the property holds for all $1 \leq m < n$. By n-linearity of the determinant,

$$\det\left(\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)\right) = (-\alpha_1)\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1-\alpha_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 1-\alpha_n \end{vmatrix} + \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1-\alpha_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\alpha_n \end{vmatrix}.$$

Applying Laplace's formula to the first column, we can see that the first determinant is, in fact, equal to $\det\left(\mathcal{M}\left((\alpha_i)_{2\leq i\leq n}\right)\right)$. The second determinant can be simplified by using n-linearity one more time:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1-\alpha_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\alpha_n \end{vmatrix} = -\alpha_2 \begin{vmatrix} 1 & 0 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1-\alpha_n \end{vmatrix} + \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\alpha_n \end{vmatrix},$$

$$= -\alpha_2 \det\left(\mathcal{M}\left(0, (\alpha_i)_{3\leq i\leq n}\right)\right) + 0,$$

where the second line follows again from Laplace's formula and from the fact that the first two rows of the second matrix in the first-line's right-hand side are colinear. Therefore,

$$\begin{aligned} \det \mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right) &= -\alpha_1 \det\left(\mathcal{M}\left((\alpha_i)_{2\leq i\leq n}\right)\right) - \alpha_2 \det\left(\mathcal{M}\left(0, (\alpha_i)_{3\leq i\leq n}\right)\right), \\ &= -\alpha_1(-1)^{n-1}\left(\left(\prod_{k=2}^{n}\alpha_k\right) - \sum_{j=2}^{n}\left(\prod_{\substack{2\leq k\leq n \\ k\neq j}}\alpha_k\right)\right) \end{aligned}$$

$$
\begin{aligned}
&- \alpha_2(-1)^{n-1}\left(0 - \prod_{k=3}^{n}\alpha_k\right),\\
&= (-1)^n\left(\left(\prod_{k=1}^{n}\alpha_k\right) - \sum_{j=2}^{n}\left(\prod_{\substack{1\leq k\leq n\\ k\neq j}}\alpha_k\right) - \prod_{k=2}^{n}\alpha_k\right),\\
&= (-1)^n\left(\left(\prod_{k=1}^{n}\alpha_k\right) - \sum_{j=1}^{n}\left(\prod_{\substack{1\leq k\leq n\\ k\neq j}}\alpha_k\right)\right).
\end{aligned}
$$

We now turn our attention to the second part of the lemma. Assume first that matrix $\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)$ is negative semi-definite. Then, all its diagonal terms have to be non-positive, i.e., $\alpha_i \geq 1$ for all i. Besides, the determinant of this matrix should be non-negative (resp. non-positive) if n is even (resp. odd). Put differently, the sign of the determinant should be $(-1)^n$ or 0. Since the α's are all different from zero, this determinant can be simplified as follows:

$$
\det\left(\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)\right) = (-1)^n\left(\prod_{k=1}^{n}\alpha_k\right)\left(1 - \sum_{k=1}^{n}\frac{1}{\alpha_k}\right).
$$

This expression has sign $(-1)^n$ or 0 if and only if $\sum_{k=1}^{n}\frac{1}{\alpha_k} \leq 1$.

Conversely, assume that the α's are all ≥ 1, and that $\sum_{k=1}^{n}\frac{1}{\alpha_k} \leq 1$. The characteristic polynomial of matrix $\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)$ is defined as

$$
P(X) = \begin{vmatrix}
1-\alpha_1-X & 1 & \cdots & 1\\
1 & 1-\alpha_2-X & \cdots & 1\\
\vdots & \vdots & \ddots & \vdots\\
1 & 1 & \cdots & 1-\alpha_n-X
\end{vmatrix}.
$$

This determinant can be calculated using the first part of the lemma. For every $X > 0$,

$$
\begin{aligned}
(-1)^n P(X) &= \underbrace{\left(\prod_{k=1}^{n}(\alpha_k + X)\right)}_{>0}\left(1 - \sum_{k=1}^{n}\frac{1}{\alpha_k + X}\right),\\
&> \underbrace{\left(\prod_{k=1}^{n}(\alpha_k + X)\right)}_{>0}\underbrace{\left(1 - \sum_{k=1}^{n}\frac{1}{\alpha_k}\right)}_{\geq 0},\\
&> 0.
\end{aligned}
$$

Therefore, $P(X)$ has no strictly positive root, matrix $\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)$ has no strictly positive eigenvalue, and this matrix is therefore negative semi-definite. $\square$

Lemma B. *Let M be a symmetric n-by-n matrix, $\lambda \neq 0$, and $1 \leq k \leq n$. Let A^k be the matrix obtained by dividing the k-th line and the k-th column of M by λ. Then, M is negative semi-definite if and only if A^k is negative semi-definite.*

Proof. Suppose M is negative semi-definite, and let $X \in \mathbb{R}^n$. Write A^k as $(a_{ij})_{1\leq i,j\leq n}$ and M as $(m_{ij})_{1\leq i,j\leq n}$. Finally, define Y as the n-dimensional vector obtained by dividing X's k-th component by λ. Then,

$$
\begin{aligned}
X'A^kX &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j, \\
&= \left(\sum_{\substack{1\leq i\leq n\\ i\neq k}}\sum_{\substack{1\leq j\leq n\\ j\neq k}} a_{ij}x_ix_j\right) + 2x_k\sum_{\substack{1\leq i\leq n\\ i\neq k}} a_{ik}x_i + x_k^2a_{kk}, \\
&= \left(\sum_{\substack{1\leq i\leq n\\ i\neq k}}\sum_{\substack{1\leq j\leq n\\ j\neq k}} m_{ij}x_ix_j\right) + 2\frac{x_k}{\lambda}\sum_{\substack{1\leq i\leq n\\ i\neq k}} m_{ik}x_i + \left(\frac{x_k}{\lambda}\right)^2 m_{kk}, \\
&= \left(\sum_{\substack{1\leq i\leq n\\ i\neq k}}\sum_{\substack{1\leq j\leq n\\ j\neq k}} m_{ij}y_iy_j\right) + 2y_k\sum_{\substack{1\leq i\leq n\\ i\neq k}} m_{ik}y_i + y^2m_{kk}, \\
&= Y'MY, \\
&\leq 0, \text{ since } M \text{ is negative semi-definite.}
\end{aligned}
$$

Therefore, A^k is negative semi-definite.

The other direction is now immediate, since M can be obtained by dividing the k-th line and the k-th column of matrix A^k by $1/\lambda$. $\square$

We now have all we need to prove Theorem A:

Proof. To simplify notation, assume without loss of generality that $\mathcal{N} = \{1,\ldots,n\}$, and let $D(.)$ be the demand system associated with the demand component under consideration. For every $p >> 0$, put $J(p) = \left(\frac{\partial D_i}{\partial p_j}(p)\right)_{1\leq i,j\leq n}$. Theorem 1 in Nocke and Schutz (2015) states that D is quasi-linearly integrable if and only if $J(p)$ is symmetric and negative semi-definite for every $p >> 0$.

We first show that matrix $J(p)$ is symmetric for every p if and only if there exists a strictly positive scalar α such that, for every $k \in \mathcal{N}$, $g_k = -\alpha h'_k$. If $J(p)$ is symmetric for every p, then, for every $1 \leq i, j \leq n$ such that $i \neq j$, for every $p >> 0$,

$$-\frac{h'_j(p_j)g_i(p_i)}{\left(\sum_{k\in\mathcal{N}} h_k(p_k)\right)^2} = J_{i,j}(p) = J_{j,i}(p) = -\frac{h'_i(p_i)g_j(p_j)}{\left(\sum_{k\in\mathcal{N}} h_k(p_k)\right)^2}.$$

It follows that, for every $1 \leq i \leq n$, for every $x > 0$,

$$\frac{h'_i(x)}{g_i(x)} = \frac{h'_1(1)}{g_1(1)} \equiv -\beta \tag{19}$$

If $\beta = 0$, then $h'_i = 0$ for every i, which violates the assumption that h_i is strictly decreasing. Therefore, $\beta \neq 0$, and we can define $\alpha \equiv 1/\beta$. It follows that $g_i = -\alpha h'_i$. Since $g_i > 0$ and $h'_i \leq 0$, we can conclude that $\alpha > 0$. Conversely, if there exists a strictly positive scalar α such that, for every $k \in \mathcal{N}$, $g_k = -\alpha h'_k$, then, for every $1 \leq i, j \leq n$, $i \neq j$, for every $p >> 0$,

$$J_{i,j}(p) = -\frac{h'_j(p_j)g_i(p_i)}{\left(\sum_{k\in\mathcal{N}} h_k(p_k)\right)^2} = \alpha\frac{h'_j(p_j)h'_i(p_i)}{\left(\sum_{k\in\mathcal{N}} h_k(p_k)\right)^2} = J_{j,i}(p),$$

and matrix $J(p)$ is therefore symmetric for every p.

Next, suppose that there exists $\alpha > 0$ such that, for every $1 \leq k \leq n$, $g_k = -\alpha h'_k$. We want to show that $J(p)$ is negative semi-definite for every $p >> 0$ if and only if $h''_k > 0$ for every $1 \leq k \leq n$, and $\sum_{k=1}^n \gamma_k \leq \sum_{k=1}^n h_k$.

Fix $p >> 0$. To ease notation, we write $h_k = h_k(p_k)$ for every k, and define $H \equiv \sum_{k\in\mathcal{N}} h_k$. We obtain the following expression for matrix $J(p)$:

$$J(p) = \frac{\alpha}{H^2}\begin{pmatrix} (h'_1)^2 - h''_1 H & h'_1 h'_2 & \cdots & h'_1 h'_n \\ h'_2 h'_1 & (h'_2)^2 - h''_2 H & \cdots & h'_2 h'_n \\ \vdots & \vdots & \ddots & \vdots \\ h'_n h'_1 & h'_n h'_2 & \cdots & (h'_n)^2 - h''_n H \end{pmatrix}.$$

$J(p)$ is negative semi-definite if and only if

$$\begin{pmatrix} (h_1')^2 - h_1''H & h_1'h_2' & \cdots & h_1'h_n' \\ h_2'h_1' & (h_2')^2 - h_2''H & \cdots & h_2'h_n' \\ \vdots & \vdots & \ddots & \vdots \\ h_n'h_1' & h_n'h_2' & \cdots & (h_n')^2 - h_n''H \end{pmatrix}$$

is negative semi-definite. Applying Lemma B n times (by dividing row k and column k by h_k', $1 \leq k \leq n$), this is equivalent to matrix

$$\begin{pmatrix} 1 - \frac{h_1''}{(h_1')^2}H & 1 & \cdots & 1 \\ 1 & 1 - \frac{h_2''}{(h_2')^2}H & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 - \frac{h_n''}{(h_n')^2}H \end{pmatrix}$$

being negative semi-definite. By Lemma A, this holds if and only if $\frac{h_k''}{(h_k')^2}H \geq 1$ for all $1 \leq k \leq n$, and $\frac{1}{H}\sum_{k=1}^n \frac{(h_k')^2}{h_k''} \leq 1$. This is equivalent to $h_k'' > 0$ for all k, and $\sum_{k=1}^n \gamma_k \leq \sum_{k=1}^n h_k$.

Finally, Nocke and Schutz (2015) show that, v is an indirect subutility function for demand system D if and only if $\nabla v = -D$. Clearly, this is equivalent to

$$v(p) = \alpha \log\left(\sum_{j \in \mathcal{N}} h_j(p_j)\right) + \beta, \quad \forall p >> 0,$$

where $\beta \in \mathbb{R}$ is an integration constant. □

B Proofs for Section 3

B.1 Proof of Lemma 1

Proof. Let $M = \left((h_j)_{j\in\mathcal{N}}, (c_j)_{j\in\mathcal{N}}\right) \in \mathcal{M}(\mathcal{H})$. Let $p >> 0$ and $H^0 > 0$, and suppose that $\nabla_p \Pi(M)(p, H^0) = 0$. For every $1 \leq k \leq n$,

$$\frac{\partial^2 \Pi(M)}{\partial p_k^2} = \frac{-h_k'}{H}\left(-\frac{\partial \nu_k}{\partial p_k} + \frac{1}{H}\left(\frac{\partial \nu_k}{\partial p_k}\gamma_k + \nu_k\gamma_k' - \nu_k\frac{\sum_{j\in\mathcal{N}}\gamma_j}{H}h_k'\right)\right),$$

$$
\begin{aligned}
&= \frac{-h_k'}{H}\left(-\frac{\partial \nu_k}{\partial p_k} + \frac{1}{H}\left(\frac{\partial \nu_k}{\partial p_k}\gamma_k + \nu_k\gamma_k' - (\nu_k - 1)h_k'\right)\right), \\
&= \frac{-h_k'}{H}\left(-\frac{\partial \nu_k}{\partial p_k} + \frac{1}{H}\left(\frac{\partial \nu_k}{\partial p_k}\gamma_k - \frac{\partial \nu_k}{\partial p_k}\gamma_k\right)\right), \\
&= \frac{h_k'}{H}\frac{\partial \nu_k}{\partial p_k}.
\end{aligned}
$$

where the first line follows from differentiating equation (5) with respect to p_k and using the fact that $\partial\Pi(M)/\partial p_k = 0$, the second line follows from equation (7), and the third line follows from equation (4). Using the same method, we find that all the off-diagonal elements of the Hessian matrix are equal to zero, which proves the lemma. □

B.2 Proof of Lemma 2

Proof. Let $M = \left((h_j)_{j\in\mathcal{N}}, (c_j)_{j\in\mathcal{N}}\right) \in \mathcal{M}(\mathcal{H})$. Let $p >> 0$ and $H^0 > 0$, and suppose that $\nabla_p \Pi(M)(p, H^0) = 0$. Then, by equation (7), $\nu_k(p_k, c_k) > 0$ for every $1 \leq k \leq n$. Since $\iota_k(p_k) > 0$, it follows that $p_k > c_k$ for every k. Therefore, by equation (3) and since $h_k \in \mathcal{H}^\iota$, $\partial\nu_k/\partial p_k > 0$. By Lemma 1, the Hessian of $\Pi(M)(., H^0)$ evaluated at price vector p is therefore negative definite. Therefore, the local second-order conditions hold, p is a local maximizer of $\Pi(M)(., H^0)$, M is well-behaved, and $\mathcal{H}^\iota$ is well-behaved. □

B.3 Proof of Lemma 3

Proof. Since $h \notin \mathcal{H}^\iota$, there exists $\hat{p} > 0$ such that $\iota(\hat{p}) > 1$ and $\iota'(\hat{p}) < 0$. Our goal is to construct a two-product firm $M = ((h_1, h_2), (c_1, c_2))$, a price vector $(p_1, p_2) \in \mathbb{R}^2_{++}$ and an $H^0 > 0$ such that $\nabla_p \Pi(M)((p_1, p_2), H^0) = 0$ and $\frac{\partial \nu_1}{\partial p_1}(p_1, c_1) < 0$. We begin by setting $h_1 = h$ and $p_1 = \hat{p}$. We will tweak h_2, p_2, c_1, c_2 and H^0 along the way.

Since $\iota_1'(p_1) < 0$, equation (3) implies that there exists $\bar{c} \in (0, p_1)$ such that $\frac{\partial \nu_1}{\partial p_1}(p_1, c_1) < 0$ whenever $c_1 < \bar{c}$.

For every $s \in (1, \iota_1(p_1))$, there exists a unique $C_1(s) \in (0, p_1)$ such that

$$
\frac{p_1 - C_1(s)}{p_1}\frac{\iota_1(p_1)}{s} = 1. \tag{20}
$$

$C_1(\cdot)$ is continuous and $\lim_{s\to\iota_1(p_1)} C_1(s) = 0$. In particular, there exists $\underline{s} \in (1, \iota_1(p_1))$ such that $C_1(s) \in (0, \bar{c})$ whenever $s \in (\underline{s}, \iota_1(p_1))$. It follows that, when $s \in (\underline{s}, \iota_1(p_1))$, condition (20) holds and $\frac{\partial \nu_1}{\partial p_1}(p_1, C_1(s)) < 0$.

Let $\sigma \in (\underline{s}, \iota_1(p_1))$, and $h_2(p_2) = p_2^{1-\sigma}$ for all $p_2 > 0$. Recall that $\iota_2(p_2) = \sigma$ and $\gamma_2(p_2) = \frac{\sigma-1}{\sigma} h_2(p_2)$ for all $p_2 > 0$.

For every $H^0 > 0$, define the following function:

$$\phi(x) = 1 - \frac{\gamma_1(p_1) + \frac{\sigma-1}{\sigma}x}{h_1(p_1) + x + H^0}, \ \forall x > 0.$$

Notice that $\lim_\infty \phi = \frac{1}{\sigma}$. Moreover,

$$\phi'(x) = \frac{\gamma_1(p_1) - \frac{\sigma-1}{\sigma}\left(h_1(p_1) + H^0\right)}{\left(h_1(p_1) + x + H^0\right)^2}.$$

Choose some H^0 such that $\gamma_1(p_1) - \frac{\sigma-1}{\sigma}\left(h_1(p_1) + H^0\right) < 0$. Then, $\phi'(x) < 0$ for all $x > 0$. Therefore, $\phi(x) > \frac{1}{\sigma}$ for all $x > 0$.

Let $(p_2, c_2) \in \mathbb{R}^2_{++}$. The first-order condition for product 2 can be written as follows:

$$\frac{p_2 - c_2}{p_2}\sigma\left(1 - \frac{\gamma_1(p_1) + \gamma_2(p_2)}{h_1(p_1) + h_2(p_2) + H^0}\right) = 1,$$

or, equivalently,

$$\frac{p_2 - c_2}{p_2} \times \underbrace{\sigma\phi\left(p_2^{1-\sigma}\right)}_{>1, \text{ since } \phi(x) > 1/\sigma} = 1.$$

Therefore, for every $p_2 > 0$, there exists a unique $C_2(p_2) \in (0, p_2)$ such that the first-order condition for product 2 holds.

The first-order condition for product 1 can be written as follows:

$$\frac{p_1 - c_1}{p_1}\frac{\iota_1(p_1)}{\phi\left(p_2^{1-\sigma}\right)^{-1}} = 1.$$

Since $\phi\left(p_2^{1-\sigma}\right)^{-1} \underset{p_2 \to 0^+}{\longrightarrow} \sigma$ and $\sigma \in (\underline{s}, \iota_1(p_1))$, there exists $P_2 > 0$ such that $\phi\left(P_2^{1-\sigma}\right)^{-1} \in (\underline{s}, \iota_1(p_1))$. Put $c_1 = C_1\left(\phi\left(P_2^{1-\sigma}\right)^{-1}\right)$. Then, the first-order condition for product 1 holds, $c_1 \in (0, \bar{c})$, and therefore, $\frac{\partial \nu_1}{\partial p_1}(p_1, c_1) < 0$.

To summarize, we have constructed a multi-product firm $M = ((h_1, h_2), (c_1, c_2))$ with $h_1 = h$, $h_2(x) = x^{1-\sigma}$, $c_1 = C_1\left(\phi\left(P_2^{1-\sigma}\right)^{-1}\right)$ and $c_2 = C_2(P_2)$, an $H^0 > 0$ and a price vector $(p_1, p_2) = (\hat{p}, P_2)$ such that $\nabla_p \Pi(M)\left((p_1, p_2), H^0\right) = 0$ and $\frac{\partial \nu_1}{\partial p_1}(p_1, c_1) < 0$. By Lemma 1, the Hessian matrix of $\Pi(M)(\cdot, H^0)$ evaluated at price vector (p_1, p_2) has a strictly positive eigenvalue. Therefore, (p_1, p_2) is not a local maximizer of $\Pi(M)(\cdot, H^0)$, and multi-product firm M is not well-behaved. It follows that $\mathcal{H}^{CES} \cup \{h\}$ is not well-behaved. $\square$

B.4 Proof of Proposition 2

Proof. Let $h \in \mathcal{H}$, $c > 0$ and $M = (h, c)$. With single-product firms, first-order condition (7) can be simplified as follows:

$$\nu \left(1 - \frac{\gamma}{h + H^0}\right) = 1. \tag{21}$$

By Lemma 1, $\partial^2 \Pi(M)/\partial p^2$ has the same sign as $\partial \nu/\partial p$ whenever condition (21) holds.

Assume that (ii) holds. We want to show that, for every $(p, c, H^0) \in \mathbb{R}^3_{++}$, $\partial \nu(p, c)/\partial p > 0$ whenever condition (21) holds. Let $p > 0$. If $\iota(p) \leq 1$, then for every $c, H^0 > 0$,

$$\nu \left(1 - \frac{\gamma}{h + H^0}\right) < 1,$$

so there is nothing to prove. Next, assume that $\iota(p) > 1$. For every $c > 0$, $\partial \nu/\partial p$ is given by equation (3). If $\iota'(p) \geq 0$, then $\partial \nu(p, c)/\partial p > 0$ for every $H^0 > 0$ and $0 < c \leq p$. In particular, $\partial \nu(p, c)/\partial p > 0$ when condition (21) holds.

Assume instead that $\iota'(p) < 0$. Then, since (ii) holds, $\rho'(p) \geq 0$. Notice that

$$\begin{aligned} \frac{\rho'}{\rho} &= \left(\log\left(\frac{h\iota}{p(-h')}\right)\right)', \\ &= \frac{h'}{h} + \frac{\iota'}{\iota} - \frac{1}{p} + \frac{h''}{-h'}. \end{aligned}$$

It follows that

$$\begin{aligned} p\frac{\rho'}{\rho} &= p\frac{\iota'}{\iota} - p\frac{-h'}{h} - 1 + \iota, \\ &= p\frac{\iota'}{\iota} - \frac{\iota}{\rho} - 1 + \iota, \\ &= p\frac{\iota'}{\iota} + \iota\left(1 - \frac{1}{\rho}\right) - 1. \end{aligned}$$

Since $\iota' < 0$ and $\rho' \geq 0$, it follows that $\iota\left(1 - \frac{1}{\rho}\right) - 1 > 0$.

For every $H^0 > 0$, there exists a unique $c\left(H^0\right)$ such that condition (21) holds. This $c\left(H^0\right)$ is given by:

$$c\left(H^0\right) = p\left(1 - \frac{1}{\iota\left(1 - \frac{\gamma}{h+H^0}\right)}\right). \tag{22}$$

Since $\iota\left(1 - \frac{1}{\rho}\right) - 1 > 0$, $c\left(H^0\right) \in (0, p)$ for every $H^0 > 0$. Notice also that $c'\left(H^0\right) > 0$. All we need to do now is check that

$$\frac{\partial \nu}{\partial p}\left(p, c\left(H^0\right)\right) = \frac{c\left(H^0\right)}{p^2}\iota + \frac{p - c\left(H^0\right)}{p}\iota'$$

is strictly positive for every $H^0 > 0$. Since the right-hand side is strictly increasing in $c\left(H^0\right)$ and $c'\left(H^0\right) > 0$, this boils down to checking that $\partial \nu\left(p, c(0)\right)/\partial p \geq 0$:

$$\begin{aligned}
\frac{\partial \nu}{\partial p}\left(p, c(0)\right) &= \frac{\iota}{p}\left(\frac{c(0)}{p}\iota + \frac{p - c(0)}{p}p\frac{\iota'}{\iota}\right), \\
&= \frac{\iota}{p}\left(\left(1 - \frac{1}{\iota\left(1 - \frac{1}{\rho}\right)}\right) + \frac{1}{\iota\left(1 - \frac{1}{\rho}\right)}p\frac{\iota'}{\iota}\right), \\
&= \frac{1}{p\left(1 - \frac{1}{\rho}\right)}\left(\iota\left(1 - \frac{1}{\rho}\right) - 1 + p\frac{\iota'}{\iota}\right), \\
&= \frac{\rho'}{\rho - 1},
\end{aligned}$$

which is indeed non-negative. Therefore, (i) holds.

Conversely, suppose that (ii) does not hold. There exists $p > 0$ such that $\iota(p) > 1$, $\iota'(p) < 0$ and $\rho'(p) < 0$. We distinguish two cases. Assume first that $\iota\left(1 - \frac{1}{\rho}\right) - 1 \geq 0$. Then, the $c\left(H^0\right)$ defined in equation (22) satisfies $c(H^0) \in (0, p)$ and

$$\frac{p - c\left(H^0\right)}{p}\iota\left(1 - \frac{\gamma}{h + H^0}\right) = 1$$

for every $H^0 > 0$. In addition, as proven above,

$$\frac{\partial \nu}{\partial p}\left(p, c(0)\right) = \frac{\rho'}{\rho - 1} < 0.$$

By continuity, there exists $\varepsilon > 0$ such that $\frac{\partial \nu}{\partial p}\left(p, c(\varepsilon)\right) < 0$. It follows that $\frac{\partial \Pi(M)}{\partial p}(p, \varepsilon) = 0$ and $\frac{\partial^2 \Pi(M)}{\partial p^2}(p, \varepsilon) > 0$. Therefore, M is not well-behaved.

Next, assume that $\iota\left(1 - \frac{1}{\rho}\right) - 1 < 0$. Then, there exists $H^0 > 0$ such that $c\left(H^0\right) = 0$. Notice that $\frac{\partial \nu}{\partial p}\left(p, 0\right) = \iota'(p) < 0$. Therefore, by continuity of $\partial \nu/\partial p$ and $c(.)$, for $\varepsilon > 0$ small enough,

$$\frac{\partial \nu}{\partial p}\left(p, c\left(H^0 + \varepsilon\right)\right) < 0,$$

and $c\left(H^0 + \varepsilon\right) > 0$. Therefore, multiproduct firm $\left(h, c\left(H^0 + \varepsilon\right)\right)$ is not well-behaved. $\square$

B.5 Proof of Lemma 4

Proof.

(i). Assume for a contradiction that $\iota(p) \leq 1$ for all $p > 0$ (we drop argument h from function ι to ease notation). Then, for all $p > 0$, $ph''(p) + h'(p) \leq 0$, i.e., $\frac{d}{dp}(ph'(p)) \geq 0$. Integrating this inequality between 1 and $p \geq 1$, we obtain that $ph'(p) - h'(1) \geq 0$. Therefore, for all $p \geq 1$, $h'(p) \leq h'(1)/p$. Integrating this inequality between 1 and p, we obtain that, for all $p \geq 1$,

$$h(p) \leq h(1) + \underbrace{h'(1)}_{<0} \log(p) \underset{p\to\infty}{\longrightarrow} -\infty.$$

This contradicts the assumption that $h > 0$.

Therefore, there exists $\hat{p} > 0$ such that $\iota(p) > 1$, and

$$\underline{p} \equiv \inf\{p \in \mathbb{R}_{++} : \quad \iota(p) > 1\} < \infty.$$

We prove two claims:

Claim 1: $\underline{p} \notin \{p > 0 : \ \iota(p) > 1\}$.

If $\underline{p} = 0$, then this is obvious. If instead $\underline{p} > 0$, then the claim follows immediately from the continuity of ι.

Claim 2: $\iota(y) \geq \iota(x)$ whenever $0 < x < y$ and $\iota(x) > 1$.

Assume for a contradiction that $\iota(y) < \iota(x)$. Put $S = \{z \in [x, y] : \ \iota(z) \leq 1\}$. If S is empty, then $\iota(z) > 1$ for every $z \in [x, y]$. Since $h \in \mathcal{H}^{\iota}$, $\iota'(z) \geq 0$ for every $z \in [x, y]$, and ι is non-decreasing on interval $[x, y]$. It follows that $\iota(y) \geq \iota(x)$, which is a contradiction.

Next, assume that S is not empty. Then, $\hat{y} \equiv \inf S \in [x, y]$ and, by continuity of ι, $\iota(\hat{y}) = 1$. In addition, $\iota(z) > 1$ for every $z \in [x, \hat{y})$. Using the same reasoning as above, it follows that

$$1 = \iota_k(\hat{y}) \geq \iota_k(x) > 1,$$

which is a contradiction.

Combining Claims 1 and 2, it follows that $\{x > 0 : \ \iota(x) > 1\} = \big(\underline{p}, \infty\big)$, and that ι is non-decreasing on $\big(\underline{p}, \infty\big)$, which proves point (i).

(ii). Since ι is monotone on $(\underline{p}, \infty)$, $\bar{\mu}$ exists. Assume for a contradiction that $\bar{\mu} \leq 1$. Then, by monotonicity, $\iota(p) \leq \bar{\mu} \leq 1$ for every $p > \underline{p}$. This contradicts point (i).

(iii). Let $p > \underline{p}$. Notice that

$$\begin{aligned}\gamma(p) &= \frac{-h'(p)}{ph''(p)} \left(p(-h'(p))\right), \\ &= \frac{-ph'(p)}{\iota(p)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\gamma'(p) &= \frac{1}{(\iota(p))^2} \left(-\left(ph''(p) + h'(p)\right) \times \iota(p) + \iota'(p) \times ph'(p)\right), \\ &= \frac{1}{(\iota(p))^2} \left(-h'(p)\left(1 - \iota(p)\right)\iota(p) + \iota'(p)ph'(p)\right) < 0,\end{aligned}$$

as $\iota' \geq 0$ and $\iota(p) > 1$ for all $p > \underline{p}$.

(iv). Notice first that

$$\frac{1}{\gamma} = \frac{h''}{h'^2} = \left(\frac{1}{-h'}\right)'.$$

γ is strictly positive and, as shown in (iii), strictly decreasing on $(\underline{p}, \infty)$. Therefore, $\lim_\infty \gamma \equiv l$ exists, and $l \in [0, \infty)$. Assume for a contradiction that $l > 0$, and let $\varepsilon > 0$. There exists $p_0 > \underline{p}$ such that

$$\forall t \geq p_0, \quad \frac{1}{\gamma(t)} \leq \frac{1}{l} + \varepsilon.$$

Integrating this inequality between p_0 and $p \geq x_0$, we get:

$$\underbrace{\left(\frac{1}{l} + \varepsilon\right)}_{\equiv M > 0}(p - p_0) \geq \int_{p_0}^{p} \frac{dt}{\gamma(t)} = \frac{1}{-h'(p)} - \underbrace{\frac{1}{-h'(p_0)}}_{\equiv A > 0}.$$

Therefore,

$$-h'(p) \geq \frac{1}{A + M(p - p_0)}, \quad \forall p \geq p_0.$$

Integrating this inequality between p_0 and $p \geq p_0$, we get:

$$\underbrace{\int_{p_0}^{p} \frac{dx}{A + M(x - p_0)}}_{\xrightarrow[p \to \infty]{} \infty} \leq -h(p) + h(p_0).$$

Therefore, $\lim_\infty h = -\infty$, which contradicts the assumption that $h > 0$.

(v). Notice that

$$\begin{aligned}\frac{d}{dp}\left(-ph'(p)\right) &= -h'(p) - ph''(p),\\ &= -h'(p)\left(1-\iota(p)\right),\end{aligned}$$

which is strictly negative for all $p > \underline{p}$. Therefore, $-ph'(p)$ is strictly decreasing for p high enough, and $\lim_{p\to\infty} -ph'(p)$ is a non-negative real number. Call this number l, and assume for a contradiction that $l > 0$. Let $p_0 > \underline{p}$. Since $-ph'(p)$ is decreasing on (p_0, ∞), $-ph'(p) \geq l$ for all $p \geq p_0$. It follows that

$$h(p_0) - h(p) = \int_{p_0}^{p} -h'(t)dt \geq \underbrace{\int_{p_0}^{p} \frac{l}{t}dt}_{\underset{p\to\infty}{\longrightarrow}\infty}.$$

Therefore, $\lim_\infty h = -\infty$, which is a contradiction.

(vi). Suppose $\bar{\mu} < \infty$ and $\lim_\infty h = 0$. For all $p > \underline{p}$,

$$\begin{aligned}\rho(p) &= \frac{h(p)h''(p)}{\left(h'(p)\right)^2},\\ &= \frac{ph''(p)}{-h'(p)}\frac{h(p)}{-ph'(p)},\\ &= \iota(p)\frac{h(p)}{-ph'(p)}.\end{aligned}$$

By assumption, $\lim_\infty h = 0$. By (v), $\lim_{p\to\infty} -ph'(p) = 0$. Moreover,

$$\begin{aligned}\lim_{p\to\infty}\frac{\frac{d}{dp}h(p)}{\frac{d}{dp}\left(-ph'(p)\right)} &= \lim_{p\to\infty}\frac{h'(p)}{-h'(p)-ph''(p)},\\ &= \lim_{p\to\infty}\frac{1}{\iota(p)-1},\\ &= \frac{1}{\bar{\mu}-1}.\end{aligned}$$

Therefore, by L'Hospital's rule,

$$\lim_{p\to\infty}\frac{h(p)}{-ph'(p)} = \frac{1}{\bar{\mu}-1},$$

and $\lim_\infty \rho = \frac{\bar{\mu}}{\bar{\mu}-1}$. $\square$

B.6 Proof of Lemma 5

Proof. We first rule out prices below marginal costs. Let $p >> 0$. Put $\mathcal{N}' \equiv \{k \in \mathcal{N} : \ p_k < c_k\}$ and $\mathcal{N}'' = \{k \in \mathcal{N} : \ c_k \leq p_k < \infty\}$. Define $\tilde{p}$ as follows: for every $k \in \mathcal{N}$, $\tilde{p}_k = \max(p_k, c_k)$. Since the h_ks are decreasing, $\sum_{k\in\mathcal{N}} h_k(p_k) > \sum_{k\in\mathcal{N}} h_k(\tilde{p}_k)$. Therefore,

$$\begin{aligned}
\Pi(M)\left(p, H^0\right) &= \sum_{k\in\mathcal{N}'} (p_k - c_k) \frac{-h_k'(p_k)}{\sum_{j\in\mathcal{N}} h_j(p_j) + H^0} + \sum_{k\in\mathcal{N}''} (p_k - c_k) \frac{-h_k'(p_k)}{\sum_{j\in\mathcal{N}} h_j(p_j) + H^0}, \\
&< 0 + \sum_{k\in\mathcal{N}''} (p_k - c_k) \frac{-h_k'(p_k)}{\sum_{j\in\mathcal{N}} h_j(p_j) + H^0}, \\
&\leq 0 + \sum_{k\in\mathcal{N}''} (\tilde{p}_k - c_k) \frac{-h_k'(\tilde{p}_k)}{\sum_{j\in\mathcal{N}} h_j(\tilde{p}_j) + H^0}, \\
&= \Pi(M)\left(\tilde{p}, H^0\right).
\end{aligned}$$

Therefore, $(p_k)_{k\in f}$ is not a solution of maximization problem (9).

For every $k \in \mathcal{N}$ and $x_k \in [0,1]$, define

$$\phi_k(x_k) = \begin{cases} c_k + \frac{x_k}{1-x_k} & \text{if } x_k < 1, \\ \infty & \text{if } x_k = 1. \end{cases}$$

Note that $\lim_1 \phi = \phi(1) = \infty$. For every $x \in [0,1]^n$, define $\phi(x) = (\phi_1(x_1), \ldots, \phi_n(x_n))$. ϕ is a bijection from $[0,1]^n$ to $\prod_{k=1}^n [c_k, \infty]$. Finally, put

$$\Psi(x) = \Pi(M)\left(\phi(x), H^0\right), \quad \forall x \in [0,1]^n.$$

Since ϕ is a bijection, maximization problem $\max_{x\in[0,1]^n} \Psi(x)$ has a solution if and only if maximization problem (9) has a solution. All we need to do now is show that Ψ is continuous on $[0,1]^n$.

Clearly, Ψ is continuous at every point x such that $x_k < 1$ for every $1 \leq k \leq n$. Next, let x such that $x_k = 1$ for some $1 \leq k \leq n$. To fix ideas, suppose that $x_k = 1$ for all $1 \leq k \leq K$, and that $x_k < 1$ for all $K+1 \leq k \leq n$, where $K \geq 1$. Then,

$$\lim_{\tilde{x}\to x} \Psi(\tilde{x}) = \lim_{\tilde{x}\to x} \sum_{k=1}^n (\phi_k(\tilde{x}_k) - c_k) \frac{-h_k'(\phi_k(\tilde{x}_k))}{\sum_{j=1}^n h_j(\phi_j(\tilde{x}_j)) + H^0},$$

$$
\begin{aligned}
&= \frac{\sum_{k=1}^{n} \lim_{\tilde{x}_k \to x_k} \left(\phi_k(\tilde{x}_k) - c_k\right)\left(-h_k'\left(\phi_k(\tilde{x}_k)\right)\right)}{\sum_{j=1}^{n} \lim_{\tilde{x}_j \to x_j} h_j\left(\phi_j(\tilde{x}_j)\right) + H^0}, \\
&= \frac{0 + \sum_{k=K+1}^{n} \left(\phi_k(x_k) - c_k\right)\left(-h_k'\left(\phi_k(x_k)\right)\right)}{\sum_{j=1}^{K} h_j(\infty) + \sum_{j=K+1}^{n} h_j\left(\phi_j(p_j)\right) + H^0}, \\
&= \Psi(x),
\end{aligned}
$$

where the third line follows by Lemma 4-(v). Therefore, Ψ is continuous. Combining this with the fact that $[0,1]^n$ implies that maximization problem $\max_{x \in [0,1]^n} \Psi(x)$ has a solution. This concludes the proof. □

B.7 Proof of Lemma 7

Proof. Let $c > 0$. Since $h \in \mathcal{H}^\iota$, $\partial \nu(h)(p,c)/\partial p > 0$ for every $p > c$ (see equation (3)). It follows from the inverse function theorem that $\nu(.,c)$ is a $\mathcal{C}^1$-diffeomorphism from (c,∞) to $\nu\left((c,\infty)\right)$, and that $\partial r/\partial \mu = \left(\partial \nu/\partial p\right)^{-1}$ (we drop argument h to ease notation). Therefore, equation (10) follows immediately from equation (4). Since $\nu(.,c)$ is strictly increasing,

$$
\begin{aligned}
\nu\left((c,\infty),c\right) &= \left(\lim_{p \to c} \nu(p,c), \lim_{p \to \infty} \nu(p,c)\right), \\
&= (0, \bar{\mu}).
\end{aligned}
$$

The fact that r is increasing in c follows immediately from the fact that ν is decreasing in c and increasing in p. Finally, let us show that $r(\mu,c) > \underline{p}$ whenever $\mu \geq 1$. Let $\mu \geq 1$. If $c \geq \underline{p}$, then this is trivial. Assume instead that $c < \underline{p}$. Then,

$$
\nu(\underline{p}, c) < \iota(\underline{p}) = 1 \leq \mu.
$$

Since r is increasing in μ, it follows that $r(\mu,c) > \underline{p}$. □

B.8 Proof of Lemma 8

Proof. Put $\mathcal{N}' = \{k \in \mathcal{N} :\ p_k < \infty\}$. Clearly, $\mathcal{N}' \neq \emptyset$, because if this set were empty, then the firm could obtain a strictly positive profit by pricing, say, product 1, at c_1+1, which would violate condition (b) in Definition 8. Assume without loss of generality that $\mathcal{N}' = \{1,\dots,K\}$, where $1 \leq K \leq n$. Define a new multiproduct firm $M' = \left((h_j)_{1 \leq j \leq K}, (c_j)_{1 \leq j \leq K}\right)$, and note that, for every $(\tilde{p}_j)_{1\leq j\leq K} \in \mathbb{R}^K_{++}$,

$$\Pi\left(M'\right)\left(\left(\tilde{p}_1,\ldots,\tilde{p}_K\right), H^0 + \sum_{j=K+1}^{n} h_j(\infty)\right) = \Pi\left(M\right)\left(\left(\tilde{p}_1,\ldots,\tilde{p}_K,\infty,\ldots,\infty\right), H^0\right).$$

Then, Condition (a) in definition 8 is equivalent to

$$\nabla_p \Pi\left(M'\right)\left(\left(p_1,\ldots,p_K\right), H^0 + \sum_{j=K+1}^{n} h_j(\infty)\right) = 0.$$

We have shown in Section 3.1.2 that this implies that $\nu_i(p_i, c_i) = \nu_j(p_j, c_j)$ for every $1 \leq i, j \leq K$, and that condition (6) holds for every $1 \leq k \leq K$. Put $\mu = \nu_1(p_1, c_1)$. Then, for every $1 \leq j \leq K$, by definition of function r_j (see Lemma 7), $p_j = r_j(\mu, c_j)$, and condition (6) can be rewritten as follows:

$$\mu = 1 + \mu \frac{\sum_{k=1}^{K} \gamma_k\left(r_k(\mu, c_k)\right)}{\sum_{j=1}^{K} h_j\left(r_j(\mu, c_j)\right) + \sum_{j=K+1}^{n} h_j(\infty) + H^0}. \tag{23}$$

In addition,

$$\begin{aligned}
\Pi(M)\left(p, H^0\right) &= \sum_{k=1}^{K} (p_k - c_k) \frac{-h'_k(p_k)}{\sum_{j=1}^{n} h_j(p_j) + H^0}, \\
&= \sum_{k=1}^{K} \frac{p_k - c_k}{p_k} \iota_k(p_k) \frac{\gamma_k(p_k)}{\sum_{j=1}^{n} h_j(p_j) + H^0}, \\
&= \sum_{k=1}^{K} \nu_k(p_k, c_k) \frac{\gamma_k(p_k)}{\sum_{j=1}^{n} h_j(p_j) + H^0}, \\
&= \mu \frac{\sum_{k=1}^{K} \gamma_k\left(r_k(\mu, c_k)\right)}{\sum_{j=1}^{K} h_j\left(r_j(\mu, c_j)\right) + \sum_{j=K+1}^{n} h_j(\infty) + H^0}, \\
&= \mu - 1, \text{ by equation (23).}
\end{aligned}$$

Next, we claim that, for every $j \geq K+1$, $r_j\left(\mu, c_j\right) = \infty$, or, equivalently, $\bar{\mu}_j \leq \mu$. Assume for a contradiction, that, for some $j \geq K+1$, $\bar{\mu}_j > \mu$. To fix ideas, assume that this j is equal to $K+1$. Define a new multiproduct firm $M'' = \left(\left(h_j\right)_{1\leq j\leq K+1}, \left(c_j\right)_{1\leq j\leq K+1}\right)$, and note that, for every $(\tilde{p}_j)_{1\leq j\leq K+1} \in (0,\infty]^{K+1}$,

$$\Pi\left(M''\right)\left(\left(\tilde{p}_1,\ldots,\tilde{p}_{K+1}\right), H^0 + \sum_{j=K+2}^{n} h_j(\infty)\right) = \Pi\left(M\right)\left(\left(\tilde{p}_1,\ldots,\tilde{p}_{K+1},\infty,\ldots,\infty\right), H^0\right) = \mu - 1.$$

Using equation (5), we see that, for every $x \in \mathbb{R}_{++}$

$$\begin{aligned}
&\frac{\partial \Pi\left(M''\right)}{\partial p_{K+1}}\left(\left(p_1,\ldots,p_K,x\right), H^0 + \sum_{j=K+2}^{n} h_j(\infty)\right)\\
=&\frac{-h'_{K+1}(x)}{\sum_{j=1}^{K} h_j(p_j) + h_{K+1}(x) + \sum_{j=K+2}^{n} h_j(\infty) + H^0}\times\\
&\left(1-\nu_{K+1}\left(x,c_{K+1}\right) + \frac{\sum_{j=1}^{K}(p_j-c_j)\left(-h'_j(p_j)\right) + (x-c_{K+1})h'_{K+1}\left(x\right)}{\sum_{j=1}^{K} h_j(p_j) + h_{K+1}(x) + \sum_{j=K+2}^{n} h_j(\infty) + H^0}\right),\\
=&\overbrace{\frac{-h'_{K+1}(x)}{\sum_{j=1}^{K} h_j(p_j) + h_{K+1}(x) + \sum_{j=K+2}^{n} h_j(\infty) + H^0}}^{>0}\times\\
&\left(1-\underbrace{\nu_{K+1}\left(x,c_{K+1}\right)}_{\xrightarrow[x\to\infty]{}\bar{\mu}_{K+1}} + \underbrace{\Pi\left(M''\right)\left(\left(p_1,\ldots,p_K,x\right), H^0 + \sum_{j=K+2}^{n} h_j(\infty)\right)}_{\xrightarrow[x\to\infty]{}\mu-1}\right).
\end{aligned}$$

Since, by assumption, $\bar{\mu}_j > \mu$, this implies that

$$\frac{\partial \Pi\left(M''\right)}{\partial p_{K+1}}\left(\left(p_1,\ldots,p_K,x\right), H^0 + \sum_{j=K+2}^{n} h_j(\infty)\right)$$

is strictly negative when x is high enough. Therefore, there exists $\tilde{p}_{K+1} \in \mathbb{R}_{++}$ such that

$$\begin{aligned}
\Pi\left(M\right)\left(\left(p_1,\ldots,p_K,\tilde{p}_{K+1},\infty,\ldots,\infty\right), H^0\right) &= \Pi\left(M''\right)\left(\left(p_1,\ldots,p_K,\tilde{p}_{K+1}\right), H^0 + \sum_{j=K+2}^{n} h_j(\infty)\right),\\
&> \Pi\left(M''\right)\left(\left(p_1,\ldots,p_K,\infty\right), H^0 + \sum_{j=K+2}^{n} h_j(\infty)\right),\\
&= \Pi\left(M\right)\left(\left(p_1,\ldots,p_K,\infty,\ldots,\infty\right), H^0\right),
\end{aligned}$$

which contradicts condition (b) in definition 8.

Therefore, $r_j(\mu, c_j) = \infty$ for all $j \geq K+1$, and we can rewrite equation (23) as

$$
\begin{aligned}
\mu &= 1 + \mu \frac{\sum_{k=1}^{K} \gamma_k \left(r_k(\mu, c_k)\right) + \overbrace{\sum_{k=K+1}^{n} \gamma_k \left(r_k(\mu, c_k)\right)}^{=0}}{\sum_{j=1}^{K} h_j \left(r_j(\mu, c_j)\right) + \sum_{j=K+1}^{n} \underbrace{h_j \left(r_j(\mu, c_j)\right)}_{=h_j(\infty)} + H^0}, \\
&= 1 + \mu \frac{\sum_{k=1}^{n} \gamma_k \left(r_k(\mu, c_k)\right)}{\sum_{j=1}^{n} h_j \left(r_j(\mu, c_j)\right) + H^0}.
\end{aligned}
$$

This concludes the proof. □

B.9 Proof of Lemma 9

Proof. Assume without loss of generality that $\bar{\mu}_1 \leq \bar{\mu}_2 \leq \ldots \bar{\mu}_n$. Put $S = \{\bar{\mu}_j\}_{1 \leq j \leq n}$. Set S contains $K \leq n$ distinct elements: $\hat{\mu}_1 < \hat{\mu}_2 < \ldots < \hat{\mu}_K$. We define the following function:

$$
\phi : \mu \in (1, \infty) \mapsto (\mu - 1) \left(\sum_{j=1}^{n} h_j \left(r_j(\mu, c_j)\right) + H^0 \right) - \mu \sum_{j=1}^{n} \gamma_j \left(r_j(\mu, c_j)\right).
$$

Note that μ solves equation (11) if and only if $\phi(\mu) = 0$. ϕ is continuous and

$$
\lim_{1^+} \phi = - \sum_{j=1}^{n} \gamma_j \left(r_j(1, c_j)\right) < 0.
$$

Next, we show that $\phi(\mu)$ is positive for μ high enough. Assume first that $\hat{\mu}_K < \infty$. Then, for all $\mu \geq \hat{\mu}_K$, $\phi(\mu) = (\mu - 1)H^0 > 0$ (recall that $h_j(\infty) = \gamma_j(\infty) = 0$ and $\lim_{\mu \to \infty} r_j(\mu, c_j) = \infty$ for all j). Next, assume instead that $\hat{\mu}_K = \infty$. Let i be the smallest $j \in \{1, \ldots, n\}$ such that $\bar{\mu}_j = \infty$. Then,

$$
\begin{aligned}
\lim_{\bar{\mu}} \phi &= \lim_{\mu \to \infty} \mu \left(\frac{\mu - 1}{\mu} \left(H^0 + \sum_{j=i}^{n} h_j \left(r_j(\mu, c)\right) \right) - \sum_{j=1}^{n} \gamma_j \left(r_j(\mu, c)\right) \right), \\
&= \left(\lim_{\mu \to \infty} \mu \right) \left(\lim_{\mu \to \infty} \left(\frac{\mu - 1}{\mu} \left(H^0 + \sum_{j=i}^{n} h_j \left(r_j(\mu, c)\right) \right) - \sum_{j=1}^{n} \gamma_j \left(r_j(\mu, c)\right) \right) \right), \\
&= \left(\lim_{\mu \to \infty} \mu \right) \left(H^0 + \sum_{j=i}^{n} \lim_{\infty} h_j \right), \\
&= \infty.
\end{aligned}
$$

It follows from the intermediate value theorem that ϕ has a zero. Moreover, all the zeros of ϕ are contained in $(1, \hat{\mu}_K)$.

Next, we claim that ϕ is strictly increasing. To see this, we divide interval $(1, \hat{\mu}_K)$ into intervals $(1, \hat{\mu}_1)$, $(\hat{\mu}_1, \hat{\mu}_2)$, ..., $(\hat{\mu}_{K-1}, \hat{\mu}_K)$. Pick one of these intervals, call it (a, b), and let i be the smallest $j \in \{1, \dots, n\}$ such that $\bar{\mu}_j = b$. Then, for all $\mu \in (a, b)$,

$$\phi(\mu) = (\mu - 1)\left(\sum_{j=i}^{n} h_j\left(r_j(\mu, c_j)\right) + H^0\right) - \mu \sum_{j=i}^{n} \gamma_j\left(r_j(\mu, c_j)\right),$$

and, by Lemma 7, ϕ is $\mathcal{C}^1$ on (a, b). $\phi'(\mu)$ is given by (we omit the arguments of functions to save space):

$$\begin{aligned}
\phi'(\mu) &= H^0 + \sum_{j=i}^{f}(h_j - \gamma_j) + (\mu - 1)\left(\sum_{j=i}^{n} \frac{\partial r_j}{\partial p_j} h'_j\right) - \mu\left(\sum_{j=i}^{n} \frac{\partial r_j}{\partial p_j}\gamma'_j\right), \\
&= H^0 + \sum_{j=i}^{n}(h_j - \gamma_j) + \sum_{j=i}^{n} \underbrace{\frac{\partial r_j}{\partial p_j}\left(\mu(-\gamma'_j) - (\mu - 1)(-h'_j)\right)}_{=\gamma_j \text{ by Lemma 7}}, \\
&= H^0 + \sum_{j=i}^{n} h_j > 0.
\end{aligned}$$

Therefore, ϕ is strictly increasing on intervals $(1, \hat{\mu}_1)$, $(\hat{\mu}_1, \hat{\mu}_2)$, ..., $(\hat{\mu}_{K-1}, \hat{\mu}_K)$. By continuity, it follows that ϕ is strictly increasing on $(1, \hat{\mu}_K)$. Therefore, equation (11) has a unique solution. □

B.10 Proof of Theorem 2

Proof. Let p^* be a solution of maximization problem (9). Such a p^* exists by Lemma 5. By Lemmas 6, p^* satisfies the generalized first-order conditions. Therefore, by Lemma 8, p^* satisfies the common ι-markup property, and the corresponding μ solves equation (11). By Lemma 9, this equation has a unique solution, which we denote μ^*. Therefore, $p^* = (r_i(\mu^*, c_i))_{1 \le i \le n}$, and maximization problem (9) has a unique solution. The fact that

$$\Pi(M)\left(p^*, H^0\right) = \mu^* - 1$$

follows immediately from Lemma 8.

Conversely, assume that the generalized first-order conditions hold at price vector $\tilde{p}$.

Then, by Lemmas 8 and 9, $\tilde{p} = (r_i(\mu^*, c_i))_{1 \le i \le n} = p^*$. It follows that generalized first-order conditions are sufficient for global optimality. $\square$

B.11 Proof of Proposition 3

Proof. Recall that μ solves the monopolist's first-order condition if and only if $\phi(\mu) = 0$, where

$$\phi(\mu) = (\mu - 1)\left(\sum_{j=1}^{n} h_j\left(r_j(\mu, c_j)\right) + H^0\right) - \mu \sum_{j=1}^{n} \gamma_j\left(r_j(\mu, c_j)\right) \quad \forall \mu > 1.$$

In the proof of Lemma 9, we established that ϕ is increasing in μ.

Suppose that H^0 increases. Since ϕ is increasing in H^0 and decreasing in μ, the optimal value of μ decreases. Therefore, all prices decrease (since r_k decreases with μ), the set of products with finite prices expands (or, at least, does not shrink), and the equilibrium value of H increases (because both H^0 and $\sum_{j \in \mathcal{N}} h_j(p_j)$ increase).

Next, suppose that c_j increases for some $j \in \mathcal{N}$. If p_j^* was initially infinite, then nothing changes, since the value of ϕ is unaffected. Assume instead that $p_j^* < \infty$. Note that[19]

$$\begin{aligned}
\frac{\partial \phi}{\partial c_j} &= \frac{\partial r_j}{\partial c_j}\left((\mu - 1) h_j'\left(r_j(\mu, c_j)\right) - \mu \gamma_j'\left(r_j(\mu, c_j)\right)\right), \\
&= \gamma_j\left(r_j(\mu, c_j)\right) \frac{\frac{\partial r_j}{\partial c_j}}{\frac{\partial r_j}{\partial \mu}}.
\end{aligned}$$

Since $\partial \phi / \partial \mu = H$, it follows from the implicit function theorem that

$$\frac{\partial \mu^*}{\partial c_j} = -\frac{\gamma_j}{H} \frac{\frac{\partial r_j}{\partial c_j}}{\frac{\partial r_j}{\partial \mu}},$$

which is strictly negative. Therefore, the optimal μ decreases and the set of products sold with finite prices expands as c_j increases. Since r_k decreases with μ, it follows that p_k^* decreases for every $k \neq j$. Finally, p_j^* increases, since

$$\begin{aligned}
\frac{\partial p_j^*}{\partial c_j} &= \frac{\partial r_j}{\partial c_j} + \frac{\partial \mu^*}{\partial c_j} \frac{\partial r_j}{\partial \mu}, \\
&= \frac{\partial r_j}{\partial c_j}\left(1 - \frac{\gamma_j}{H}\right),
\end{aligned}$$

[19] Cases where ϕ is not differentiable can be handled as in the proof of Lemma 9.

which is strictly positive since $h_j \geq \gamma_j$. $\square$

C Proofs for Section 4

C.1 Proof of Lemma 10

Proof. μ^f solves equation (12) if and only if

$$\psi(\mu^f, H) \equiv \mu^f \left(1 - \frac{\sum_{j\in f} \gamma_j \left(r_j(\mu^f, c_j)\right)}{H}\right) = 1.$$

Note that $\psi(., H)$ is continuous on $(1, \infty) \times \mathbb{R}_{++}$. In addition, if $\bar{\mu}^f < \infty$, then $\psi(\bar{\mu}^f, H) = \mu^f > 1$ for all $\mu^f \geq \bar{\mu}^f$. Therefore, equation 12 does not have a solution on $[\bar{\mu}^f, \infty)$.

Next, note that $\psi(1) = 1 - \frac{\sum_{j\in f} \gamma_j(r_j(1,c_j))}{H} < 1$. Suppose that there exists $\mu^f > 1$ such that $\sum_{j\in f} \gamma_j \left(r_j(\mu^f, c_j)\right) \geq H$. Then, since $\gamma_j(\cdot)$ is decreasing and $r_j(\cdot, c_j)$ is increasing (see Lemmas 4 and 7), $\sum_{j\in f} \gamma_j \left(r_j(\mu^{f\prime}, c_j)\right) \geq H$ for all $\mu^{f\prime} \leq \mu^f$. Define

$$\underline{\mu}^f = \sup\left\{\mu^f \in (1,\infty): \quad \sum_{j\in f} \gamma_j \left(r_j(\mu^f, c_j)\right) \geq H\right\},$$

and note that $\underline{\mu}^f < \infty$, since $\sum_{j\in f} \gamma_j \left(r_j(\mu^f, c_j)\right)$ goes to zero as μ^f goes to infinity. For all $\mu^f \leq \underline{\mu}^f$, $\psi(\mu^f, H) < 1$.

Therefore, equation 12 does not have solutions outside interval $(\underline{\mu}^f, \bar{\mu}^f)$ (put $\underline{\mu}^f = 1$ if $\sum_{j\in f} \gamma_j \left(r_j(\mu^f, c_j)\right) < H$ for all $\mu^f > 1$). In addition, $\psi\left(\underline{\mu}^f, H\right) < 1$ and $\lim_{\bar{\mu}^f} \psi > 1$. By the intermediate value theorem, equation (12) has a solution on interval $(\underline{\mu}^f, \bar{\mu}^f)$. Since ψ is strictly increasing in μ^f (see Lemmas 4 and 7), this solution is unique. This establishes the existence and uniqueness of $m^f(H)$. In addition, since ψ is strictly increasing in μ^f and H, m^f is strictly decreasing in H.

Let $H > 0$. Assume for a contradiction that m^f is not continuous at H. There exists $\varepsilon_0 > 0$ and $(H^n)_{n\in\mathbb{N}} \in \mathbb{R}_{++}^{\mathbb{N}}$ such that $H^n \underset{n\to\infty}{\longrightarrow} H$ and $\left|m^f\left(H^n\right) - m^f(H)\right| > \varepsilon_0$ for every $n \geq 0$. Let $0 < \eta < H$. Then, for high enough n, $H^n \in [H-\eta, H+\eta]$. By monotonicity of m^f, it follows that $m^f\left(H^n\right) \in \left[m^f\left(H+\eta\right), m^f\left(H-\eta\right)\right]$. Therefore, $\left(m^f\left(H^n\right)\right)_{n\geq 0}$ is bounded, and we can extract a convergent subsequence: there exists $\xi : \mathbb{N} \longrightarrow \mathbb{N}$ strictly increasing and $\mu^f \in (1, \bar{\mu}^f)$ such that $m^f\left(H^{\xi(n)}\right) \underset{n\to\infty}{\longrightarrow} \mu^f$. Since $m^f\left(H^n\right)$ is bounded away from $m^f(H)$, it follows that $\mu \neq m^f(H)$. By definition of m^f, for every $n \geq 0$, $\psi\left(m^f\left(H^{\xi(n)}\right), H^{\xi(n)}\right) = 1$. Since ψ is continuous, we can take limits, and obtain that $\psi(\mu, H) = 1$. By uniqueness, it follows that $\mu = m^f(H)$, which is a contradiction. Therefore, m^f is continuous.

Finally, we study the differentiability properties of m^f. Let $H^0 > 0$ such that $m^f(H^0) \neq \bar{\mu}_k$ for all $k \in f$, and choose $\varepsilon > 0$ such that $\big(m^f(H^0) - \varepsilon, m^f(H^0) + \varepsilon\big) \cap \{\bar{\mu}_k\}_{k \in f} = \emptyset$ and $m^f(H^0) > \underline{\mu}^f + \varepsilon$. We introduce the following notation: for every $H > 0$, put

$$\hat{f} = \left\{k \in f : \quad m^f(H^0) < \bar{\mu}_k\right\}.$$

Note that if $\hat{f}$ were empty, then $\psi\left(m^f(H^0), H^0\right)$ would be equal to $m^f(H^0) > 1$, a contradiction. Define

$$\hat{\psi} : \ (\mu^f, H) \in \big(m^f(H^0) - \varepsilon, m^f(H^0) + \varepsilon\big) \times \mathbb{R}_{++} \mapsto \mu^f \left(1 - \frac{\sum_{j \in \hat{f}} \gamma_j \left(r_j(\mu^f, c_j)\right)}{H}\right),$$

and note that $\hat{\psi}(\mu^f, H) = \psi(\mu^f, H)$ for all $(\mu^f, H) \in \big(m^f(H^0) - \varepsilon, m^f(H^0) + \varepsilon\big)$. In addition, $\hat{\psi}\left(m^f(H^0), H^0\right) = 1$, $\hat{\psi}$ is $\mathcal{C}^1$,

$$\begin{aligned}\frac{\partial \hat{\psi}}{\partial \mu^f}\left(m^f(H^0), H^0\right) &= 1 - \frac{\sum_{k \in \hat{f}} \gamma_k}{H^0} + m^f(H^0)\left(-\frac{\sum_{k \in \hat{f}} \frac{\partial r_k}{\partial \mu^f} \gamma_k'}{H^0}\right), \\ &= \frac{1}{m^f(H^0)} + \left(m^f(H^0) - 1\right)\frac{\sum_{k \in \hat{f}} \frac{\partial r_k}{\partial \mu^f}\left(-\gamma_k'\right)}{H^0},\end{aligned}$$

which is strictly positive, and

$$\begin{aligned}\frac{\partial \hat{\psi}}{\partial H}\left(m^f(H^0), H^0\right) &= m^f(H^0)\frac{\sum_{k \in \hat{f}} \gamma_k}{\left(H^0\right)^2}, \\ &= \frac{m^f(H^0) - 1}{H^0}.\end{aligned}$$

By the implicit function theorem, there exist $\eta > 0$ and a $\mathcal{C}^1$ function

$$\hat{m}^f : (H^0 - \eta, H^0 + \eta) \longrightarrow \big(m^f(H^0) - \varepsilon, m^f(H^0) + \varepsilon\big)$$

such that $\hat{\psi}\left(\hat{m}^f(H), H\right) = 1$ for all $H \in (H^0 - \eta, H^0 + \eta)$. In addition,

$$\hat{m}^{f\prime}(H^0) = -\frac{1}{H^0}\frac{m^f(H^0)\left(m^f(H^0) - 1\right)}{1 + m^f(H^0)\left(m^f(H^0) - 1\right)\frac{\sum_{k \in \hat{f}} \frac{\partial r_k}{\partial \mu^f}\left(-\gamma_k'\right)}{\sum_{k \in \hat{f}} \gamma_k}},$$

which is indeed strictly negative. Since functions ψ and $\hat{\psi}$ coincide on $\big(m^f(H^0) - \varepsilon, m^f(H^0) + \varepsilon\big) \times$

$\mathbb{R}_{++}$, and by uniqueness of m^f, it follows that m^f and $\hat{m}^f$ coincide on $(H^0-\eta, H^0+\eta)$. Therefore, m^f is $\mathcal{C}^1$ in an open neighborhood of H^0, and $m^{f\prime}(H^0) = \hat{m}^{f\prime}(H^0)$. $\square$

C.2 Proof of Lemma 11

Proof. By Lemmas 7 and 10, Ω is continuous on $\mathbb{R}_{++}$. In addition, when H goes to ∞, the numerator of Ω goes to

$$\sum_{f\in\mathcal{F}}\sum_{k\in f} h_k\left(r_k(1)\right),$$

which is finite. Therefore, $\lim_\infty \Omega = 0$. If we show that Ω is strictly greater than 1 in the neighborhood of 0^+, then we can apply the intermediate value theorem to obtain the existence of H^*.

Assume first that there exists $j \in \mathcal{N}$ such that $\lim_\infty h_j = l > 0$. Since h_j is decreasing, $h_j(x) \geq l$ for all $x > 0$. Let $f \in \mathcal{F}$ such that $j \in f$. Then, for all $H > 0$,

$$\begin{aligned}\Omega(H) &\geq \frac{h_j\left(r_j\left(m^f(H)\right)\right)}{H},\\ &\geq \frac{l}{H} \underset{H\to 0^+}{\longrightarrow} \infty.\end{aligned}$$

Therefore, $\lim_{0^+} \Omega = \infty > 1$.

Next, assume that $h_k(x) \underset{x\to\infty}{\longrightarrow} 0$ for all $k \in \mathcal{N}$. Let $f \in \mathcal{F}$, we define threshold $\underline{H}^{f\prime} > 0$ as follows. If $\bar{\mu}_k = \bar{\mu}^f$ for all $k \in f$, then put $\underline{H}^{f\prime} = 1$. If $\bar{\mu}_k < \bar{\mu}^f$ for some $k \in f$, then, since $\lim_{0^+} m^f = \bar{\mu}^f$, there exists $\hat{H} > 0$ such that $m^f(H) > \max\left((\bar{\mu}_k)_{k\in f} \setminus \{\bar{\mu}^f\}\right)$ whenever $H < \hat{H}$. In this case, put $\underline{H}^{f\prime} \equiv \hat{H}$. Having done that for every $f \in \mathcal{F}$, put $\underline{H}' = \min_{f\in\mathcal{F}} \underline{H}^{f\prime}$. Then, for every $H \in (0, \underline{H}')$,

$$\Omega(H) = \frac{1}{H}\sum_{f\in\mathcal{F}}\sum_{\substack{j\in f\\ \bar{\mu}_j=\bar{\mu}^f}} h_j\left(r_j\left(m^f(H), c_j\right)\right).$$

We partition the set of firms into two sets: $\mathcal{F}'$ and $\mathcal{F}''$, where

$$\mathcal{F}' = \left\{f \in \mathcal{F} :\ \bar{\mu}^f = \infty\right\}$$

and $\mathcal{F}'' = \mathcal{F}\setminus\mathcal{F}'$.

Let $f \in \mathcal{F}''$. Then, by Lemma 4-(vi), $\lim_\infty \rho_k = \frac{\bar{\mu}^f}{\bar{\mu}^f-1}$ for every $k \in f$ such that $\bar{\mu}_k = \bar{\mu}^f$. In addition, by Lemmas 7 and 10, for every $k \in f$, $r_k\left(m^f(H)\right) \underset{H\to 0^+}{\longrightarrow} \infty$. Therefore, there exists $\underline{H}^{f\prime\prime} > 0$ such that

$$\rho_k\left(r_k\left(m^f(H)\right)\right) \geq \frac{\bar{\mu}^f}{\bar{\mu}^f - 1}\left(1 - \frac{1}{2|\mathcal{F}|}\right), \quad \forall H < \underline{H}^{f\prime\prime},\ \forall k \in f \text{ s.t. } \bar{\mu}_k = \bar{\mu}^f.$$

Let $\underline{H}'' = \min_{f\in\mathcal{F}''} \underline{H}^{f\prime\prime}$ (or any strictly positive real number if $\mathcal{F}''$ is empty), and $\underline{H} = \min\left(\underline{H}', \underline{H}''\right)$. For every $H < \underline{H}$,

$$\begin{aligned}
\Omega(H) &= \frac{1}{H}\left(\sum_{f\in\mathcal{F}'}\sum_{\substack{k\in f\\ \bar{\mu}_k=\bar{\mu}^f}} h_k\left(r_k\left(m^f(H), c_k\right)\right) + \sum_{f\in\mathcal{F}''}\sum_{\substack{k\in f\\ \bar{\mu}_k=\bar{\mu}^f}} h_k\left(r_k\left(m^f(H), c_k\right)\right)\right), \\
&\geq \frac{1}{H}\left(\sum_{f\in\mathcal{F}'}\sum_{\substack{k\in f\\ \bar{\mu}_k=\bar{\mu}^f}} \gamma_k\left(r_k\left(m^f(H), c_k\right)\right) + \sum_{f\in\mathcal{F}''}\sum_{\substack{k\in f\\ \bar{\mu}_k=\bar{\mu}^f}} \gamma_k\left(r_k\left(m^f(H), c_k\right)\right)\rho_k\left(r_k\left(m^f(H)\right)\right)\right), \\
&\geq \sum_{f\in\mathcal{F}'}\frac{1}{H}\sum_{\substack{k\in f\\ \bar{\mu}_k=\bar{\mu}^f}} \gamma_k\left(r_k\left(m^f(H), c_k\right)\right) + \sum_{f\in\mathcal{F}''}\frac{\bar{\mu}^f}{\bar{\mu}^f - 1}\left(1 - \frac{1}{2|\mathcal{F}|}\right)\frac{1}{H}\sum_{\substack{k\in f\\ \bar{\mu}_k=\bar{\mu}^f}} \gamma_k\left(r_k\left(m^f(H), c_k\right)\right), \\
&= \sum_{f\in\mathcal{F}'}\frac{\sum_{k\in f}\gamma_k\left(r_k\left(m^f(H), c_k\right)\right)}{H} + \sum_{f\in\mathcal{F}''}\frac{\bar{\mu}^f}{\bar{\mu}^f - 1}\left(1 - \frac{1}{2|\mathcal{F}|}\right)\frac{\sum_{k\in f}\gamma_k\left(r_k\left(m^f(H), c_k\right)\right)}{H}, \\
&= \sum_{f\in\mathcal{F}'}\frac{m^f(H) - 1}{m^f(H)} + \sum_{f\in\mathcal{F}''}\frac{m^f(H) - 1}{m^f(H)}\frac{\bar{\mu}^f}{\bar{\mu}^f - 1}\left(1 - \frac{1}{2|\mathcal{F}|}\right), \text{ using equation (12).}
\end{aligned}$$

When H goes to 0^+, the right-hand side term on the last line goes to

$$|\mathcal{F}'| + |\mathcal{F}''|\left(1 - \frac{1}{2|\mathcal{F}|}\right) \geq |\mathcal{F}| - \frac{1}{2},$$

which is strictly greater than 1. Therefore, $\Omega(H) > 1$ when H is small enough. This concludes the proof. □

C.3 Proof of Proposition 4

Proof. The first part of the proposition follows immediately from equation (2), Theorem 3 and Lemma 10.

Next, we prove that largest and smallest (in terms of the value of H) equilibria exist. If there is a finite number of equilibrium aggregators, then this is trivial. Next, assume that there is an infinite number of equilibria. We have shown in the proof of Lemma 11 that $\Omega(H) > 1$ for H low enough and $\Omega(H) < 1$ for H high enough. Therefore, the set of equilibrium aggregators is contained in a closed interval $[\underline{H}, \overline{H}]$, with $\underline{H} > 0$. Put

$$\overline{H}^* \equiv \sup\left\{H \in [\underline{H}, \overline{H}] : \quad \Omega(H) = 1\right\}.$$

Let $(H^n)_{n\geq 0}$ be a sequence such that $\Omega(H^n) = 1$ for all n and $H^n \underset{n\to\infty}{\longrightarrow} \overline{H}^*$. Since Ω is continuous on $[\underline{H}, \overline{H}]$, we can take limits and obtain that $\Omega(\overline{H}^*) = 1$. Therefore,

$$\overline{H}^* = \max\left\{H \in [\underline{H}, \overline{H}] : \quad \Omega(H) = 1\right\}$$

is the highest equilibrium aggregator level. The existence of a lowest equilibrium aggregator follows from the same line of argument. □

C.4 Proof of Proposition 5

Proof. Let $H^0 > 0$. Given outside option $H^0 \geq 0$, $H > 0$ is an equilibrium aggregator level if and only if $\Omega(H, H^0) = 1$, where

$$\Omega(H, H^0) = \frac{H^0 + \sum_{f\in\mathcal{F}} \sum_{j\in f} h_j\left(r_j\left(m^f(H), c_j\right)\right)}{H}.$$

Let $H^{0\prime} > H^0 \geq 0$, and note that $\Omega(H, H^{0\prime}) > \Omega(H, H^0)$ for all $H > 0$. Let $\overline{H}$ and $\underline{H}$ (resp. $\overline{H}'$ and $\underline{H}'$) be the highest and lowest equilibrium aggregator levels when the outside option is H^0 (resp. $H^{0\prime}$). We know from the proof of Lemma 11 that $\Omega(H, H^0) \geq 1$ for all $H \leq \underline{H}$. Therefore, for all $H \leq \underline{H}$,

$$\Omega(H, H^{0\prime}) > \Omega(H, H^0) \geq 1.$$

It follows that, when the outside option is $H^{0\prime}$, there is no equilibrium aggregator level weakly below $\underline{H}$. Therefore, $\underline{H} < \underline{H}'$. The fact that $\overline{H} < \overline{H}'$ follows from the same line of argument. This establishes point (iii) in the proposition.

Points (i), (ii) and (iv) follow from the fact that a firm's profit is equal to its ι-markup minus one (Theorem 3), m^f is decreasing (Lemma 10), and $r_j(\cdot, c_j)$ is increasing (Lemma 7). □

C.5 Proof of Proposition 7

Proof. It is straightforward to show, using standard differential equation techniques, that $-x\frac{h''(x)}{h'(x)} = \widetilde{\iota}(x)$ for all x if and only if $h = h^{\alpha,\beta}$ for some $\alpha \neq 0$ and $\beta \in \mathbb{R}$. All we need to do now is look for the set of pairs (α, β) such that $h^{\alpha,\beta} \in \mathcal{H}^\iota$. Note first that $h^{\alpha,\beta}$ cannot be in $\mathcal{H}^\iota$ if $\alpha \leq 0$. In addition, if $h^{\alpha,\beta} \in \mathcal{H}^\iota$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$, then $h^{\alpha',\beta} \in \mathcal{H}^\iota$ for all $\alpha' > 0$. Therefore, we can set α equal to 1 without loss of generality.

The problem now boils down to finding the set of β's such that $h^\beta \equiv h^{1,\beta}$ is strictly positive, decreasing and log-convex. For all $\beta \in \mathbb{R}$,

$$h^{\beta\prime}(x) = -\exp\left(-\int_1^x \frac{\iota(u)}{u} du\right),$$

which is strictly negative for all x. So the fact that h^β has to be decreasing does not impose any constraint on β.

Next, we show that $\lim_\infty h^0$ (which exists, since h^0 is monotone) is finite and strictly negative. It is trivial to see that this limit is strictly negative. Let $x^0 > 0$ such that $\widetilde{\iota}(x^0) > 1$. Proving that $\lim_\infty h^0$ is finite is equivalent to showing that function $t \mapsto \exp\left(-\int_1^t \frac{\widetilde{\iota}(u)}{u} du\right)$ is integrable on $[x^0, \infty)$. For every $t \geq x^0$,

$$\begin{aligned}
\exp\left(-\int_1^t \frac{\widetilde{\iota}(u)}{u} du\right) &\leq \exp\left(-\int_1^{x^0} \frac{\widetilde{\iota}(u)}{u} du - \int_{x^0}^t \frac{\iota(x^0)}{u} du\right), \\
&= \exp\left(-\int_1^{x^0} \frac{\widetilde{\iota}(u)}{u} du\right) \exp\left(-\widetilde{\iota}(x^0) \log\left(\frac{t}{x^0}\right)\right), \\
&= \exp\left(-\int_1^{x^0} \frac{\widetilde{\iota}(u)}{u} du\right) \left(\frac{t}{x^0}\right)^{-\widetilde{\iota}(x^0)}.
\end{aligned} \tag{24}$$

The last expression is integrable on $[x^0, \infty)$, since $\widetilde{\iota}(x^0) > 1$. Therefore, $t \mapsto \exp\left(-\int_1^t \frac{\widetilde{\iota}(u)}{u} du\right)$ is integrable on $[x^0, \infty)$ and $\hat{\beta} \equiv \lim_\infty h^0$ is finite and strictly negative. It follows that function h^β is strictly positive if and only if $\beta \geq \hat{\beta}$.

Let $\beta \geq \hat{\beta}$. Then,

$$\frac{d}{dx} \frac{h^{\beta\prime}(x)}{h(x)} = \frac{h^{\beta\prime\prime}(x) h^\beta(x) - \left(h^{\beta\prime}(x)\right)^2}{h^\beta(x)^2} = \frac{1}{x} \frac{-h^{\beta\prime}(x)}{h^\beta(x)} \left(\widetilde{\iota}(x) - x \frac{-h^{\beta\prime}(x)}{h^\beta(x)}\right).$$

Therefore, h^β is log-convex if and only if $\widetilde{\iota}(x) \geq x \frac{-h^{\beta\prime}(x)}{h^\beta(x)}$ for all $x > 0$. Since $h^\beta(x)$ increases with β and $h^{\beta\prime}(x)$ does not depend on β, it follows that, if h^β is log-convex and $\beta' > \beta$, then $h^{\beta'}$ is also log-convex.

Moreover, using (24), we see that, for every $x > \hat{x}$,

$$\begin{aligned}-xh^{\beta\prime}(x) &\leq x\exp\left(-\int_1^{x^0}\frac{\widetilde{\iota}(u)}{u}du\right)\left(\frac{x}{x^0}\right)^{-\widetilde{\iota}(x^0)},\\ &= \exp\left(-\int_1^{x^0}\frac{\widetilde{\iota}(u)}{u}du\right)\left(x^0\right)^{\widetilde{\iota}(x^0)}x^{1-\widetilde{\iota}(x^0)} \underset{x\to\infty}{\longrightarrow} 0,\end{aligned}$$

where the last line follows from the fact that $\widetilde{\iota}(x^0) > 1$.

Let $\beta > \hat{\beta}$. Then, $\lim_\infty h^\beta > 0$, and therefore, $\lim_{x\to\infty} x\frac{-h^{\beta\prime}(x)}{h^\beta(x)} = 0$. Since $\lim_\infty \widetilde{\iota} > 0$, it follows that there exists $\hat{x}$ such that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta\prime}(x)}{h^\beta(x)}$ whenever $x \geq \hat{x}$. In addition, since h^β increases with β, we also have that, for all $\beta' \geq \beta$, $\widetilde{\iota}(x) \geq x\frac{-h^{\beta'}{}'(x)}{h^{\beta'}(x)}$ whenever $x \geq \hat{x}$.

Next, we turn our attention to $\lim_{x\to 0^+}\frac{-xh^{\beta\prime}(x)}{h^\beta(x)}$. Note that

$$\frac{d}{dx}(-xh^{\beta\prime}(x)) = -h'(x)\left(1-\widetilde{\iota}(x)\right).$$

Therefore, if $\lim_{0^+}\widetilde{\iota} > 1$ or $\lim_{0^+}\widetilde{\iota} < 1$, then $x \mapsto (-xh^{\beta\prime}(x))$ is monotone in the neighborhood of zero, and $\lim_{x\to 0^+} -xh^{\beta\prime}(x)$ exists. If instead $\lim_{0^+}\widetilde{\iota} = 1$, then either there exists $\varepsilon > 0$ such that $\widetilde{\iota}(x) < 1$ for all $x \in (0,\varepsilon)$, or $\widetilde{\iota}(x) > 0$ for all $x > 0$. In both cases, $x \mapsto (-xh^{\beta\prime}(x))$ is still monotone in the neighborhood of zero, and $\lim_{x\to 0^+} -xh^{\beta\prime}(x)$ therefore exists. Note that $\lim_{0^+} h^\beta$ trivially exists, since h^β is monotone.

We distinguish two cases. Suppose first that $\lim_{x\to 0^+} -xh^{\beta\prime}(x)$ is finite, and denote this limit by l. If $\lim_{0^+} h^\beta = \infty$, then

$$\widetilde{\iota}(x) - x\frac{-h^{\beta\prime}(x)}{h^\beta(x)} \underset{x\to 0^+}{\longrightarrow} \lim_{0^+}\widetilde{\iota} > 0.$$

Therefore, there exists $\tilde{x} > 0$ such that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta\prime}(x)}{h^\beta(x)}$ for all $x \in (0,\tilde{x}]$. In addition, the inequality also holds if we replace β by $\beta' \geq \beta$. If, instead, $\lim_{0^+} h^\beta < \infty$, then

$$\widetilde{\iota}(x) - x\frac{-h^{\beta\prime}(x)}{h^\beta(x)} \underset{x\to 0^+}{\longrightarrow} \underbrace{\lim_{0^+}\widetilde{\iota}}_{>0} - \frac{l}{\lim_{0^+} h^{\hat{\beta}} + \beta - \hat{\beta}},$$

which is strictly positive for β high enough. For such a high enough β, we again get the existence of an $\tilde{x}$ such that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta\prime}(x)}{h^\beta(x)}$ for all $x \in (0,\tilde{x}]$.

Next, assume instead that $\lim_{x\to 0^+} -xh^{\beta\prime}(x) = \infty$. Let $M > 0$. There exists $\varepsilon > 0$ such that $h^{\beta\prime}(x) < -M/x$ whenever $x \leq \varepsilon$. Integrating this inequality between x and ε, we see that

$$h^\beta(x) > h^\beta(\varepsilon) + M \log \frac{\varepsilon}{x} \underset{x\to 0^+}{\longrightarrow} \infty.$$

Therefore, $\lim_{0^+} h^\beta = \infty$, and we can apply l'Hospital's rule:

$$\lim_{x\to 0^+} \frac{-xh^{\beta\prime}(x)}{h^\beta(x)} = \lim_{x\to 0^+} \frac{-xh^{\beta\prime\prime}(x) - h^{\beta\prime}(x)}{h^{\beta\prime}(x)} = \lim_{0^+} \widetilde{\iota} - 1.$$

Therefore,

$$\widetilde{\iota}(x) - x\frac{-h^{\beta\prime}(x)}{h^\beta(x)} \underset{x\to 0^+}{\longrightarrow} 1 > 0.$$

Again, this gives us the existence of an $\tilde{x}$ such that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta\prime}(x)}{h^\beta(x)}$ for all $x \in (0, \tilde{x}]$.

To summarize, we have found a $\beta > \hat{\beta}$ and two strictly positive reals $\tilde{x}$ and $\hat{x}$ such that for all $\beta' \geq \beta$, $\widetilde{\iota}(x) \geq x\frac{-h^{\beta'}{}'(x)}{h^{\beta'}(x)}$ whenever $x \geq \hat{x}$ or $x \leq \tilde{x}$. If $\tilde{x} \geq \hat{x}$, then we are done: there exists $\beta > \hat{\beta}$ such that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta\prime}(x)}{h^\beta(x)}$ for all $x > 0$. Assume instead that $\tilde{x} < \hat{x}$. Let $x_1 \in \arg\min_{x\in[\tilde{x},\hat{x}]} \widetilde{\iota}(x)$ and $x_2 \in \arg\max_{x\in[\tilde{x},\hat{x}]} x\frac{-h^{\beta\prime}(x)}{h^\beta(x)}$. Note that x_1 and x_2 exist, since $[\tilde{x}, \hat{x}]$ is compact and the functions being optimized are continuous. If $\widetilde{\iota}(x_1) \geq x_2\frac{-h^{\beta\prime}(x_2)}{h^\beta(x_2)}$, then we are done, since this would imply that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta\prime}(x)}{h^\beta(x)}$ for all $x \in [\tilde{x}, \hat{x}]$. Suppose instead that $\widetilde{\iota}(x_1) < x_2\frac{-h^{\beta\prime}(x_2)}{h^\beta(x_2)}$. Since $h^\beta(x_2) \underset{\beta\to\infty}{\longrightarrow} \infty$ and $\widetilde{\iota}(x_1) > 0$, there exists $\beta' > \beta$ such that $x_2\frac{-h^{\beta'}{}'(x_2)}{h^{\beta'}(x_2)} < \widetilde{\iota}(x_1)$. By definition of x_1 and x_2, and since h^β increases with β, we can conclude that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta'}{}'(x)}{h^{\beta'}(x)}$ for all $x \in [\tilde{x}, \hat{x}]$. It follows that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta'}{}'(x)}{h^{\beta'}(x)}$ for all $x > 0$.

This implies that set

$$B \equiv \left\{\beta \geq \hat{\beta} : \quad h^\beta \text{ is log-convex}\right\}$$

is non-empty. In addition, we also know that if $\beta' > \beta$ and $\beta \in B$, then $\beta' \in B$. Put $\underline{\beta} = \inf B$. Assume for a contradiction that $\underline{\beta} \notin B$. Then, there exists $x > 0$ such that

$$\widetilde{\iota}(x) < x\frac{-h^{\underline{\beta}\prime}(x)}{h^{\underline{\beta}}(x)}.$$

Then, by continuity of h^β in β, there exists $\beta' > \underline{\beta}$ such that

$$\widetilde{\iota}(x) < x\frac{-h^{\beta'}{}'(x)}{h^{\beta'}(x)}.$$

But then, $\beta' \in B$ and $h^{\beta'}$ is not log-convex, a contradiction. Therefore, the set of β's such that h^β is positive, decreasing and log-convex is $[\underline{\beta}, \infty)$. $\square$

D Proofs for Section 4.3

In this section, we fix a pricing game $\left((h_j)_{j\in\mathcal{N}}, \mathcal{F}, (c_j)_{j\in\mathcal{N}}\right)$.

D.1 Preliminaries

In this section, we prove several technical lemmas, which will allow us to derive firm-level conditions for equilibrium uniqueness.

We introduce the following notation. For every $f \in \mathcal{F}$, for every $\mu^f \in (1, \bar{\mu}^f)$,

$$\omega^f = \frac{\mu^f - 1}{\mu^f},$$
$$\bar{\omega}^f = \lim_{\mu^f \to \bar{\mu}^f} \frac{\mu^f - 1}{\mu^f},$$

and for every $k \in \mathcal{N}$, for every $x > \underline{p}_k$,

$$\chi_k(x) = \frac{\iota_k(x) - 1}{\iota_k(x)}.$$

The following lemma is useful to understand our uniqueness conditions:

Lemma C. *For every $f \in \mathcal{F}$, $\omega^f \in (0, \bar{\omega}^f)$ and $k \in f$:*

- *For every x such that $\chi_k(x) > \omega^f$, $1 - \omega^f \theta_k(x) > 0$.*
- *In particular, for every $c_k > 0$, for every $x \geq r_k\left(\frac{1}{1-\omega^f}, c_k\right)$, $1 - \omega^f \theta_k(x) > 0$.*
- *In particular, for every $x > \underline{p}_k$, $\chi_k(x)\theta_k(x) \leq 1$.*

Proof. Let $f \in \mathcal{F}$, $k \in f$, $\omega^f \in \left(0, \bar{\omega}^f\right)$, and x such that $\chi_k(x) > \omega^f$. Put $\mu^f = \frac{1}{1-\omega^f}$. Then, $\iota_k(x) > \mu^f$. Therefore, there exists $c > 0$ such that $\nu_k(x, c) = \mu^f$. We know from Lemma 7 that

$$\begin{aligned}\frac{\partial r_k}{\partial \mu^f}(\mu^f, c) &= \frac{\gamma_k(r_k(\mu^f, c_k))}{\mu^f\left(-\gamma_k'\left(r_k(\mu^f, c_k)\right)\right) - (\mu^f - 1)\left(-h_k'\left(r_k(\mu^f, c_k)\right)\right)},\\ &= \frac{\gamma_k(x)}{-\gamma_k'(x)\mu^f}\frac{1}{1 - \omega^f\theta_k(x)} > 0.\end{aligned}$$

In addition, by Lemmas 4-(iii) and 7, $\gamma_k'(x) < 0$. Therefore, $1 - \omega^f \theta_k(x) > 0$. This establishes the first bullet point in the statement of the lemma.

Next, let $c > 0$ and $x \geq r_k(\mu^f, c)$. Then, since $\nu_k(., c)$ is increasing, $\nu_k(x, c) \geq \mu^f$. Since $c > 0$, it follows that $\iota_k(x) > \mu^f$, and that $\chi_k(x) > \omega^f$. It follows from the first part of the lemma that $1 - \omega^f \theta_k(x) > 0$.

Finally, let $x > \underline{p}_k$. Put $\omega^f = \chi_k(x)$. Then, for every y such that $\chi_k(y) > \omega^f$, $1 - \omega^f \theta_k(y) > 0$. By monotonicity of χ_k, this implies that, for every $y > x$, $1 - \chi_k(x)\theta_k(y) > 0$. Therefore, by continuity of θ_k, $\chi_k(x)\theta_k(x) \leq 1$. $\square$

Lemma D. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. If, for every $f \in \mathcal{F}$,*

$$\forall \omega^f \in (0, \bar{\omega}^f), \ \left(\sum_{k \in f} \frac{\omega^f \theta_k}{1 - \omega^f \theta_k} \gamma_k \right) \left(\frac{1}{\sum_{k \in f} h_k} - \frac{\omega^f}{\sum_{k \in f} \gamma_k} \right) < 1, \tag{25}$$

where, for every k, functions θ_k, γ_k and h_k are all evaluated at point $p_k = r_k \left(\frac{1}{1-\omega^f}, c_k \right)$, then pricing game $\left((h_j)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}} \right)$ has a unique equilibrium.

Proof. By Theorem 3, $\left((h_j)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}} \right)$ has a pricing equilibrium. To prove that there is only one equilibrium, we show that $\Omega(.)$ is strictly decreasing. Let $H > 0$, and, for every $f \in \mathcal{F}$, $\mu^f = m^f(H)$ and $\omega^f = \frac{\mu^f - 1}{\mu^f}$. Then,

$$\begin{aligned} H^2 \Omega'(H) &= H \sum_{f \in \mathcal{F}} m^{f\prime(H)} \sum_{k \in f} r'_k \left(\mu^f \right) h'_k \left(r_k \left(\mu^f \right) \right) - \sum_{f \in \mathcal{F}} \sum_{k \in \mathcal{N}} h_k \left(r_k \left(\mu^f \right) \right), \\ &= \sum_{f \in \mathcal{F}} \left(\frac{\mu^f (\mu^f - 1)}{1 + \mu^f (\mu^f - 1) \frac{\sum_{k \in f} r'_k (-\gamma'_k)}{\sum_{k \in f} \gamma_k}} \left(\sum_{k \in f} r'_k \left(-h'_k \right) \right) - \sum_{k \in f} h_k \right), \text{ by Lemma 10.} \end{aligned}$$

Therefore, a sufficient condition for this derivative to be strictly negative is that, for all $f \in \mathcal{F}$,

$$\frac{\mu^f (\mu^f - 1) \frac{\sum_{k \in f} r'_k (-h'_k)}{\sum_{k \in f} h_k}}{1 + \mu^f (\mu^f - 1) \frac{\sum_{k \in f} r'_k (-\gamma'_k)}{\sum_{k \in f} \gamma_k}} < 1. \tag{26}$$

Let $f \in \mathcal{F}$. Then,

$$\begin{aligned} (26) &\iff (\mu^f - 1) \left(\frac{\sum_{k \in f} \mu^f r'_k \left(-h'_k \right)}{\sum_{k \in f} h_k} - \frac{\sum_{k \in f} \mu^f r'_k \left(-\gamma'_k \right)}{\sum_{k \in f} \gamma_k} \right) < 1, \\ &\iff (\mu^f - 1) \left(\frac{\sum_{k \in f} \mu^f \frac{\gamma_k (-h'_k)}{\mu^f (-\gamma'_k) - (\mu^f - 1)(-h'_k)}}{\sum_{k \in f} h_k} - \frac{\sum_{k \in f} \mu^f \frac{\gamma_k (-\gamma'_k)}{\mu^f (-\gamma'_k) - (\mu^f - 1)(-h'_k)}}{\sum_{k \in f} \gamma_k} \right) < 1, \end{aligned}$$

$$\iff (\mu^f - 1)\left(\frac{\sum_{k\in f}\frac{\theta_k}{1-\omega^f\theta_k}\gamma_k}{\sum_{k\in f} h_k} - \frac{\sum_{k\in f}\frac{1}{1-\omega^f\theta_k}\gamma_k}{\sum_{k\in f}\gamma_k}\right) < 1,$$

$$\iff (\mu^f - 1)\left(\frac{\sum_{k\in f}\frac{\theta_k}{1-\omega^f\theta_k}\gamma_k}{\sum_{k\in f} h_k} - 1 - \frac{\sum_{k\in f}\frac{\omega^f\theta_k}{1-\omega^f\theta_k}\gamma_k}{\sum_{k\in f}\gamma_k}\right) < 1,$$

$$\iff (\mu^f - 1)\left(-1 + \sum_{k\in f}\frac{\theta_k}{1-\omega^f\theta_k}\gamma_k\left(\frac{1}{\sum_{k\in f} h_k} - \frac{\omega^f}{\sum_{k\in f}\gamma_k}\right)\right) < 1,$$

$$\iff \left(\sum_{k\in f}\frac{\omega^f\theta_k}{1-\omega^f\theta_k}\gamma_k\right)\left(\frac{1}{\sum_{k\in f} h_k} - \frac{\omega^f}{\sum_{k\in f}\gamma_k}\right) < 1,$$

where, for every $k \in f$, functions θ_k, γ_k and h_k are evaluated at point $p_k = r_k(\mu^f) = r_k\left(\frac{1}{1-\omega^f}\right)$. Since condition (25) holds by assumption, Ω is strictly decreasing. Therefore, the pricing game has a unique equilibrium. $\square$

Lemma E. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Let $(\underline{c}_k)_{k\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$. If, for every $f \in \mathcal{F}$,*

$$\begin{gathered}\forall \omega^f \in (0, \bar{\omega}^f),\ \forall\, (x_k)_{k\in f} \in \prod_{k\in f}\left[r_k\left(\frac{1}{1-\omega^f}, \underline{c}_k\right), \infty\right), \\ \left(\sum_{k\in f}\frac{\omega^f\theta_k(x_k)}{1-\omega^f\theta_k(x_k)}\gamma_k(x_k)\right)\left(\frac{1}{\sum_{k\in f} h_k(x_k)} - \frac{\omega^f}{\sum_{k\in f}\gamma_k(x_k)}\right) < 1,\end{gathered} \tag{27}$$

or, equivalently, if

$$\begin{gathered}\forall \omega^f \in (0, \bar{\omega}^f),\ \forall\, (x_k)_{k\in f} \in \prod_{k\in f}\left[r_k\left(\frac{1}{1-\omega^f}, \underline{c}_k\right), \infty\right), \\ \sum_{i,j\in f}\gamma_i(x_i)\gamma_j(x_j)\left(\omega^f\theta_i(x_i)\frac{1-\omega^f\rho_j(x_j)}{1-\omega^f\theta_i(x_i)} - \rho_i(x_i)\right) < 0,\end{gathered} \tag{28}$$

then pricing game $\left((h_j)_{j\in\mathcal{N}}, \mathcal{F}, (c_j)_{j\in\mathcal{N}}\right)$ has a unique equilibrium for every $(c_j)_{j\in\mathcal{N}} \in \prod_{j\in\mathcal{N}}[\underline{c}_j, \infty)$.

Proof. Assume that condition (27) holds, and let $(c_k)_{k\in\mathcal{N}} \in \prod_{k\in\mathcal{N}}[\underline{c}_k, \infty)$. We want to show that condition (25) holds, so let $f \in \mathcal{F}$, $\omega^f \in (0, \bar{\omega}^f)$ and $\mu^f = \frac{1}{1-\omega^f}$. Let $k \in f$ and $p_k = r_k(\mu^f, c_k)$. Since $c_k \geq \underline{c}_k$ and r_k is increasing in its second argument, it follows that $p_k \geq r_k(\mu^f, \underline{c}_k)$. Therefore, $(p_k)_{k\in f} \in \prod_{k\in f}\left[r_k\left(\mu^f, \underline{c}_k\right), \infty\right)$. It follows that condition (25) holds. By Lemma D, pricing game $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$ has a unique equilibrium.

Finally, we show that conditions (27) and (28) are equivalent. Let $f \in \mathcal{F}$, $\omega^f \in (0, \bar{\omega}^f)$, and $(x_k)_{k\in f} \in \prod_{k\in f} \left[r_k\left(\frac{1}{1-\omega^f}, \underline{c}_k\right), \infty\right)$. Then,

$$\begin{aligned}
&\left(\sum_{k\in f} \frac{\omega^f\theta_k}{1-\omega^f\theta_k}\gamma_k\right)\left(\frac{1}{\sum_{k\in f} h_k} - \frac{\omega^f}{\sum_{k\in f}\gamma_k}\right) < 1 \\
\iff &\left(\sum_{i\in f} \frac{\omega^f\theta_i}{1-\omega^f\theta_i}\gamma_i\right)\left(\sum_{j\in f}\left(\gamma_j - \omega^f h_j\right)\right) - \left(\sum_{i\in f} h_i\right)\left(\sum_{j\in f}\gamma_j\right) < 0, \\
\iff &\left(\sum_{i\in f} \frac{\omega^f\theta_i}{1-\omega^f\theta_i}\gamma_i\right)\left(\sum_{j\in f}\gamma_j\left(1 - \omega^f \rho_j\right)\right) - \left(\sum_{i\in f} \rho_i\gamma_i\right)\left(\sum_{j\in f}\gamma_j\right) < 0, \\
\iff &\sum_{i,j\in f}\gamma_i\gamma_j\left(\omega^f\theta_i\frac{1-\omega^f\rho_j}{1-\omega^f\theta_i} - \rho_i\right) < 0.
\end{aligned}$$

□

Lemma F. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. If, for every $f \in \mathcal{F}$,*

$$\begin{aligned}
&\forall \omega^f \in (0, \bar{\omega}^f),\ \forall (x_k)_{k\in f} \in \left\{(x_k)_{k\in f} \in \mathbb{R}^f_{++} :\ \forall k \in f,\ \chi_k(x_k) > \omega^f\right\}, \\
&\left(\sum_{k\in f} \frac{\omega^f\theta_k(x_k)}{1-\omega^f\theta_k(x_k)}\gamma_k(x_k)\right)\left(\frac{1}{\sum_{k\in f} h_k(x_k)} - \frac{\omega^f}{\sum_{k\in f}\gamma_k(x_k)}\right) < 1,
\end{aligned} \tag{29}$$

or, equivalently, if[20]

$$\begin{aligned}
&\forall \omega^f \in (0, \bar{\omega}^f),\ \forall (x_k)_{k\in f} \in \left\{(x_k)_{k\in f} \in \mathbb{R}^f_{++} :\ \forall k \in f,\ \chi_k(x_k) > \omega^f\right\}, \\
&\sum_{i,j\in f}\gamma_i(x_i)\gamma_j(x_j)\left(\omega^f\theta_i(x_i)\frac{1-\omega^f\rho_j(x_j)}{1-\omega^f\theta_i(x_i)} - \rho_i(x_i)\right) < 0,
\end{aligned} \tag{30}$$

then pricing game $\left((h_j)_{j\in\mathcal{N}}, \mathcal{F}, (c_j)_{j\in\mathcal{N}}\right)$ has a unique equilibrium for every $(c_j)_{j\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$.

Proof. Let $(c_k)_{k\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$, and assume that condition (29) holds. Let $f \in \mathcal{F}$, $\omega^f \in (0, \bar{\omega}^f)$ and $\mu^f = 1/(1-\omega^f)$. Let $(x_k)_{k\in f} \in \prod_{k\in f}\left[r_k(\mu^f, c_k), \infty\right)$. Then, for every $k \in f$,

$$\iota_k(x_k) > \nu_k(x_k, c_k) = \mu^f.$$

Therefore,

$$(x_k)_{k\in f} \in \left\{(x_k)_{k\in f} \in \mathbb{R}^f_{++} :\ \forall k \in f,\ \chi_k(x_k) > \omega^f\right\},$$

and, by condition (29), condition (27) holds for $(\underline{c}_k)_{k\in\mathcal{N}} = (c_k)_{k\in\mathcal{N}}$. By Lemma E, pricing game $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$ has a unique equilibrium. In addition, as shown in the proof

[20] Recall that $\rho_k = \frac{h_k}{\gamma_k}$.

of Lemma E, conditions (29) and (30) are equivalent. $\square$

All we need to do to prove Theorem 4 is show that, for every $f \in \mathcal{F}$, conditions (i), (ii) in Theorem 4 imply condition (29) (or, equivalently, condition (30)), and that condition (iii) implies condition (25).

D.2 Sufficiency of condition (i)

We prove the following lemma:

Lemma G. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Let $f \in \mathcal{F}$. If $\min_{j \in f} \inf_{p_j > \underline{p}_j} \rho_j(p_j) \geq \max_{j \in f} \sup_{p_j > \underline{p}_j} \theta_j(p_j)$, then condition (29) holds for firm f.*

Proof. We show that condition (30) holds for firm f. Let $f \in \mathcal{F}$, $\omega^f \in (0, \bar{\omega}^f)$, and

$$(x_k)_{k \in f} \in \left\{ (x_k)_{k \in f} \in \mathbb{R}^f_{++} : \ \forall k \in f, \ \chi_k(x_k) > \omega^f \right\}.$$

Since for every $k \in f$, $\chi_k(x_k) > \omega^f$, it follows that $\iota_k(x_k) > 1$. Therefore, $x_k > \underline{p}_k$ for every k, and

$$\max_{k \in f} \theta_k(x_k) \leq \min_{k \in f} \rho_k(x_k).$$

Therefore,

$$\begin{aligned} \sum_{i,j \in f} \gamma_i \gamma_j \left(\omega^f \theta_i \frac{1 - \omega^f \rho_j}{1 - \omega^f \theta_i} - \rho_i \right) &\leq \sum_{i,j \in f} \gamma_i \gamma_j \left(\omega^f \rho_i - \rho_i \right), \\ &\leq (\omega^f - 1) \sum_{i,j \in f} \gamma_i \gamma_j \rho_i < 0, \end{aligned}$$

where the first inequality follows by Lemma C and $\max_{k \in f} \theta_k(x_k) \leq \min_{k \in f} \rho_k(x_k)$. Therefore, condition (30) holds for firm f. $\square$

D.3 Sufficiency of condition (ii)

The aim of this section is to prove the following lemma:

Lemma H. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Let $f \in \mathcal{F}$. Suppose that, $\bar{\mu}^f \leq \mu^*(\simeq 2.78)$, and for every $j \in f$, $\lim_\infty h_j = 0$ and ρ_j is non-decreasing on $\left(\underline{p}_j, \infty\right)$. Then, condition (29) holds for firm f.*

This lemma is proven in several steps. Start with the following technical lemmas:

Lemma I. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Let $f \in \mathcal{F}$. Suppose that $\bar{\mu}^f < \infty$, and for every $j \in f$, $\lim_\infty h_j = 0$ and ρ_j is non-decreasing on $\left(\underline{p}_j, \infty\right)$. Then, for every $\omega^f \in (0, \bar{\omega}^f)$, for every $k \in f$, for every $x > 0$ such that $\chi_k(x) > \omega^f$,*

$$\frac{1-\bar{\omega}^f}{\bar{\omega}^f}\frac{1}{1-\omega^f} \leq \rho_k(x) \leq \frac{1}{\bar{\omega}^f}.$$

Proof. Let $k \in f$ and $\omega^f \in (0, \bar{\omega}^f)$. By Lemma 4-(vi), $\lim_\infty \rho_k = \frac{\bar{\mu}^f}{\bar{\mu}^f-1} = \frac{1}{\bar{\omega}^f}$. In addition, ρ_k is non-decreasing. Therefore, $\rho_k(x) \leq \frac{1}{\bar{\omega}^f}$ for all $x > \underline{p}_k$. In particular, this inequality is also satisfied if x is such that $\chi_k(x) > \omega^f$.

In addition, as shown in the proof of Lemma 4-(vi),

$$\rho_k(x) = \iota_k(x)\frac{h_k(x)}{-xh_k'(x)}.$$

Therefore,

$$\begin{aligned}
\frac{d\log\rho_k(x)}{dx} &= \frac{\iota_k'(x)}{\iota_k(x)} + \left(\frac{h_k'(x)}{h_k(x)} - \frac{1}{x} + \frac{h_k''(x)}{-h_k'(x)}\right), \\
&= \frac{\iota_k'(x)}{\iota_k(x)} + \frac{1}{x}\left(-\frac{\iota_k(x)}{\rho_k(x)} - 1 + \iota_k(x)\right), \\
&= \frac{\iota_k'(x)}{\iota_k(x)} + \frac{\iota_k(x)}{x\rho_k(x)}\left(\rho_k(x)\chi_k(x) - 1\right), \\
&\leq \frac{\iota_k'(x)}{\iota_k(x)},
\end{aligned}$$

where the last inequality follows from the fact that $\chi_k(x) \leq \bar{\omega}^f$ and $\rho_k(x) \leq \frac{1}{\bar{\omega}^f}$. Therefore, for all $x > \underline{p}_k$,

$$\begin{aligned}
\log\left(\frac{1}{\bar{\omega}^f\rho_k(x)}\right) &= \int_x^\infty \frac{\rho_k'(t)}{\rho_k(t)}dt, \\
&\leq \int_x^\infty \frac{\iota_k'(t)}{\iota_k(t)}dt, \\
&= \log\left(\frac{\bar{\mu}^f}{\iota_k(x)}\right), \\
&= \log\left(\frac{1-\chi_k(x)}{1-\bar{\omega}^f}\right).
\end{aligned}$$

Therefore,

$$\rho_k(x) \geq \frac{1-\bar{\omega}^f}{\bar{\omega}^f}\frac{1}{1-\chi_k(x)}, \ \forall x > \underline{p}_k.$$

In particular, if $\chi_k(x) > \omega^f$, then

$$\rho_k(x) \geq \frac{1-\bar{\omega}^f}{\bar{\omega}^f}\frac{1}{1-\omega^f}. \qquad \square$$

Lemma J. *For every $\bar{\omega} \in (0,1]$, for every $\omega \in (0,\bar{\omega})$, define*

$$\phi_{\omega,\bar{\omega}} : (y,z) \in \left[\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right]^2 \mapsto \omega y\frac{1-\omega z}{1-\omega y} + \omega z\frac{1-\omega y}{1-\omega z} - y - z.$$

There exists a threshold $\omega^* \in (0,1)$ ($\omega^* \simeq 0.64$) such that if $\bar{\omega} \leq \omega^*$, then $\phi_{\omega,\bar{\omega}} \leq 0$ for all $\omega \in (0,\bar{\omega})$.

Proof. Let $\bar{\omega} \in (0,1)$ and $\omega \in (0,\bar{\omega})$. Define

$$M(\omega,\bar{\omega}) = \max_{(y,z)\in\left[\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega},\frac{1}{\bar{\omega}}\right]^2} \phi_{\omega,\bar{\omega}}(y,z).$$

Notice that $\phi_{\omega,\bar{\omega}}(y,z) = \phi_{\omega,\bar{\omega}}(z,y)$ for every y and z. It follows that

$$M(\omega,\bar{\omega}) = \max_{\substack{(y,z)\in\left[\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega},\frac{1}{\bar{\omega}}\right]^2\\ y\leq z}} \phi_{\omega,\bar{\omega}}(y,z).$$

Let $\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega} \leq y \leq z \leq \frac{1}{\bar{\omega}}$. Then,

$$\begin{aligned}
\frac{\partial\phi_{\omega,\bar{\omega}}}{\partial y} &= \frac{\omega(1-\omega z)}{(1-\omega y)^2} - \frac{\omega^2 z}{1-\omega z} - 1, \\
&= \frac{1}{1-\omega z}\left(\omega\left(\frac{1-\omega z}{1-\omega y}\right)^2 - \omega^2 z - (1-\omega z)\right), \\
&\leq \frac{1}{1-\omega z}\left(\omega - \omega^2 z - (1-\omega z)\right), \text{ since } y \leq z, \\
&= \omega - 1 < 0.
\end{aligned}$$

It follows that, for every $(y,z) \in \left[\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right]^2$ such that $y \leq z$,

$$\phi_\omega(y,z) \leq \phi_\omega\left(\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, z\right) \equiv \psi_{\omega,\bar{\omega}}(z).$$

Therefore,

$$M(\omega,\bar{\omega}) = \max_{z\in\left[\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega},\frac{1}{\bar{\omega}}\right]} \psi_{\omega,\bar{\omega}}(z).$$

Since

$$\psi''_{\omega,\bar{\omega}}(z) = \left(1 - \frac{\omega}{1-\omega}\frac{1-\bar{\omega}}{\bar{\omega}}\right)\frac{2\omega^2}{(1-\omega z)^3} > 0,$$

function $\psi_{\omega,\bar{\omega}}(.)$ is strictly convex. Therefore,

$$M(\omega,\bar{\omega}) = \max\left\{\phi_{\omega,\bar{\omega}}\left(\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}\right), \phi_{\omega,\bar{\omega}}\left(\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right)\right\}.$$

Since $\phi_{\omega,\bar{\omega}}(z,z) = 2(\omega-1)z < 0$ for every z, it follows that $M(\omega,\bar{\omega}) \leq 0$ if and only if $\zeta(\omega,\bar{\omega}) \leq 0$, where

$$\begin{aligned}
\zeta(\omega,\bar{\omega}) &\equiv \phi\left(\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right), \\
&= \left(1-\frac{\omega}{\bar{\omega}}\right)\frac{\frac{\omega}{1-\omega}\frac{1-\bar{\omega}}{\bar{\omega}}}{1-\frac{\omega}{1-\omega}\frac{1-\bar{\omega}}{\bar{\omega}}} + \frac{\omega}{\bar{\omega}-\omega}\left(1-\frac{\omega}{1-\omega}\frac{1-\bar{\omega}}{\bar{\omega}}\right) - \frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega} - \frac{1}{\bar{\omega}}, \\
&= \frac{\omega(1-\bar{\omega})}{\bar{\omega}} + \frac{\omega}{(1-\omega)\bar{\omega}} - \frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega} - \frac{1}{\bar{\omega}}, \\
&= \frac{1}{1-\omega} + \frac{\omega-2}{\bar{\omega}} - \omega.
\end{aligned}$$

For every $\omega \in (0,\bar{\omega})$,

$$\frac{\partial\zeta}{\partial\omega} = \frac{1}{(1-\omega)^2} + \frac{1}{\bar{\omega}} - 1 > 0.$$

Therefore, ζ is strictly increasing in ω on interval $(0,\bar{\omega})$. It follows that $M(\omega,\bar{\omega}) \leq 0$ for every $\omega \in (0,\bar{\omega})$ if and only if $\xi(\bar{\omega}) \leq 0$, where

$$\begin{aligned}
\xi(\bar{\omega}) &\equiv \zeta(\bar{\omega},\bar{\omega}), \\
&= \frac{1}{1-\bar{\omega}} + 1 - \bar{\omega} - \frac{2}{\bar{\omega}}.
\end{aligned}$$

For every $\bar{\omega} \in (0,1)$,

$$\xi'(\bar{\omega}) = \frac{1}{(1-\bar{\omega})^2} + \frac{2}{(\bar{\omega})^2} - 1 > 0.$$

Therefore, ξ is strictly increasing on $(0,1)$. Since $\lim_{0^+}\xi = -\infty$ and $\lim_{1^-} = +\infty$, there exists a unique threshold $\omega^* \in (0,1)$ such that $\xi(\bar{\omega}) \leq 0$ if and only if $\bar{\omega} \leq \omega^*$. Numerically, we find that $\omega^* \simeq 0.64$. This concludes the proof. □

We can now prove Lemma H:

Proof. Assume that $\bar{\omega}^f < \omega^*$ (or, equivalently, that $\bar{\mu}^f < \mu^* \simeq 2.78$). Splitting the sum in two terms, condition (30) can be rewritten as follows:

$$
\begin{aligned}
&\forall \omega^f \in (0, \bar{\omega}^f),\ \forall\, (x_k)_{k \in f} \in \left\{ (x_k)_{k \in f} \in \mathbb{R}^f_{++} :\ \forall k \in f,\ \chi_k(x_k) > \omega^f \right\}, \\
&\frac{1}{2} \sum_{\substack{i,j \in f \\ i \neq j}} \gamma_i(x_i)\gamma_j(x_j) \left(\omega^f \theta_i(x_i) \frac{1 - \omega^f \rho_j(x_j)}{1 - \omega^f \theta_i(x_i)} + \omega^f \theta_j(x_j) \frac{1 - \omega^f \rho_i(x_i)}{1 - \omega^f \theta_j(x_j)} - \rho_i(x_i) - \rho_j(x_j) \right) \\
&+ \left(\sum_{i \in f} \gamma_i(x_i)^2 \left(\omega^f \theta_i(x_i) \frac{1 - \omega^f \rho_i(x_i)}{1 - \omega^f \theta_i(x_i)} - \rho_i(x_i) \right) \right) < 0.
\end{aligned}
\tag{31}
$$

Let us first show that the second sum is strictly negative. Let $\omega^f \in (0, \bar{\omega}^f)$, $i \in f$ and x_i such that $\chi_i(x_i) > \omega^f$. Therefore,

$$
\omega^f \theta_i(x_i) \frac{1 - \omega^f \rho_i(x_i)}{1 - \omega^f \theta_i(x_i)} - \rho_i(x_i) \leq \omega^f \theta_i(x_i) - \rho_i(x_i) < 0,
$$

where the first inequality follows from the fact that ρ_i is non-decreasing ($\theta_i(x_i) \leq \rho_i(x_i)$) and Lemma C ($1 - \omega^f \theta_i(x_i) > 0$).

Next, we turn our attention to the first sum. Let $\omega^f \in (0, \bar{\omega}^f)$ and $(x_k)_{k \in f}$ such that $\chi_k(x_k) > \omega^f$ for every $k \in f$. By Lemma I,

$$
\forall k \in f,\ \rho_k(x) \in \left[\frac{1 - \bar{\omega}^f}{\bar{\omega}^f} \frac{1}{1 - \omega^f}, \frac{1}{\bar{\omega}^f} \right].
$$

In addition, as shown above, for every $k \in f$, $\theta_k(x_k) \leq \rho_k(x_k)$ $\left(< \frac{1}{\omega^f}\right)$. Therefore,

$$
\begin{aligned}
&\frac{1}{2} \sum_{\substack{i,j \in f \\ i \neq j}} \gamma_i(x_i)\gamma_j(x_j) \left(\omega^f \theta_i(x_i) \frac{1 - \omega^f \rho_j(x_j)}{1 - \omega^f \theta_i(x_i)} + \omega^f \theta_j(x_j) \frac{1 - \omega^f \rho_i(x_i)}{1 - \omega^f \theta_j(x_j)} - \rho_i(x_i) - \rho_j(x_j) \right) \\
&\leq \frac{1}{2} \sum_{\substack{i,j \in f \\ i \neq j}} \gamma_i(x_i)\gamma_j(x_j) \phi_{\omega^f, \bar{\omega}^f} \left(\rho_i(x_i), \rho_j(x_j) \right), \\
&\leq 0, \text{ by Lemma J.}
\end{aligned}
$$

This concludes the proof. □

D.4 Sufficiency of condition (iii)

We prove a slightly more general result:

Lemma K. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Let $f \in \mathcal{F}$. Assume that* *there exist $h^f \in \mathbb{R}_{++}^{\mathbb{R}_{++}}$, $c^f > 0$ and $(\alpha_k)_{k\in f} \in \mathbb{R}_{++}^f$ such that for every k in f, $c_k = c^f$, and for every $x > 0$, $h_k(x) = \alpha_k h^f(x)$. Assume in addition that ρ^f is non-decreasing. Then, condition* (25) *holds.*

Proof. Let $k \in f$. It is straightforward to show that $\theta_k = \theta^f$, $\rho_k = \rho^f$, $\gamma_k = \alpha_k\gamma^f$, $\iota_k = \iota^f$, and $\chi_k = \chi^f$. In addition, $\nu_k = \nu^f$. Therefore, $r_k = r^f$. Condition (25) is equivalent to

$$\forall\omega^f \in (0,\bar{\omega}^f),\ \left(\sum_{k\in f}\frac{\omega^f\theta^f}{1-\omega^f\theta^f}\alpha_k\gamma\right)\left(\frac{1}{\sum_{k\in f}\alpha_k h^f} - \frac{\omega^f}{\sum_{k\in f}\alpha_k\gamma^f}\right) < 1,$$

where all functions are evaluated at $r^f\left(\frac{1}{1-\omega^f}\right)$. This is equivalent to

$$\frac{1-\omega^f\rho^f}{1-\omega^f\theta^f}\frac{\omega^f\theta^f}{\rho^f} < 1,$$

which clearly holds, since $\theta^f \leq \rho^f$. □

D.5 Condition (ii) when $\lim_\infty h_j \geq 0$

In this section, we extend condition (ii) in Theorem 4 to cases where $\lim_\infty h_j$ is not necessarily equal to zero. We start with the following technical lemma:

Lemma L. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Let $f \in \mathcal{F}$. Assume that ρ_j is non-decreasing on $\left(\underline{p}_j, \infty\right)$ for every $j \in f$. Then, for every $k \in f$,*

$$S_k = \left\{\omega \in (0,\bar{\omega}^f):\ \exists x > \underline{p}_k,\ \omega = \chi_k(x) = \frac{1}{\rho_k(x)}\right\}$$

contains at most one element. If S_k is empty, then, either $\chi_k(x)\rho_k(x) > 1$ for every $x > \underline{p}_k$, or $\chi_k(x)\rho_k(x) < 1$ for every $x > \underline{p}_k$. If, instead, $S_k = \{\hat{\omega}\}$, then, for every $x > \underline{p}_k$,

- *$\theta_k(x) \leq \frac{1}{\hat{\omega}}$, and*
- *if $\rho_k(x) < \frac{1}{\hat{\omega}}$, then $\rho_k(x) \geq \frac{1-\hat{\omega}}{\hat{\omega}}\frac{1}{1-\chi_k(x)}$.*

Proof. Let $k \in f$, and assume for a contradiction that S_k contains two distinct elements. There exist $x, y > \underline{p}_k$ such that $\chi_k(x)\rho_k(x) = 1$, $\chi_k(y)\rho_k(y) = 1$ and $\chi_k(x) \neq \chi_k(y)$. To fix ideas, assume $\chi_k(y) > \chi_k(x)$. Then, since χ_k is non-decreasing, $y > x$. Since ρ_k is non-decreasing, $\rho_k(x) \leq \rho_k(y)$. Therefore, $\chi_k(x)\rho_k(x) < \chi_k(y)\rho_k(y) = 1$, which is a contradiction.

Let $\kappa : x \in (\underline{p}_k, \infty) \mapsto \rho_k(x)\chi_k(x)$, and notice that κ is continuous and non-decreasing. If $S_k = \emptyset$, then, there is no x such that $\kappa(x) = 1$. Since κ is continuous, either $\kappa > 1$, or $\kappa < 1$.

Next, let $x > \underline{p}_k$. If $\rho_k(x) \leq \frac{1}{\hat{\omega}}$, then, $\theta_k(x) \leq \rho_k(x) \leq \frac{1}{\hat{\omega}}$. Assume instead that $\rho_k(x) > \frac{1}{\hat{\omega}}$. Let $\hat{x}$ such that $\chi_k(\hat{x}) = \hat{\omega} = \frac{1}{\rho_k(\hat{x})}$. Then, $\rho_k(x) > \rho_k(\hat{x}) = \frac{1}{\hat{\omega}}$ and, by monotonicity, $x > \hat{x}$. Therefore, $\chi_k(x) \geq \chi_k(\hat{x}) = \hat{\omega}$. Next, we claim that $\theta_k(x) \leq \frac{1}{\chi_k(x)}$. To see this, notice that $\iota_k(x) = x\frac{-h_k'(x)}{\gamma_k(x)}$. Therefore,

$$\begin{aligned}
\frac{\iota_k'(x)}{\iota_k(x)} &= \frac{1}{x} + \frac{h_k''(x)}{h_k'(x)} - \frac{\gamma_k'(x)}{\gamma_k(x)}, \\
&= \frac{1}{x}\left(1 - \iota_k(x) + \frac{\gamma_k'(x)}{h_k'(x)} x \frac{-h_k'(x)}{\gamma_k(x)}\right), \\
&= \frac{1}{x}\left(1 - \iota_k(x) + \frac{\iota_k(x)}{\theta_k(x)}\right).
\end{aligned}$$

Therefore,

$$\theta_k(x) = \frac{\iota_k(x)}{\iota_k(x) - 1 + x\frac{\iota_k'(x)}{\iota_k(x)}} \leq \frac{\iota_k(x)}{\iota_k(x) - 1} = \frac{1}{\chi_k(x)}.$$

Therefore, $\theta_k(x) \leq \frac{1}{\chi_k(x)} \leq \frac{1}{\hat{\omega}}$.

Next, assume that $\rho_k(x) < \frac{1}{\hat{\omega}}$. We know from the proof of Lemma I that for every $t \in [x, \hat{x}]$,

$$\begin{aligned}
\frac{\rho_k'(t)}{\rho_k(t)} &= \frac{\iota_k'(t)}{\iota_k(t)} + \frac{\iota_k(t)}{t\rho_k(t)}\left(\rho_k(t)\chi_k(t) - 1\right), \\
&\leq \frac{\iota_k'(t)}{\iota_k(t)} + \frac{\iota_k(t)}{t\rho_k(t)}\left(\rho_k(\hat{x})\chi_k(\hat{x}) - 1\right), \text{ by monotonicity,} \\
&= \frac{\iota_k'(t)}{\iota_k(t)}, \text{ since } \rho_k(\hat{x})\chi_k(\hat{x}) = 1.
\end{aligned}$$

Integrating this inequality between x and $\hat{x}$, we obtain that $\frac{\rho_k(\hat{x})}{\rho_k(x)} \leq \frac{\iota_k(\hat{x})}{\iota_k(x)}$. Therefore,

$$\begin{aligned}
\rho_k(x) &\geq \rho_k(\hat{x})\frac{\iota_k(x)}{\iota_k(\hat{x})}, \\
&= \frac{1-\hat{\omega}}{\hat{\omega}}\frac{1}{1-\chi_k(x)}.
\end{aligned}$$

□

Proposition A. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Let $f \in \mathcal{F}$. Assume that ρ_j is non-decreasing on $\left(\underline{p}_j, \infty\right)$ for every $j \in f$ and that $\bar{\omega}^f \leq \omega^*$. Assume also, using the notation introduced in Lemma L that, for every $i \in f$, $S_i = \{\hat{\omega}\}$. Then, condition* (30) *holds for firm f.*

Proof. As in the proof of Theorem H, the expression in condition (30) can be split in two terms (see equation (31)). Since ρ_j is non-decreasing for every $j \in f$ and by Lemma C, the second sum is strictly negative. Next, we turn our attention to the first sum. Let $\omega^f \in (0, \bar{\omega}^f)$, $i, j \in f$, and x_i, x_j such that $\chi_i(x_i) > \omega^f$ and $\chi_j(x_j) > \omega^f$. We want to show that

$$\Psi = \omega^f \theta_i(x_i) \frac{1 - \omega^f \rho_j(x_j)}{1 - \omega^f \theta_i(x_i)} + \omega^f \theta_j(x_j) \frac{1 - \omega^f \rho_i(x_i)}{1 - \omega^f \theta_j(x_j)} - \rho_i(x_i) - \rho_j(x_j) \leq 0. \tag{32}$$

To fix ideas, assume that $\rho_i(x_i) \leq \rho_j(x_j)$. If $\rho_i(x_i) \geq \frac{1}{\omega^f}$, then condition (32) is clearly satisfied, since, by Lemma C, $1 - \omega^f \theta_i(x_i)$ and $1 - \omega^f \theta_j(x_i)$ are strictly positive. Assume instead that $\rho_i(x_i) < \frac{1}{\omega^f}$. Then, we claim that $\omega^f < \hat{\omega}$. Assume for a contradiction that $\hat{\omega} \leq \omega^f$. Since $S_i = \{\hat{\omega}\}$, there exists $\hat{x}_i > \underline{p}_i$ such that $\chi_i(\hat{x}_i) = \hat{\omega} = \frac{1}{\rho_i(\hat{x}_i)}$. Therefore, $\rho_i(x_i) < \rho_i(\hat{x}_i)$ and, by monotonicity, $x_i < \hat{x}_i$. Since χ_i is non-decreasing, it follows that

$$\omega^f < \chi_i(x_i) \leq \chi_i(\hat{x}_i) = \hat{\omega},$$

which is a contradiction. Therefore, $\omega^f < \hat{\omega}$.

We distinguish three cases. Assume first that $\rho_j(x_j) < \frac{1}{\hat{\omega}}$. Then, by Lemma L,

$$\rho_k(x_k) \geq \frac{1 - \hat{\omega}}{\hat{\omega}} \frac{1}{1 - \chi_k(x_k)} \geq \frac{1 - \hat{\omega}}{\hat{\omega}} \frac{1}{1 - \omega^f},$$

for $k \in \{i, j\}$. In addition, $\frac{\theta_i(x_i)}{1 - \omega^f \theta_i(x_i)} \leq \frac{\rho_i(x_i)}{1 - \omega^f \rho_i(x_i)}$ and $\frac{\theta_j(x_j)}{1 - \omega^f \theta_j(x_j)} \leq \frac{\rho_j(x_j)}{1 - \omega^f \rho_j(x_j)}$. Therefore,

$$\Psi \leq \phi_{\omega^f, \hat{\omega}} \left(\rho_i(x_i), \rho_j(x_j) \right),$$

which, by Lemma J, is non-positive, since $\hat{\omega} < \bar{\omega}^f \leq \omega^*$.

Next, assume that $\rho_i(x_i) < \frac{1}{\hat{\omega}} \leq \rho_j(x_j)$. Then, by Lemma L,

$$\rho_i(x_i) \geq \frac{1 - \hat{\omega}}{\hat{\omega}} \frac{1}{1 - \chi_i(x_i)} \geq \frac{1 - \hat{\omega}}{\hat{\omega}} \frac{1}{1 - \omega^f},$$

and $\theta_j(x_j) \leq \frac{1}{\hat{\omega}}$. Therefore,

$$\begin{aligned}
\Psi &\leq \frac{\omega^f \theta_i(x_i)}{1 - \omega^f \theta_i(x_i)} \left(1 - \frac{\omega^f}{\hat{\omega}}\right) + \frac{\frac{\omega^f}{\hat{\omega}}}{1 - \frac{\omega^f}{\hat{\omega}}} (1 - \omega^f \rho_i(x_i)) - \rho_i(x_i) - \frac{1}{\hat{\omega}}, \\
&\leq \frac{\omega^f \rho_i(x_i)}{1 - \omega^f \rho_i(x_i)} \left(1 - \frac{\omega^f}{\hat{\omega}}\right) + \frac{\frac{\omega^f}{\hat{\omega}}}{1 - \frac{\omega^f}{\hat{\omega}}} (1 - \omega^f \rho_i(x_i)) - \rho_i(x_i) - \frac{1}{\hat{\omega}},
\end{aligned}$$

$$
\begin{aligned}
&= \phi_{\omega^f,\hat{\omega}}\left(\rho_i(x_i), \frac{1}{\hat{\omega}}\right), \\
&\leq 0 \text{ by Lemma J.}
\end{aligned}
$$

Finally, assume that $\rho_i(x_i) \geq \frac{1}{\hat{\omega}}$. By Lemma L, $\theta_i(x_i) \leq \frac{1}{\hat{\omega}}$ and $\theta_j(x_j) \leq \frac{1}{\hat{\omega}}$. Therefore,

$$
\begin{aligned}
\Psi &\leq \frac{\omega^f \theta_i(x_i)}{1-\omega^f \theta_i(x_i)}\left(1-\frac{\omega^f}{\hat{\omega}}\right) + \frac{\omega^f \theta_j(x_j)}{1-\omega^f \theta_j(x_j)}\left(1-\frac{\omega^f}{\hat{\omega}}\right) - \frac{1}{\hat{\omega}} - \frac{1}{\hat{\omega}}, \\
&\leq \frac{\frac{\omega^f}{\hat{\omega}}}{1-\frac{\omega^f}{\hat{\omega}}}\left(1-\frac{\omega^f}{\hat{\omega}}\right) + \frac{\frac{\omega^f}{\hat{\omega}}}{1-\frac{\omega^f}{\hat{\omega}}}\left(1-\frac{\omega^f}{\hat{\omega}}\right) - \frac{1}{\hat{\omega}} - \frac{1}{\hat{\omega}}, \\
&= \phi_{\omega^f,\hat{\omega}}\left(\frac{1}{\hat{\omega}}, \frac{1}{\hat{\omega}}\right), \\
&\leq 0 \text{ by Lemma J.}
\end{aligned}
$$

This concludes the proof. $\square$

Condition $S_i = \{\hat{\omega}\}\ \forall i$ in Proposition A may look a little bit arcane. The following corollary is easier to understand:

Corollary A. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Let $f \in \mathcal{F}$. Assume that ρ_j is non-decreasing on $\left(\underline{p}_j, \infty\right)$ for every $j \in f$ and that $\bar{\omega}^f \leq \omega^*$. Assume also that there exist $h \in \mathbb{R}_{++}^{\mathbb{R}_{++}}$ and $(\alpha_k, \beta_k)_{k\in f} \in \left(\mathbb{R}_{++}^2\right)^f$ such that for every $k \in f$, for every $x > 0$, $h_k(x) = \alpha_k h(\beta_k x)$. Then, condition* (30) *holds for firm f.*

Proof. Let us first show that $S_i \subseteq S_j$ for all $i, j \in f$. Let $i, j \in f$. If S_i is empty, then, trivially, $S_i \subseteq S_j$. Assume instead that $S_i \neq \emptyset$, and let $\hat{\omega} \in S_i$. There exists $\hat{x}_i > \underline{p}_i$ such that

$$
\chi_i(\hat{x}_i) = \hat{\omega} = \frac{1}{\rho_i(\hat{x}_i)}.
$$

Since $h_i(x_i) = \alpha_i h(\beta_i x)$, it is straightforward to show that $\rho_i(\hat{x}_i) = \rho(\beta_i \hat{x}_i)$ and $\chi_i(\hat{x}_i) = \chi(\beta_i \hat{x}_i)$. Let $\hat{x}_j = \frac{\beta_i}{\beta_j}\hat{x}_i$. Then,

$$
\chi_j\left(\hat{x}_j\right) = \chi\left(\beta_j \frac{\beta_i}{\beta_j}\hat{x}_i\right) = \chi_i(\hat{x}_i) = \hat{\omega} = \frac{1}{\rho_i(\hat{x}_i)} = \frac{1}{\rho(\beta_i \hat{x}_i)} = \frac{1}{\rho_j(\hat{x}_j)}.
$$

Therefore, $\hat{\omega} \in S_j$, and $S_i \subseteq S_j$. It follows that $S_i = S_j$ for all $i, j \in f$.

If $S_i \neq \emptyset$, then, by Proposition A, condition (30) holds for firm f. Assume instead that $S_i = \emptyset$ for all i. Let $i \in f$. By Lemma L, either $\chi_i(x_i)\rho_i(x_i) < 1$ for all x_i, or $\chi_i(x_i)\rho_i(x_i) > 1$

for all x_i. Assume first that $\chi_i(x_i)\rho_i(x_i) < 1$ for all x_i. Let $j \in f$ and $x_j > \underline{p}_j$. Then,

$$\chi_j(x_j)\rho_j(x_j) = \chi_i\left(\frac{\beta_j}{\beta_i}x_j\right)\rho_i\left(\frac{\beta_j}{\beta_i}x_j\right) < 1.$$

Therefore, $\chi_j\rho_j < 1$ for every j in f. It follows that

$$\lim_{\infty} \rho_j \leq \lim_{\infty} \frac{1}{\chi_j} = \frac{1}{\bar{\omega}^f} < \infty.$$

Therefore, $\lim_\infty h_j = 0$ for every $j \in f$ (if $\lim_\infty h_j$ were strictly positive, then $\rho_j(x_j)$ would go to ∞ as x_j goes to ∞). By Lemma H, condition (30) holds for firm f.

Finally, assume that $\chi_i(x_i)\rho_i(x_i) > 1$ for all x_i. Then, using the same argument as above, $\chi_j\rho_j > 1$ for every $j \in f$. Let $i \in f$, and assume for a contradiction that $\underline{p}_i > 0$. Since $1/\chi_i$ is non-increasing, and since, by continuity, $\iota_i(\underline{p}_i) = 1$, it follows that $\lim_{\underline{p}_i^+} \frac{1}{\chi_i} = \infty$. Therefore, $\lim_{\underline{p}_i^+} \rho_i = \infty$, which is a contradiction, since ρ_i is non-decreasing. Therefore, $\underline{p}_i = 0$.

Assume for a contradiction that $\lim_{0^+} \iota_i = 1$. Then, using the same reasoning as in the previous paragraph, $\lim_{0^+} \rho_i = \infty$, which is again a contradiction, since ρ_i is non-decreasing. Therefore, $\lim_{0^+} \iota_i > 1$, and $\hat{\omega} \equiv \lim_{0^+} \chi_i$ is strictly positive. In addition, since

$$\chi_j(x) = \chi_i\left(\frac{\beta_j}{\beta_i}x\right),$$

$\lim_{0^+} \chi_j = \hat{\omega}$ for every $j \in f$. Notice that, for every $j \in f$, for every $x > 0$,

$$\rho_j(x) \geq \lim_{0^+} \rho_j \geq \lim_{0^+} \frac{1}{\chi_j} = \frac{1}{\hat{\omega}},$$

and that, by Lemma C,

$$\theta_j(x) \leq \frac{1}{\chi_j(x)} \leq \lim_{0^+} \frac{1}{\chi_j} = \frac{1}{\hat{\omega}}.$$

It follows that

$$\max_{i \in f} \sup \theta_i \leq \frac{1}{\hat{\omega}} \leq \min_{i \in f} \inf \rho_i,$$

i.e., condition (i) in Theorem 4 holds. By Lemma G, condition (30) is therefore satisfied for firm f. □

Proposition B. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Let $f \in \mathcal{F}$. Assume that ρ_j is non-decreasing on $\left(\underline{p}_j, \infty\right)$ for every $j \in f$, that $\bar{\omega}^f \leq \omega^*$, and that $\theta_k \leq \frac{1}{\bar{\omega}^f}$ for every k in f. Then, condition* (30) *holds for firm f.*

Proof. Let $i, j \in f$, $\omega^f \in (0, \bar{\omega}^f)$ and $x_i, x_j > 0$ such that $\chi_i(x_i) > \omega^f$ and $\chi_j(x_j) > \omega^f$. Define

$$\Psi = \frac{\omega^f \theta_i(x_i)}{1 - \omega^f \theta_i(x_i)} \left(1 - \omega^f \rho_j(x_j)\right) + \frac{\omega^f \theta_j(x_j)}{1 - \omega^f \theta_j(x_j)} \left(1 - \omega^f \rho_i(x_i)\right) - \rho_i(x_i) - \rho_j(x_j).$$

As in the previous proofs, all we need to do is show that $\Psi \leq 0$. Assume first that $\rho_i(x_i) \geq \frac{1}{\bar{\omega}^f}$ and $\rho_j(x_j) \geq \frac{1}{\bar{\omega}^f}$. Then,

$$\max\left(\theta_i(x_i), \theta_j(x_j)\right) \leq \min\left(\rho_i(x_i), \rho_j(x_j)\right).$$

Therefore, $\Psi < 0$.

Next, assume that $\rho_i(x_i) < \frac{1}{\bar{\omega}^f}$ and $\rho_j(x_j) \geq \frac{1}{\bar{\omega}^f}$. Then, we claim that

$$\rho_i(x_i) \geq \frac{1 - \bar{\omega}^f}{\bar{\omega}^f} \frac{1}{1 - \omega^f}. \tag{33}$$

To see this, assume first that $S_i = \{\hat{\omega}_i\}$, where $\hat{\omega}_i \in (0, \bar{\omega}^f)$. Since $\rho_i(x_i) < \frac{1}{\bar{\omega}^f} < \frac{1}{\hat{\omega}^i}$, by Lemma L,

$$\rho_i(x_i) \geq \frac{1 - \hat{\omega}_i}{\hat{\omega}_i} \frac{1}{1 - \chi_i(x_i)} \geq \frac{1 - \bar{\omega}^f}{\bar{\omega}^f} \frac{1}{1 - \omega^f}.$$

Assume instead that $S_i = \emptyset$. By Lemma L, either $\chi_i \rho_i < 1$ or $\chi_i \rho_i > 1$. If $\chi_i \rho_i > 1$, then we know from the proof of Corollary A that

$$\rho_i \geq \sup \frac{1}{\chi_i} \geq \frac{1}{\bar{\omega}^f}.$$

This contradicts our assumption that $\rho_i(x_i) < \frac{1}{\bar{\omega}^f}$. If, instead, $\chi_i \rho_i < 1$, then we know from the proof of Corollary A that $\lim_\infty h_i = 0$. Therefore, by Lemma I, inequality (33) holds.

Therefore,

$$\begin{aligned}
\Psi &\leq \frac{\omega^f \theta_i(x_i)}{1 - \omega^f \theta_i(x_i)} \left(1 - \frac{\omega^f}{\bar{\omega}^f}\right) + \frac{\frac{\omega^f}{\bar{\omega}^f}}{1 - \frac{\omega^f}{\bar{\omega}^f}} \left(1 - \omega^f \rho_i(x_i)\right) - \rho_i(x_i) - \frac{1}{\bar{\omega}^f}, \\
&\leq \frac{\omega^f \rho_i(x_i)}{1 - \omega^f \rho_i(x_i)} \left(1 - \frac{\omega^f}{\bar{\omega}^f}\right) + \frac{\frac{\omega^f}{\bar{\omega}^f}}{1 - \frac{\omega^f}{\bar{\omega}^f}} \left(1 - \omega^f \rho_i(x_i)\right) - \rho_i(x_i) - \frac{1}{\bar{\omega}^f}, \\
&= \phi_{\omega^f, \bar{\omega}^f}\left(\rho_i(x_i), \frac{1}{\bar{\omega}^f}\right), \\
&\leq 0 \text{ by Lemma J.}
\end{aligned}$$

Finally, assume that $\rho_i(x_i) < \frac{1}{\bar{\omega}^f}$ and $\rho_j(x_j) < \frac{1}{\bar{\omega}^f}$. Then, as above,

$$\rho_k(x_k) \geq \frac{1-\bar{\omega}^f}{\bar{\omega}^f}\frac{1}{1-\omega^f}$$

for $k \in \{i, j\}$. Therefore,

$$\Psi \leq \phi_{\omega^f,\bar{\omega}^f}\left(\rho_i(x_i), \rho_j(x_j)\right),$$

which is non-positive by Lemma J. □

Corollary B. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Let $f \in \mathcal{F}$. Assume that ρ_j is non-decreasing on $\left(\underline{p}_j, \infty\right)$ for every $j \in f$, that $\bar{\omega}^f \leq \omega^*$, and that θ_k is non-decreasing for every k in f. Then, condition* (30) *holds for firm f.*

Proof. Let $k \in f$. Since θ_k is non-increasing, for every $x > \underline{p}_k$,

$$\theta_k(x) \leq \sup\theta_k = \lim_{\infty}\theta_k \leq \lim_{\infty}\frac{1}{\chi_k} = \frac{1}{\bar{\omega}^f},$$

where the second inequality follows from Lemma C. Therefore, by Proposition B, condition (30) holds for firm f. □

D.6 Proof of Proposition 8.

Proof. Let $j \in f$. Then, for all $x > 0$,

$$\begin{aligned}
h_j'(x) &= \alpha_j\beta_j h'(\beta_j x + \delta_j) < 0,\\
h_j''(x) &= \alpha_j\beta_j^2 h''(\beta_j x + \delta_j) > 0,\\
\gamma_j(x) &= \alpha_j\gamma(\beta_j x + \delta_j),\\
\gamma_j'(x) &= \alpha_j\beta_j\gamma'(\beta_j x + \delta_j),\\
\rho_j(x) &= \rho(\beta_j x + \delta_j) + \frac{\epsilon_j}{\alpha_j\gamma(\beta_j x + \delta_j)} \geq \rho(\beta_j x + \delta_j),\\
\theta_j(x) &= \theta(\beta_j x + \delta_j),\\
\iota_j(x) &= \frac{\beta_j x}{\beta_j x + \delta_j}\iota(\beta_j x + \delta_j).
\end{aligned}$$

Therefore, h_j is positive, decreasing and log-convex, ι_j is non-decreasing whenever ι_j is > 1, and $\bar{\mu}_j = \lim_\infty \iota$. In addition, for every $x > \underline{p}_j$,

$$1 < \iota_j(x) \leq \iota(\beta_j x + \delta_j).$$

Therefore, $\beta_j x + \delta_j > \underline{p}$, and

$$\theta_j(x) \leq \sup_{y>\underline{p}} \theta(y).$$

It follows that $\sup_{y>\underline{p}_j} \theta_j(y) \leq \sup_{y>\underline{p}} \theta(y)$. Using the same reasoning, we also obtain that $\inf_{y>\underline{p}_j} \rho_j(y) \geq \inf_{y>\underline{p}} \rho(y)$. Therefore,

$$\begin{aligned}
\max_{j\in f} \sup_{x>\underline{p}_j} \theta_j(x) &\leq \max_{j\in f} \sup_{x>\underline{p}} \theta(x), \\
&\leq \sup_{x>\underline{p}} \theta(x), \\
&\leq \inf_{x>\underline{p}} \rho(x), \\
&\leq \min_{j\in f} \inf_{x>\underline{p}} \rho(x), \\
&\leq \min_{j\in f} \inf_{x>\underline{p}_j} \rho_j(x).
\end{aligned}$$

□

D.7 Proof of Proposition 9

In this section, we let $m^f\left(H, (c_j)_{j\in f}\right)$ be firm f's fitting-in function when its costs are given by $(c_j)_{j\in f}$. It is straightforward to adapt the proof of Lemma 10 to show that m^f is non-increasing in $(c_j)_{j\in f}$, and that

$$\lim_{c^f\to\infty} m^f\left(H, \left(c^f, \ldots, c^f\right)\right) = 1.$$

We introduce the following notation: For every $f \in \mathcal{F}$, put $\underline{\mu}^f = \min_{j\in f} \bar{\mu}_j$ and $\underline{\omega}^f = \frac{\underline{\mu}^f - 1}{\underline{\mu}^f}$ (or $\underline{\omega}^f = 1$ if $\underline{\mu}^f = \infty$). For every $\underline{c} > 0$, define

$$\underline{H}(\underline{c}) = \min_{f\in\mathcal{F}} \inf\left\{H > 0 : \quad m^f(H, (\underline{c}, \ldots, \underline{c})) < \underline{\mu}^f\right\}.$$

By Lemma 10, $\underline{H}(\underline{c})$ is finite, and $m^f(H, (\underline{c}, \ldots, \underline{c})) < \underline{\mu}^f$ for all $f \in \mathcal{F}$ whenever $H > \underline{H}(\underline{c})$. In addition, since m^f is decreasing in $(c_j)_{j\in f}$, $m^f\left(H, (c_j)_{j\in f}\right) < \underline{\mu}^f$ for all $H > \underline{H}(\underline{c})$, $f \in \mathcal{F}$ and $(c_j)_{j\in f} \in [\underline{c}, \infty)^f$. Note also that $\underline{H}$ is non-increasing in $\underline{c}$, and that $\lim_{\underline{c}\to\infty} \underline{H}(\underline{c}) = 0$.

We prove the following preliminary technical lemma:

Lemma M. *Let $\underline{c} > 0$. If, for every $f \in \mathcal{F}$,*

$$\begin{aligned}
&\forall \omega^f \in (0, \underline{\omega}^f),\ \forall \left(x_k\right)_{k\in f} \in \prod_{k\in f} \left[r_k\left(\frac{1}{1-\omega^f}, \underline{c}\right), \infty\right), \\
&\left(\sum_{k\in f} \frac{\omega^f \theta_k(x_k)}{1-\omega^f\theta_k(x_k)} \gamma_k(x_k)\right)\left(\frac{1}{\sum_{k\in f} h_k(x_k)} - \frac{\omega^f}{\sum_{k\in f}\gamma_k(x_k)}\right) < 1,
\end{aligned} \tag{34}$$

or, equivalently, if

$$\begin{aligned}
&\forall \omega^f \in (0, \underline{\omega}^f),\ \forall \left(x_k\right)_{k\in f} \in \prod_{k\in f} \left[r_k\left(\frac{1}{1-\omega^f}, \underline{c}\right), \infty\right), \\
&\sum_{i,j\in f} \gamma_i(x_i)\gamma_j(x_j)\left(\omega^f\theta_i(x_i)\frac{1-\omega^f\rho_j(x_j)}{1-\omega^f\theta_i(x_i)} - \rho_i(x_i)\right) < 0,
\end{aligned} \tag{35}$$

then, for every $\left(c_j\right)_{j\in\mathcal{N}} \in [\underline{c}, \infty)^{\mathcal{N}}$, pricing game $\left(\left(h_j\right)_{j\in\mathcal{N}}, \mathcal{F}, \left(c_j\right)_{j\in\mathcal{N}}\right)$ has at most one equilibrium aggregator level in $(\underline{H}(\underline{c}), \infty)$.

Proof. The proof is exactly the same as the proof of Lemma E. $\square$

We can now prove Proposition 9:

Proof. We only prove the first bullet point. The proof of the second bullet point is similar, and therefore omitted.

Let $\underline{H}^0 > 0$, and $H^0 \geq \underline{H}^0$. Recall that pricing game $\left(\left(h_j\right)_{j\in\mathcal{N}}, \mathcal{F}, \left(c_j\right)_{j\in\mathcal{N}}\right)$ with outside option H^0 is equivalent to pricing game $\left(\left(h_j^{H^0}\right)_{j\in\mathcal{N}}, \mathcal{F}, \left(c_j\right)_{j\in\mathcal{N}}\right)$ with outside option 0, where

$$h_j^{H^0} = h_j + \frac{H^0}{|\mathcal{N}|} \quad \forall j \in \mathcal{N}.$$

Note that, for every $H^0 \geq \underline{H}^0$ and $j \in \mathcal{N}$, $\rho_j^{H^0} \geq \rho_j^{\underline{H}^0}$ and $\lim_{p\to\infty} \rho_j^{H^0} = \infty$.

Fix some $c > \max_{j\in\mathcal{N}} \underline{p}_j$. For every $j \in \mathcal{N}$ and $x \geq c$,[21]

$$\theta_j(x) \leq \frac{1}{\chi_j(x)} \leq \frac{1}{\chi_j(c)} \leq \max_{k\in\mathcal{N}} \frac{1}{\chi_k(c)} \equiv \bar{\theta},$$

where the first inequality follows by Lemma C. Since $\lim_\infty \rho_j^{\underline{H}^0} = \infty$ for every $j \in \mathcal{N}$, there exists $c' > c$ such that, for every $j \in \mathcal{N}$, $\rho_j^{\underline{H}^0}(x) \geq \bar{\theta}$ whenever $x \geq c'$. Therefore, for every

[21]Since neither θ_j nor χ_j depend on H^0, we drop superscript H^0 to ease notation.

$H^0 \geq \underline{H}^0$, $f \in \mathcal{F}$, $i, j \in f$, $x_i \geq c'$ and $x_j \geq c'$, $\rho_i^{H^0}(x_i) \geq \theta_j^{H^0}(x_j)$, and, in particular,

$$\forall \omega^f \in (0, \underline{\omega}^f), \quad \frac{\omega^f \theta_i^{H^0}(x_i)}{1 - \omega^f \theta_i^{H^0}(x_i)} \left(1 - \omega^f \rho_j^{H^0}(x_j)\right) - \rho_i^{H^0}(x_i) < 0.$$

Therefore, condition (35) holds, and, for every $H^0 \geq \underline{H}^0$ and $(c_j)_{j \in \mathcal{N}} \in [c', \infty)^{\mathcal{N}}$, pricing game $\left(\left(h_j^{H^0}\right)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}}\right)$ has at most one equilibrium aggregator level in $(\underline{H}(c'), \infty)$.

Next, choose $c'' > 0$ such that $\underline{H}(c'') < \underline{H}^0$. Since $\lim_\infty \underline{H} = 0$, such a c'' exists. Put $\underline{c} = \max(c', c'')$. Since $\underline{H}(\cdot)$ is non-increasing, $\underline{H}(\underline{c}) < \underline{H}^0$. Combining this with our previous findings, we can conclude that for every $H^0 \geq \underline{H}^0$ and $(c_j)_{j \in \mathcal{N}} \in [\underline{c}, \infty)^{\mathcal{N}}$, pricing game $\left((h_j)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}}\right)$ with outside option H^0 has at most one equilibrium aggregator level in (H^0, ∞). Since this pricing game has an equilibrium (Theorem 3), and since no equilibrium aggregator level can be less than H^0, it follows that this pricing game has a unique equilibrium. □

D.8 Establishing Equilibrium Uniqueness Using an Index Approach

The reader may wonder whether we could obtain weaker uniqueness conditions by using more standard approaches. Uniqueness of a fixed point is usually established by using the contraction mapping approach, the univalence approach or the index (Poincaré-Hopf) approach. It is well known that the index approach is more general than the others, and that it provides an "almost if and only if" condition for uniqueness. We will therefore focus on the index approach. Since we will be working with matrices, we will sometimes assume that $\mathcal{F} = \{1, \ldots, F\}$, and that firm f's set of products is $\mathcal{N}^f$.

We know that establishing uniqueness in the pricing game is equivalent to establishing uniqueness in the auxiliary game in which firms are simultaneously choosing their μ^f's. We also know that a profile $\mu = (\mu^f)_{f \in \mathcal{F}}$ is an equilibrium of the auxiliary game if and only if for every $f \in \mathcal{F}$,

$$\phi^f(\mu) \equiv (\mu^f - 1)\left(\left(\sum_{k \in \mathcal{N}^f} h_k\right) + \left(\sum_{\substack{g \in \mathcal{F} \\ g \neq f}} \sum_{k \in \mathcal{N}^f} h_k\right)\right) - \mu^f \sum_{k \in \mathcal{N}^f} \gamma_k = 0.$$

Therefore, all we need to do is show that map ϕ has a unique zero. By the index theorem, this holds if the determinant of the Jacobian matrix of ϕ evaluated at μ is strictly positive whenever $\phi(\mu) = 0$. We have shown in the proof of Lemma 9 that

$$\frac{\partial \phi^f}{\partial \mu^f} = \sum_{f \in \mathcal{F}} \sum_{k \in \mathcal{N}^f} h_k \equiv H(\mu).$$

Moreover, if $g \neq f$, then

$$\frac{\partial \phi^f}{\partial \mu^g} = (\mu^f - 1) \sum_{k \in \mathcal{N}^g} r'_k h'_k.$$

Therefore,

$$\begin{aligned}
\det J(\phi) &= \begin{vmatrix} H(\mu) & (\mu_1 - 1) \sum_{k \in \mathcal{N}^2} r'_k h'_k & \cdots & (\mu_1 - 1) \sum_{k \in \mathcal{N}^F} r'_k h'_k \\ (\mu_2 - 1) \sum_{k \in \mathcal{N}^1} r'_k h'_k & H(\mu) & \cdots & (\mu_2 - 1) \sum_{k \in \mathcal{N}^F} r'_k h'_k \\ \vdots & \vdots & \ddots & \vdots \\ (\mu^F - 1) \sum_{k \in \mathcal{N}^1} r'_k h'_k & (\mu^F - 1) \sum_{k \in \mathcal{N}^2} r'_k h'_k & \cdots & H(\mu) \end{vmatrix}, \\
&= \left(\prod_{f \in \mathcal{F}} (\mu^f - 1) \sum_{k \in \mathcal{N}^f} r'_k h'_k \right) \det \mathcal{M} \left(\left(1 + \frac{H(\mu)}{(\mu^f - 1) \sum_{k \in \mathcal{N}^f} r'_k (-h'_k)} \right)_{1 \leq f \leq F} \right),
\end{aligned}$$

where the second line has been obtained by dividing row f by $\mu^f - 1$ and dividing column f by $\sum_{k \in \mathcal{N}^f} r'_k h'_k$ for every f in $\{1, \ldots, F\}$ and by using the F-linearity of the determinant. By Lemma A,

$$\begin{aligned}
\det\left(J(\phi)\right) = &\left(\prod_{f \in \mathcal{F}} (\mu^f - 1) \sum_{k \in \mathcal{N}^f} r'_k h'_k \right) (-1)^F \left(\left(\prod_{f \in \mathcal{F}} \left(1 + \frac{H(\mu)}{(\mu^f - 1) \sum_{k \in \mathcal{N}^f} r'_k (-h'_k)} \right) \right) \right. \\
&\left. - \sum_{g \in \mathcal{F}} \prod_{f \neq g} \left(1 + \frac{H(\mu)}{(\mu^f - 1) \sum_{k \in \mathcal{N}^f} r'_k (-h'_k)} \right) \right), \\
= &\left(\prod_{f \in \mathcal{F}} (\mu^f - 1) \sum_{k \in \mathcal{N}^f} r'_k h'_k \right) (-1)^F \left(\prod_{f \in \mathcal{F}} \left(1 + \frac{H(\mu)}{(\mu^f - 1) \sum_{k \in \mathcal{N}^f} r'_k (-h'_k)} \right) \right) \\
&\times \left(1 - \sum_{f \in \mathcal{F}} \frac{1}{1 + \frac{H(\mu)}{(\mu^f - 1) \sum_{k \in \mathcal{N}^f} r'_k (-h'_k)}} \right), \\
= &\underbrace{\left(\prod_{f \in \mathcal{F}} \left(H(\mu) + (\mu^f - 1) \sum_{k \in \mathcal{N}^f} r'_k (-h'_k) \right) \right)}_{>0} \left(1 - \sum_{f \in \mathcal{F}} \frac{1}{1 + \frac{H(\mu)}{(\mu^f - 1) \sum_{k \in \mathcal{N}^f} r'_k (-h'_k)}} \right).
\end{aligned}$$

Therefore, we need to show that

$$\sum_{f\in\mathcal{F}} \frac{\frac{\mu^f-1}{H(\mu)} \sum_{k\in\mathcal{N}^f} r_k'(-h_k')}{1+\frac{\mu^f-1}{H(\mu)} \sum_{k\in\mathcal{N}^f} r_k'(-h_k')} < 1 \tag{36}$$

whenever $\phi(\mu)=0$. Notice that

$$\begin{aligned}
(36) \iff & \sum_{f\in\mathcal{F}} \left(\frac{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r_k'(-h_k')}{1+\frac{\mu^f-1}{H(\mu)}\sum_{k\in\mathcal{N}^f} r_k'(-h_k')} - \sum_{k\in\mathcal{N}^f} h_k \right) < 0 \\
\iff & \sum_{f\in\mathcal{F}} \left(\frac{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r_k'(-h_k')}{1+\frac{(\mu^f-1)^2}{\mu^f}\frac{\sum_{k\in\mathcal{N}^f} r_k'(-h_k')}{\sum_{k\in\mathcal{N}^f}\gamma_k}} - \sum_{k\in\mathcal{N}^f} h_k \right) < 0, \text{ since } \phi(\mu)=0, \\
\iff & \sum_{f\in\mathcal{F}} \left(\frac{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r_k'(-h_k')}{1+\frac{\mu^f-1}{\mu^f}\frac{\sum_{k\in\mathcal{N}^f} r_k'\left((\mu^f-1)(-h_k')-\mu^f(-\gamma_k')+\mu^f(-\gamma_k')\right)}{\sum_{k\in\mathcal{N}^f}\gamma_k}} - \sum_{k\in\mathcal{N}^f} h_k \right) < 0, \\
\iff & \sum_{f\in\mathcal{F}} \left(\frac{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r_k'(-h_k')}{1-\frac{\mu^f-1}{\mu^f}+(\mu^f-1)\frac{\sum_{k\in\mathcal{N}^f} r_k'(-\gamma_k')}{\sum_{k\in\mathcal{N}^f}\gamma_k}} - \sum_{k\in\mathcal{N}^f} h_k \right) < 0, \text{ by Lemma 7}, \\
\iff & \sum_{f\in\mathcal{F}} \left(\frac{\mu^f(\mu^f-1)\sum_{k\in\mathcal{N}^f} r_k'(-h_k')}{1+\mu^f(\mu^f-1)\frac{\sum_{k\in\mathcal{N}^f} r_k'(-\gamma_k')}{\sum_{k\in\mathcal{N}^f}\gamma_k}} - \sum_{k\in\mathcal{N}^f} h_k \right) < 0, \\
\iff & \Omega'\left(H(\mu)\right) < 0 \text{ (see the proof of Lemma D).}
\end{aligned}$$

Therefore, the index approach gives us the exact same condition as the aggregative game approach.

E Proofs for Section 5

E.1 Proof of Proposition E.1

Using the definition of function S and equation (15), it is easy to see that $m(x)$ and $S(x)$ are jointly pinned down by:

$$m(x) = \frac{1}{\sigma-(\sigma-1)S(x)},$$
$$S(x) = x\left(1-m(x)\right)^{\sigma-1}.$$

Differentiating wrt x, we get:

$$\begin{aligned} m'(x) &= (\sigma-1)\,(m(x))^2\,S'(x), \\ S'(x) &= (1-m(x))^{\sigma-1} - (\sigma-1)xm'(x)\,(1-m(x))^{\sigma-2}. \end{aligned}$$

Solving out for S' and m' yields:

$$\begin{aligned} S'(x) &= \frac{(1-m(x))^{\sigma-1}}{1+(\sigma-1)^2 x\,(1-m(x))^{\sigma-2}\,(m(x))^2} > 0, \\ m'(x) &= \frac{(\sigma-1)\,(m(x))^2\,(1-m(x))^{\sigma-1}}{1+(\sigma-1)^2 x\,(1-m(x))^{\sigma-2}\,(m(x))^2} > 0. \end{aligned}$$

Since $\pi = mS$, it follows that $\pi' > 0$.

Applying the implicit function theorem to equation $\Omega(H) = 1$ yields:

$$\frac{dH^*}{dT^f} = \frac{S'\left(\frac{T^f}{H^*}\right)}{\sum_{g\in\mathcal{F}} \frac{T^g}{H^*} S'\left(\frac{T^g}{H^*}\right)} > 0.$$

Next, notice that

$$\begin{aligned} \frac{d\left(\frac{T^f}{H^*}\right)}{d\theta^f} &= \frac{1}{H^*}\left(1 - \frac{T^f}{H^*}\frac{dH^*}{dT^f}\right), \\ &= \frac{1}{H^*}\left(1 - \frac{\frac{T^f}{H^*} S'\left(\frac{T^f}{H^*}\right)}{\sum_{g\in\mathcal{F}} \frac{T^g}{H^*} S'\left(\frac{T^g}{H^*}\right)}\right) > 0, \end{aligned}$$

and that, for $g \neq f$,

$$\frac{d\left(\frac{T^g}{H^*}\right)}{d\theta^f} = -\frac{T^g}{H^{*2}}\frac{dH^*}{dT^f} < 0.$$

Therefore, points (ii) and (iii) follow immediately by applying the chain rule.

E.2 Proof of Proposition 13

Proof. Applying the implicit function theorem to equation (17), we see that, for every $x > 0$,

$$m'(x) = \frac{m(x)e^{-m(x)}}{\frac{1}{m(x)} + m(x)\,xe^{-m(x)}} > 0.$$

Notice also that $S(x) = x\,\mathrm{e}^{-m(x)} = 1 - \frac{1}{m(x)}$. Therefore, $S' > 0$ and $\pi' > 0$.

Points (ii) and (iii) follow by applying the implicit function theorem to equation

$$\sum_{f \in \mathcal{F}} S\left(\frac{T^f}{H}\right) = 1$$

as we did in E.1. $\square$

F Proofs for Section 7

F.1 Proof of Proposition 14

Proof. Using the notation introduced in Section 5, let

$$\hat{T}^M \equiv H^* S^{-1}\left(S\left(\frac{T^f}{H^*}\right) + S\left(\frac{T^g}{H^*}\right)\right). \tag{37}$$

If $T^M = \hat{T}^M$, we have:

$$\begin{aligned} 1 &= \sum_{l \in \mathcal{F}} S\left(\frac{T^l}{H^*}\right) \\ &= S\left(\frac{T^M}{H^*}\right) + \sum_{l \in \mathcal{F} \setminus (f \cup g)} S\left(\frac{T^l}{H^*}\right), \end{aligned}$$

where the first equality is the pre-merger equilibrium condition whereas the second equality follows from $T^M = \hat{T}^M$. As equilibrium is unique when demand is CES or multinomial logit, we have $\hat{H}^* = H^*$. That is, the merger is CS-neutral if $T^M = \hat{T}^M$. As $S'(\cdot) > 0$, if $T^M > \hat{T}^M$, we have

$$S\left(\frac{T^M}{H^*}\right) + \sum_{l \in \mathcal{F} \setminus (f \cup g)} S\left(\frac{T^l}{H^*}\right) > 1,$$

implying that $\hat{H}^* > H^*$, so the merger is CS-increasing. Similarly, if $T^M < \hat{T}^M$, then $\hat{H}^* < H^*$, so the merger is CS-decreasing.

Next, we note that a CS-neutral merger involves synergies in that $\hat{T}^M > T^f + T^g$. Suppose otherwise that $\hat{T}^M \leq T^f + T^g$. Then,

$$\begin{aligned} S\left(\frac{\hat{T}^M}{H^*}\right) &\leq S\left(\frac{T^f + T^g}{H^*}\right) \\ &< S\left(\frac{T}{H^*}\right) + S\left(\frac{T^g}{H^*}\right), \end{aligned}$$

where the first inequality follows from $S'(\cdot) > 0$ and the second from $S''(\cdot) < 0$, as can be verified to hold under both CES and multinomial logit demands. But then the merger would be CS-decreasing, a contradiction. Hence, $\hat{T}^M > T^f + T^g$.

To see that a CS-neutral merger is profitable, note that:

$$\begin{aligned}
\pi\left(\frac{\hat{T}^M}{H^*}\right) &> m\left(\frac{\hat{T}^M}{H^*}\right) S\left(\frac{\hat{T}^M}{H^*}\right) \\
&= m\left(\frac{\hat{T}^M}{H^*}\right)\left[S\left(\frac{T^f}{H^*}\right) + S\left(\frac{T^g}{H^*}\right)\right] \\
&> m\left(\frac{T^f}{H^*}\right) S\left(\frac{T^f}{H^*}\right) + m\left(\frac{T^g}{H^*}\right) S\left(\frac{T^g}{H^*}\right) \\
&= \pi\left(\frac{T}{H^*}\right) + \pi\left(\frac{T^g}{H^*}\right),
\end{aligned}$$

where the second equality follows because the merger is CS-neutral, and the second inequality follows because $\hat{T}^M > T^f + T^g > \max\left(T^f, T^g\right)$ and $m'(\cdot) > 0$, both under CES and multinomial logit demands. Hence, merger M is profitable if $T^M = \hat{T}^M$.

Next, consider the effect of an increase in firm type T^M on the equilibrium level of the aggregator H^*. Applying the implicit function theorem to $\Omega(H^*) \equiv \sum_{u\in\mathcal{F}} S\left(\frac{T^u}{H^*}\right) = 1$, we obtain:

$$\frac{dH^*}{dT^M} = \frac{S'\left(\frac{T^M}{H^*}\right)}{\sum_{u\in\mathcal{F}} \frac{T^u}{H^*} S'\left(\frac{T^u}{H^*}\right)} > 0,$$

where the inequality follows as $S'(\cdot) > 0$. The effect of an increase in T^M on M's equilibrium profit is thus given by

$$\begin{aligned}
\frac{d\pi\left(\frac{\hat{T}^M}{H^*}\right)}{T^M} &= \frac{\pi'\left(\frac{T^M}{H^*}\right)}{H^*}\left(1 - \frac{T^M}{H^*}\frac{dH^*}{dT^M}\right) \\
&= \frac{\pi'\left(\frac{T^M}{H^*}\right)}{H^*}\left(1 - \frac{T^M}{H^*}\frac{S'\left(\frac{T^M}{H^*}\right)}{\sum_{u\in\mathcal{F}} \frac{T^u}{H^*} S'\left(\frac{T^u}{H^*}\right)}\right) > 0,
\end{aligned}$$

where the inequality follows as $\pi(\cdot) = m(\cdot)S(\cdot)$, $m'(\cdot) > 0$ and $S'(\cdot) > 0$, implying that $\pi'(\cdot) > 0$. That is, an increase in a firm's type induces an increase in that firm's equilibrium profit. It follows that merger M is profitable if $T^M \geq \hat{T}^M$, i.e., it is CS-nondecreasing. $\square$

F.2 Proof of Proposition 15

We first prove the following lemma:

Lemma N. *In a multiproduct-firm pricing game with CES or Logit demands, $\varepsilon' < 0$, where $\varepsilon(x) = x\frac{S'(x)}{S(x)}$ for all $x > 0$.*

Proof. Under CES demands,

$$\begin{aligned}\varepsilon(x) &= \frac{1}{1 + (\sigma - 1)^2 x(1 - m(x))^{\sigma-2} m(x)^2},\\ &= \frac{1}{1 + (\sigma - 1)^2 S(x)\frac{m(x)^2}{1-m(x)}},\end{aligned}$$

which is indeed decreasing in x, since m and S are decreasing.

Under Logit demands,

$$\varepsilon(x) = \frac{1}{1 + m(x)^2 S(x)},$$

which is also decreasing in x, since m and S are decreasing. □

We can now prove Proposition 15:

Proof. We first show that $d\hat{T}^M/dH^* < 0$. Differentiating equation (37), we obtain

$$\begin{aligned}S'\left(\frac{\hat{T}^M}{H^*}\right)\frac{d\hat{T}^M}{dH^*} &= \frac{\hat{T}^M}{H^*}S'\left(\frac{\hat{T}^M}{H^*}\right) - \frac{T^f}{H^*}S'\left(\frac{T^f}{H^*}\right) - \frac{T^g}{H^*}S'\left(\frac{T^g}{H^*}\right),\\ &= \varepsilon\left(\frac{\hat{T}^M}{H^*}\right)S\left(\frac{\hat{T}^M}{H^*}\right) - \varepsilon\left(\frac{T^f}{H^*}\right)S\left(\frac{T^f}{H^*}\right) - \varepsilon\left(\frac{T^g}{H^*}\right)S\left(\frac{T^g}{H^*}\right),\\ &= \varepsilon\left(\frac{\hat{T}^M}{H^*}\right)\left(S\left(\frac{T^f}{H^*}\right) + S\left(\frac{T^g}{H^*}\right)\right) - \varepsilon\left(\frac{T^f}{H^*}\right)S\left(\frac{T^f}{H^*}\right) - \varepsilon\left(\frac{T^g}{H^*}\right)S\left(\frac{T^g}{H^*}\right),\\ &< 0,\end{aligned}$$

where the third line follows by definition of $\hat{T}^M$ and the last line follows from Lemma N and from the fact that $\hat{T}^M > T^f + T^g$.

Suppose M_i is CS-nondecreasing in isolation, which means that $T^{M_1} \geq \hat{T}^{M_1}$. If the CS-nondecreasing merger M_j takes place, the equilibrium value of the aggregator H^* weakly increases, and so the cutoff $\hat{T}^{M_1}$ weakly decreases. As T^{M_1} was initially above the cutoff, it therefore remains so after M_j has taken place, i.e., M_i is still CS-nondecreasing. A similar argument can be used to show the sign-preserving complementarity for mergers that are CS-decreasing in isolation. □

References

Anderson, S. P., A. De Palma, and J.-F. Thisse (1987): "The CES is a discrete choice model?," *Economics Letters*, 24(2), 139–140.

Anderson, S. P., A. de Palma, and J.-F. Thisse (1992): *Discrete Choice Theory of Product Differentiation*. Cambridge, MA: MIT Press.

Armstrong, M., and J. Vickers (2010): "Competitive Non-linear Pricing and Bundling," *Review of Economic Studies*, 77(1), 30–60.

Bernard, A. B., S. J. Redding, and P. K. Schott (2010): "Multiple-Product Firms and Product Switching," *American Economic Review*, 100(1), 70–97.

——— (2011): "Multiproduct Firms and Trade Liberalization," *The Quarterly Journal of Economics*, 126(3), 1271–1318.

Berry, S., J. Levinsohn, and A. Pakes (1995): "Automobile Prices in Market Equilibrium," *Econometrica*, 63(4), 841–90.

Berry, S. T. (1994): "Estimating Discrete-Choice Models of Product Differentiation," *RAND Journal of Economics*, 25(2), 242–262.

Breinlich, H., V. Nocke, and N. Schutz (2015): "Merger Policy in a Quantitative Model of International Trade," CEPR Discussion Papers 10851, C.E.P.R. Discussion Papers.

Caplin, A., and B. Nalebuff (1991): "Aggregation and Imperfect Competition: On the Existence of Equilibrium," *Econometrica*, 59(1), 25–59.

Chan, T. Y. (2006): "Estimating a continuous hedonic-choice model with an application to demand for soft drinks," *RAND Journal of Economics*, 37(2), 466–482.

Chen, Y., and M. H. Riordan (2013): "Profitability Of Product Bundling," *International Economic Review*, 54(1), 35–57.

Dhingra, S. (2013): "Trading Away Wide Brands for Cheap Brands," *American Economic Review*, 103(6), 2554–84.

Dubin, J. A., and D. L. McFadden (1984): "An Econometric Analysis of Residential Electric Appliance Holdings and Consumption," *Econometrica*, 52(2), 345–62.

Eckel, C., and J. P. Neary (2010): "Multi-Product Firms and Flexible Manufacturing in the Global Economy," *Review of Economic Studies*, 77(1), 188–217.

Farrell, J., and C. Shapiro (1990): "Horizontal Mergers: An Equilibrium Analysis," *American Economic Review*, 80(1), 107–26.

Hanemann, W. M. (1984): "Discrete-Continuous Models of Consumer Demand," *Econometrica*, 52(3), 541–61.

Hanson, W., and K. Martin (1996): "Optimizing Multinomial Logit Profit Functions," *Management Science*, 42(7), 992–1003.

Mayer, T., M. J. Melitz, and G. I. P. Ottaviano (2014): "Market Size, Competition, and the Product Mix of Exporters," *American Economic Review*, 104(2), 495–536.

McAfee, R. P., J. McMillan, and M. D. Whinston (1989): "Multiproduct Monopoly, Commodity Bundling, and Correlation of Values," *The Quarterly Journal of Economics*, 104(2), 371–383.

Milgrom, P., and J. Roberts (1990): "Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities," *Econometrica*, 58(6), 1255–77.

Nevo, A. (2001): "Measuring Market Power in the Ready-to-Eat Cereal Industry," *Econometrica*, 69(2), 307–42.

Nocke, V., and N. Schutz (2015): "Quasilinear Integrability," Unpublished manuscript.

Nocke, V., and M. D. Whinston (2010): "Dynamic Merger Review," *Journal of Political Economy*, 118(6), 1201 – 1251.

Nocke, V., and S. Yeaple (2014): "Globalization And Multiproduct Firms," *International Economic Review*, 55, 993–1018.

Novshek, W., and H. Sonnenschein (1979): "Marginal Consumers and Neoclassical Demand Theory," *Journal of Political Economy*, 87(6), 1368–76.

Schmalensee, R. (1984): "Gaussian Demand and Commodity Bundling," *The Journal of Business*, 57(1), S211–30.

Selten, R. (1970): *Preispolitik der Mehrproduktenunternehmung in der Statischen Theorie.* Springer Verlag: Berlin, Germany.

Smith, H. (2004): "Supermarket Choice and Supermarket Competition in Market Equilibrium," *Review of Economic Studies*, 71(1), 235–263.

Spady, R. H. (1984): "Non-Cooperative Price-Setting by Asymmetric Multiproduct Firms," Unpublished manuscript.

Topkis, D. M. (1998): *Supermodularity and Complementarity.* Princeton, New Jersey: Princeton University Press.

Vives, X. (2000): *Oligopoly Pricing: Old Ideas and New Tools.* Cambridge, MA: MIT Press.

Williamson, O. E. (1968): "Economies as an Antitrust Defense: The Welfare Tradeoffs," *American Economic Review*, 58(1), 18–36.