Revenue Ranking in Hybrid Auctions with Speculative Resale

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Preliminary, please do not distribute.
This version is intended to give a background for the
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Abstract

In this paper, We compare revenues in hybrid auctions with speculative resale. We consider a hybrid auction in the first stage with symmetric independent private values (IPV) among N regular bidders and one speculator who has no value for the object. The winner pays a weighted sum of the winning bid and the highest losing bid with $\theta \in (0, 1]$ being the weight for the winning bid. In the second stage there is resale among the bidders. The winner in the first stage auction uses an optimal mechanism to sell the object to the losing bidders. There is revenue equivalence when $N = 1$. When $N > 1$, revenue is increasing in θ. We conjecture that this ranking result is independent of the bid revelation policy. We discuss why the ranking differs from that of common-value auctions even though the auction with resale model here is strategically equivalent to a common-value auction.

Keywords: Speculative Resale, Mechanism Design, Revenue Ranking, Bid Disclosure, Hybrid Auction,

1 Introduction

In this paper, we compare auction revenues when speculative resale is allowed. There are N regular buyers with independently distributed use values for the single object for sale. We assume the virtual value function is increasing. There is one speculator who has no value for the object, but is interested in bidding for resale to other regular buyers. There are two stages. In the first stage, the auction can be quite arbitrary, but we focus on the hybrid auctions in this paper. A hybrid auction is a mixture of the first-price and second-price auctions. The winner of the object pays a convex combination of the winning bid and the highest losing bid with the weight $\theta \in [0, 1]$ placed on the winning bid. We shall refer to it as a $\theta-$hybrid auction. First-price auctions and second-price auctions are special cases of $\theta-$hybrid auctions with $\theta = 1, 0$ respectively. In the second stage, the winner, who may be a speculator or a regular buyer, may sell to the other losing bidders. We assume that the second-stage auction is an optimal auction with commitment. We also assume that the original seller, the auctioneer, also has commitment to his auction in the sense that if the object is not sold in the first stage due to a reserve price, the game ends, and there is no second-stage auction. We assume no discounting in this model.

At the end of the first-stage auction, and before the start of the resale auction, some bids may be announced. There are essentially two different bid revelation policies. In one case, only the winning bid is announced, and this is equivalent to no bid revelation, as the winner of the first stage auction has no

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new information, and the optimal mechanism outcome at the resale stage is the same whether the winning bid is known by the losing bidders or not. We refer to this case as no bid revelation. In the second case, both the winning bid and the highest losing bid is announced. The revelation of all other lower bids have no impact on the resale outcome due to the independence assumption on use values. Hence, we shall refer to this case as full bid revelation. In a hybrid auction with $\theta < 1$, we face a full bid revelation case. The winner of the first-stage auction knows the highest winning and losing bids from the payment he makes. The losing bidder with the highest losing bid also knows both bids before the start of the resale stage. It makes no difference whether they are announced or not. The knowledge may allow the winner to extract all the surplus from the losing bidder with the highest use value when his bidding strategy is strictly increasing. Knowing this, the regular buyers may need to use mixed strategies to hide their true values. There may not exist a pure strategy equilibrium. Therefore we need to consider mixed strategy equilibria in the case of full bid revelation. In this paper, we focus on the no bid revelation case for two reasons. First it is easier to understand the intuition of our results with the pure strategy equilibrium. Second, the equilibrium revenue of the auction with no bid revelation can be replicated by an auction with full bid revelation, using the ideas in Lebrun (2010). We conjecture that the equilibrium revenue is independent of the bid revelation policy, as the full bid revelation induces the regular buyers to adopt a strategy to hide their values so that the revenue is the same as if there is no bid revelation.

In this framework of hybrid auctions with speculative resale, the revenue equivalence theorem no longer holds. This is not unexpected, as we have asymmetry between the speculator and the regular buyers. We show that the first-price auction yields the highest revenue among all $\theta > 0$ for any given reserve price ρ. The equilibrium and revenue ranking for second-price or Vickrey auctions (when $\theta = 0$) is treated elsewhere in Cheng (2015 a,b) and there is no difference in the revenue ranking result.

To establish this result, we take the following steps: (1) We show that the speculator makes zero profit in an equilibrium; (2) We establish a revenue formula in our model based on the virtual value concept of Myerson (1981). This extension makes use of the holding probabilities of the regular buyers. The holding probability of a buyer is the probability that the object ends in the hand of the person at the end of the game; (3) We show that regular buyers bid lower in equilibrium, and have lower holding probabilities when θ is higher; (4) From the revenue formula, we show that the lower holding probability of the regular buyers (thus higher holding probability of the speculator) leads to higher revenue.

In the first-step, the main intuition is related to the zero profit property in Bertrand competition. In the classical Bertrand competition by firms or sellers, any positive profit is driven away by slightly lower prices from the rival buyers. In our case, we are dealing with buyers' competition through bids. Any positive profit is driven away by slightly higher bids from competing buyers. This is more complicated to show than the classical result, and is done in Cheng and Tan (2014) for first-price auctions. The proof applies to hybrid auctions with $\theta > 0$.

In the second step, the formula we establish allows us to compute the total expected payments by the regular buyers in both stages of the auction. This includes payments to the auctioneer as well as the speculator. Since the speculator makes zero profit, the payments to the speculator end up in the hand of the auctioneer. Hence the total payment is the same as the auction revenue. Hence the holding probability of the speculator is not needed for the revenue formula.

In the third step, the intuition behind the result is that the first-price auction demands the highest payment (being the highest bid) after winning, and hence regular buyers respond by submitting the lowest bids. If there is no speculator, the revenue equivalence theorem implies that the effect from higher payment cancels the effect from lower bids, and there is no difference in revenue. When there is an active speculator in equilibrium, and $N > 1$, the revenue equivalence no longer holds. With lower bids from regular buyers in first-price auctions, the speculator makes the same revenue from resale while paying less after winning the first-stage auction with lower bids. Hence the higher profit opportunity induces the speculator to win more often. We can show that the holding probability of the regular buyers of any value v in first-price auctions is the lowest. In the final step, we show that this stochastic dominance property implies that the revenue is the highest when it is a first-price auction. When $N > 1$, we show that the revenue is strictly increasing in θ. When $N = 1$, we have revenue equivalence, and the revenue is independent of θ. It can be shown that the holding probability of a regular buyer is independent of θ.

Note that when $\theta = 1$ (i.e. the first-price auctions), the speculator has the highest holding probability, and the auction outcome is the least efficient among all $\theta > 0$, but has the highest revenue. The revenue ranking result reflects the well-known trade-off between revenue and efficiency.

Although our focus is on revenue comparisons, it is still important to establish certain results in the equilibrium bidding behavior. For $\theta > 0$, bidding behavior in $\theta-$hybrid auctions with speculative resale is similar to that in first-price auctions with resale, which has been examined, and characterized in Cheng and Tan (2014). For more detailed understanding about the equilibrium bidding strategy, we refer to that article. Here we only adopt the tools there for the analysis of hybrid auctions and establish important equilibrium properties related to the ranking results. Cheng and Tan (2014) also studied the optimal auctions in this framework. However, the result here is not about optimal auctions, but rather the comparison of different auctions given a reserve price.

Although the equilibrium with no bid revelation for hybrid auctions may involve impractical assumptions, it is theoretically sound and possible to implement in principle. First, the resale auction does not require the winner of the first stage auction to know the winning price to pay. Second, the winning price can be announced after the end of the resale auction. Third, the buyers can commit to a pricing strategy in the resale auction (mapping the winning bid to a reserve price) in the beginning. This leads to the same outcome of the model. Fourth, the winning price can be randomly simulated so that the winner pays the simulated price rather than the actual highest losing bid without affecting the equilibrium outcome of the model.

When there are two asymmetric buyers in an auction with resale, one weak, one strong, it has been shown by Hafalir and Krishna (2008) that first-price auction yields higher revenue than the truthful bidding equilibrium in the second-price auction with no reserve price ($\rho = 0$). In a two-bidder framework, Cheng and Tan (2010) have provided rather extensive analysis of the revenue ranking result in the setting of weak-strong pair of buyers.

Although we allow only one speculator, the auction revenue does not change when we have more speculators. Hence our result remains the same when there are many speculators. We also expect the first-price auction to be the dominant auction format when compared with other more general auctions.

It can be shown that the bidding behavior in our model is equivalent to that of a closely related common value auction. This observation seems to bring up an interesting question. We know that for common value auctions, the truthful bidding equilibrium in the second-price auction yields higher revenue than the first-price auction equilibrium due to bid revelation (the Linkage Principle). There are two reasons why the result is different here. First, our model starts with private values, and we have argued how the bid revelation has no effect on the revenue ranking result. The associated common value auctions are endogenously determined, and we are dealing with different common values when θ varies. Hence it is not surprising that we have different results in our setting.

Section two describes the model, and provides preliminary results needed for equilibrium analysis. Section three gives the equilibrium strategies when the speculator is absent or inactive. Section four deals with the equilibrium bidding of a regular buyer when a speculator is actively bidding. Section five gives the ranking result. Full equilibrium analysis is only given for pure strategies under no bid revelation in Section six.

2 Model and Preliminaries

We describe the model of hybrid auction with resale and speculation in section 2.1. Then we give explicit expressions of the payoffs for symmetric strategies in section 2.2. We will describe the supermodularity property for symmetric strategies in section 2.3.

2.1 Hybrid Auctions with Resale and Speculation

There are N regular symmetric buyers (referred to with a male pronoun) and one speculator[1] (referred to with a female pronoun) bidding for one object sold by the auctioneer (sometimes called the original seller to distinguish from the seller in the resale market). Regular buyers have use value for the object, but the

[1] An earlier version allows several speculators. The speculators can be combined into one with no change in the outcome of the equilibrium or the behavior of the regular buyers. For simplicity of exposition, we have only one speculator with no loss of generality in our result.

speculator has no use value for the object and participates in the auction only for resale. A regular buyer has a use value distribution $F(v)$ over $[0, \beta]$, which is assumed to be C^2 smooth[2]. Let $f(v)$ denote the density function of $F(.)$. We allow $F(0) \geq 0$, so that F may have an atom at 0. Note that a speculator is a bidder who has 0 use value for sure. A regular buyer may have 0 use value with a positive probability, but when it occurs, it does not make him a speculator. It is convenient to use the notation $F(.|v) = \frac{F(.)}{F(v)}$ for the conditional distribution of $F(.)$ if the use value has an upper bound v. The regular buyers are indexed by $i = 1, 2, ..., N$.

We study $\theta-$hybrid auctions with resale and speculation[3]. It is a two-stage game. In the first stage, the auctioneer sets a reserve price ρ for the speculator and regular buyers. A bid b is called fictitious if $b < \rho$ or $b = 0$. The highest bidder is the winner, and the winner pays

$$\theta b + (1-\theta)b^{(1)},$$

where b is the winning bid, and $b^{(1)}$ is the highest losing bid or ρ when the highest losing bid is below ρ.

A winner of the auction in the first stage may sell to the losers in the second stage when it is profitable. The seller in the resale stage chooses an optimal mechanism. By the revelation principle, we can assume that the resale mechanism is a modified second-price auction with optimally chosen reserve prices. At the end of the first-stage auction and before the beginning of the resale stage, the winning bid[4] (the highest bid) is announced. We will assume no discounting between the first stage and the second stage.

A bidding strategy of a regular buyer i is a random bid $b_i = b_i(\tilde{v}, \tilde{\omega})$. A regular buyer with use value v uses random bid $b_i(v, \tilde{\omega})$, whose cumulative bid distribution function (c.d.f.) is denoted by $G_i(v, .)$. We let $H_i(.), i = 1, 2, ..., N$ be the c.d.f. (of the marginal bid distribution) of a regular buyer i in the first stage. By definition, $H_i(.)$ is weakly increasing, continuous from the right. We assume $H_i(.)$ is C^1 smooth (or piecewise smooth) wherever it is continuous in $[\rho, \beta]$. There may be atoms for $H_i(.)$ so that a bid at ρ or above may have positive probability of being used by buyer i. Let $\phi_i(b)$ denote $\max\{v :$ the support of $G_i(v, .) \leq b\}$. The speculator uses a mixed strategy of bidding, represented by a c.d.f. $H(b)$. We allow $H(.)$ to be degenerate at 0, or have an atom at fictitious as well as non-fictitious bids. By definition, $H(b)$ is weakly increasing, and continuous from the right. The speculator may become inactive over some interval, and $H(b)$ is a constant over such an interval. The support of $H(b)$ may not be connected. Let $\bar{b}_i$,$\underline{b}_i$ be the maximum and minimum of the support of the bid distribution of regular buyer i above ρ, respectively. Let $\underline{b}_s$ be the minimum non-fictitious bid of the speculator if she is active in equilibrium. Let b_{0s} be the minimum bid of the support of $H(.)$.

The winner of the first-stage auction, i.e. the seller in the resale auction, will revise his or her belief about the losers' value distributions in the resale stage after observing all the bids submitted. When the speculator or buyer i wins the first-stage auction with a bid b, and the observed bid is b_j for buyer j, the updated belief of the winner regarding the value distribution of a losing regular bidder j is $G_j(.|b_j)$ which has a support bounded above by $\phi_j(b_j)$. A losing bidder also updates his belief about the winner i as $G_i(.|b)$. Updating of belief of a losing bidder regarding other losing buyers is not needed when the winner uses an optimal auction in the resale stage as he just bids truthfully during resale. Such revision of beliefs is common knowledge as all bids are public information. The game in the resale auction is well-known, and we can summarize the result of the resale auction by a profit function, so that we can focus on the first-stage bidding behavior anticipating the optimal resale outcome in the second stage.

[2] We can allow $F(.)$ to be piecewise C^2 smooth. This is a more natural framework for the analysis, as the equilibrium strategy in general is a maximum of two smooth functions. A typical example of a piecewise C^2 smooth function is the maximum or minimum of two smooth C^2 functions. Formally a function is called piecewise C^2 smooth if (i) it is C^2 except at a countable closed set D, (ii) At a point in D, the left and right derivatives (up to the second order) exist and are continuous from the right and left respectively.

[3] Haile (2000,2001,2003) did the pioneering works on auctions with resale. However, in his models, resale arises out of new information. Our models are fashioned along the lines of Hafalir and Krishna (2008), Garratt and Troger (2006), Cheng and Tan (2010) and Lebrun (2010), in which there is an incentive for resale due to asymmetry between buyers in the first stage auction. Second-price auctions with resale are studied in Pagnozzi (2007,2010), Garratt, Tröger, and Zheng (2009), Lebrun (2012).

[4] For the second-price auction with $N > 1$, the model is studied separately elsewhere. For $\theta \in (0, 1)$, the revelation of the highest losing bid may not allow any strictly increasing pure strategy equilibrium. Since the winner pays the expected second highest price, this payment can be generated stochastically, without informing the winner the actual second highest price. Another alternative is to assume that the resale trade is planned ahead of time without knowing the actual winning price.

To define an equilibrium strategy of the auction with resale, we shall focus on the bidding strategies in the first stage, taking the anticipated outcome of the second-stage as given. Let σ denote the strategy profile $H(.), b_i(.,.), i = 1, 2, ...N$, in the first stage. First consider the objective of the speculator's bidding behavior. If the speculator is the winner of the first-stage auction, the expected payment after winning at the bid b and observing the highest losing bid b_1,is

$$x_s(b, b_1, \sigma_{-s}) = \theta b \prod_{i=1}^{N} H_i(b) + (1-\theta)\max(b_1, \rho).$$

and the (unconditional) expected payment is

$$x_s(b, \sigma_{-s}) = \theta b \prod_{i=1}^{N} H_i(b) + (1-\theta) E\max(b_1, \rho)$$

with the expectation over the distribution of the highest losing bid. If only the winning bid is announced, we have

$$x_s(b, \sigma_{-s}) = \theta b \prod_{i=1}^{N} H_i(b) + (1-\theta)\int_0^b \max(\rho, y) d\left(\prod_{i=1}^{N} H_i(y)\right)$$

The speculator has zero payoff during resale if she loses the first-stage auction. If she wins, let $\pi_s(b, \sigma_{-s})$ denote the optimal (unconditional) expected revenue of the speculator during the resale stage. The speculator chooses b to maximize the overall expected profit

$$u_s(b, \sigma) = \pi_s(b, \sigma_{-s}) - x_s(b, \sigma_{-s}).$$

Given σ, we say that b is an optimal bid for the speculator if b is an optimal solution of the above maximization problem. Since the speculator uses a mixed strategy, all bids in the support of the equilibrium bid distribution $H(b)$ must yield the same payoff to the speculator.

Now we consider a regular buyer's objective function. When a regular buyer i with use value v_i wins the first-stage auction with the bid b, the expected payment with the highest losing bid b_1 is

$$x_i(b, b_1, \sigma_{-i}) = \theta b H(b) \prod_{j \neq i} H_j(b) + (1-\theta)\max(b_1, \rho),$$

and the (unconditional) expected payment for the object is

$$x_i(b, \sigma_{-i}) = \theta b H(b) \prod_{j \neq i} H_j(b) + (1-\theta) E\max(b_1, \rho).$$

When only the winning bid is announced, we have

$$x_i(b, \sigma_{-i}) = \theta b H(b) \prod_{j \neq i} H_j(b) + (1-\theta)\int_0^b \max(\rho, y) d\left(H(b)\prod_{j \neq i} H_j(y)\right).$$

Let

$$\pi_{1i}(v_i, b, \sigma_{-i}) = v_i H(b) \prod_{j \neq i} H_j(b) - x_i(b, \sigma_{-i}).$$

be his payoff in the first-stage auction. The updating of information is similar. Buyer i may sell the object to buyer j during the resale stage if $\phi_j(b) > v_i$. Let $\pi_{wi}(v_i, b, \sigma_{-i})$ be his (unconditional) expected payoff in the resale market after winning.

If buyer i loses the auction, and the winner is the speculator or some regular buyer j with $\phi_j(b) < v_i$, buyer i may bid for the object and buy it from the winner during resale. Since the winner will use the winning bid and the bidding strategy of the losing buyer i in this case to update belief and determine the reserve price, the payoff of buyer i after losing the auction will depend not only on the strategy profile of the other buyers, but also on his own choice of the bidding strategy. When resale fails to materialize after a winner wins the object in the first stage, the winner keeps the object. Let $\pi_{li}(v_i, b, \sigma)$ be his (unconditional) expected payoff in the resale market after losing. Then the overall payoff from bidding b is

$$u_i(v_i, b, \sigma) = \pi_{1i}(v_i, b, \sigma_{-i}) + \pi_{wi}(v_i, b, \sigma_{-i}) + \pi_{li}(v_i, b, \sigma). \tag{1}$$

When all regular buyers use the same bidding strategy, and only the winning bid is announced, more explicit formulas for the payoffs are given below. Buyer i chooses b to maximize $u_i(v_i, b, \sigma)$ in (1). We say that b is an optimal or equilibrium bid for the regular buyer i with use value v_i if it is a solution to the maximization problem. We say that $\sigma = (H(.), b_i(.,.))$ is a perfect Bayesian equilibrium of the auction with resale if (i) for the speculator, any bid b in the support of $H(.)$ is an optimal bid, (ii) for each regular buyer i with value v_i, any bid b in the support of $G_i(v_i, .)$ is an optimal bid maximizing (1).

A perfect Bayesian equilibrium of the auction with resale should describe strategies in both stages of the game. For convenience, we shall abuse the language somewhat and refer to σ as a perfect Bayesian equilibrium. We say that the equilibrium has symmetry property if all regular buyers have the same bid distribution $G(v, .)$ for all v.. When symmetry holds, the equilibrium strategy of a regular buyer will be denoted by one single bidding strategy. We say that the speculator is inactive in equilibrium if she submits fictitious bids for sure, otherwise we say that she is active in equilibrium. We take the convention that when the speculator is inactive in equilibrium, she bids 0 for sure (although bidding below ρ yields the same outcome). We also take the convention that regular buyers with $v < \rho$ all bid 0.

2.2 Payoffs Under Symmetry

When the symmetry property holds, the resale auction by the speculator is a symmetric auction with a single reserve price. We will write down the detailed payoff expression in this case when the bidding strategy is a pure strategy and only the winning bid is announced. It is convenient for the presentation to assume that the virtual value is increasing, so that there is a unique optimal reserve price in the resale stage. Let

$$J(x, w) = x - \frac{F(w) - F(x)}{f(x)}, 0 \le x \le w \le \beta.$$

denote the conditional virtual value at x when the buyer value upper bound is w. If $J(x, \beta)$ is strictly increasing in x, then $J(x, w)$ is also strictly increasing for all w. If the seller has use value v_0, the optimal reserve price $r(v_0, w)$ conditioned on the upper bound w is determined by the solution of the following equation in x

$$J(x, w) = v_0.$$

We let $v_0 = 0$ when the seller is the speculator. The increasing virtual value property of $J(x, \beta)$ is made to insure the uniqueness of the optimal reserve price. When the seller is the speculator, we may also use the simpler notation $r(w)$, or $r(v)$ when $w = v$. This optimal reserve price $r(v)$ is independent of the number of buyers, and satisfies the following equation

$$r(v)f(r(v)) + F(r(v)) = F(v). \tag{2}$$

Note also that we have $r(v) < v$ for all $v > 0$.

When a speculator sells to N regular buyers with the value distribution $F(.|v)$, the optimal revenue, according to Myerson (1981), is given by

$$B(v) = \int_{r(v)}^{v} J(x, v) dF^N(x|v) = \frac{N}{F^N(v)} \int_{r(v)}^{v} [xf(x) + F(x) - F(v]F^{N-1}(x)dx. \tag{3}$$

We shall call this the revenue function of the speculator. When $N = 1$, we have $B(v) = r(v)(1 - \frac{F(r(v))}{F(v)}), B^t(v) = r(v)$. The function $B(.)$ is not defined at $v = 0$, but we can let $B(0) = 0$, which will make $B(v)$ a continuous function over $[0, \beta]$. A regular buyer with use value $z \leq v$ contributes the following revenue in the optimal auction to the speculator

$$b_{r(v)}(z) = z - \int_{r(v)}^{z} F^{N-1}(x|z)dx.$$

Let

$$B^t(v) = b_{r(v)}(v) = v - \int_{r(v)}^{v} F^{N-1}(x|v)dx.$$

We have the following useful alternative formula, by the revenue equivalence theorem, for $B(v)$:

$$B(v) = \int_0^v b_{r(v)}(x)dF^N(x|v) = \int_0^v B^t(x)dF^N(x|v). \tag{4}$$

The speculator's objective function for $b \geq \rho$ is

$$u_s(b, \phi) = [B(\phi(b)) - \theta b] F^N(\phi(b)) - (1-\theta)\left[\rho F^N(\rho) + \int_\rho^b y dF^N(\phi(y))\right]. \tag{5}$$

Without the speculator and without resale, $N > 1$, the payoff function of a regular buyer can be written as

$$u_0(v_i, b, \phi) = F^{N-1}(\phi(b))v_i - x_i(b, \phi). \tag{6}$$

To write the explicit payoffs for a regular buyer when there is a speculator, we divide into two cases.

Case 1) (over bidding) $v_i \leq \phi(b)$.
In this case, when the regular buyer wins, he sets the optimal reserve price $r(v_i, \phi(b))$. The payoff is

$$\begin{aligned} & u_{1i}(v_i, b, \sigma) + \pi_{wi}(v_i, b, \sigma) \\ = {} & v_i H(b) F^{N-1}(\phi(b)) - x_i(b, \sigma_{-i}) + H(b) \int_{r(v_i, \phi(b))}^{\phi(b)} (J(x, \phi(b)) - v_i) dF^{N-1}(x|\phi(b)). \end{aligned} \tag{7}$$

When the buyer loses the auction, there is positive payoff after losing only if the speculator wins. When the speculator wins the first-stage auction with a bid y, the optimal reserve price set by her during resale is $r(\phi(y))$. Let

$$\begin{aligned} \tilde{y}(v_i) &= b(r^{-1}(v_i)), \text{ when } v_i < r(\beta), \\ &= \bar{b}_s, \text{ when } v_i \geq r(\beta). \end{aligned}$$

This is the highest winning bid of the speculator at which the buyer may buy from her during resale. Since $r^{-1}(v_i) > v_i$, when $v_i \in (0, r(\beta))$, we have $\tilde{y}(v_i) > b(v_i)$ for all $v_i \in (0, \beta)$. When buyer i bids $b \geq \tilde{y}(v_i)$, $\pi_{li}(v_i, b, \sigma) = 0$ as any winner will set a reserve price above his use value. We have $u_i(v_i, b, \sigma)$ given by (7). When $b \in [b(v_i), \tilde{y}(v_i))$, we have $\pi_{li}(v_i, b, \sigma) = \pi_{lis}(v_i, b, \sigma)$, given by

$$\pi_{lis}(v_i, b, \sigma) = \int_b^{\tilde{y}(v_i)} \left(\int_{r(\phi(y))}^{v_i} F^{N-1}(x)dx \right) dH(y). \tag{8}$$

Hence, $u_i(v_i, b, \sigma)$ is given by the sum of (7) and (8).

Case 2) (under bidding) $v_i > \phi(b)$.
There is zero payoff from resale after winning the first-stage auction ($\pi_{wi}(v_i, b, \sigma) = 0$), but there is payoff after losing. If $N > 1$, the winner of the first-stage auction may be a regular buyer. The winner believes

that he has the highest value, and is indifferent between no resale or offering it for resale at the price equal to his own value. Hence the payoff after losing, then buying from a regular buyer, denoted by $\pi_{li0}(v_i, b, \sigma)$, is either zero, or given by

$$\pi_{li0}(v_i, b, \sigma) = \int_b^{b(v_i)} (v_i - \phi(y))H(y)dF^{N-1}(\phi(y)). \tag{9}$$

If the regular buyer i deviates to a bid $b_i < b_i(v_i)$, and loses the auction to a speculator with the winning bid $y > b$, the speculator sets the optimal reserve price $r(\phi(y))$ based on her updated belief. When $y \in [b(v_i), \tilde{y}(v_i))$, the payoff during resale buying from the speculator is given by

$$\int_{b(v_i)}^{\tilde{y}(v_i)} \left(\int_{r(\phi(y))}^{v_i} F^{N-1}(x)dx \right) dH(y) \tag{10}$$

When $y \in (b, b(v_i))$, the payoff of the regular buyer buying from the speculator during resale is

$$\int_b^{b(v_i)} \left((v - r(\phi(y)))F^{N-1}(r(\phi(y))) + \int_{r(\phi(y))}^{\phi(y)} (v - x)dF^{N-1}(x) \right) dH(y).$$

From integration by parts, it can be written as

$$\int_b^{b(v_i)} \left((v - \phi(y))F^{N-1}(\phi(y)) + \int_{r(\phi(y))}^{\phi(y)} F^{N-1}(x)dx \right) dH(y). \tag{11}$$

The expected payoff after losing, then buying from a speculator, is the sum of (10) and (11) is given by

$$\pi_{lis}(v_i, b, \sigma) = \int_b^{b(v_i)} (v - \phi(y))F^{N-1}(\phi(y))dH(y) + \int_b^{\tilde{y}(v_i)} \left(\int_{r(\phi(y))}^{\min(v_i, \phi(y))} F^{N-1}(x)dx \right) dH(y). \tag{12}$$

Hence, we have

$$\pi_{li}(v_i, b, \sigma) = \pi_{lis}(v_i, b, \sigma) + \pi_{li0}(v_i, b, \sigma). \tag{13}$$

and the total payoff expression

$$u_i(v_i, b, \sigma) = v_i H(b)F^{N-1}(\phi(b)) - x_i(b, \sigma_{-i}) + \pi_{li}(v_i, b, \sigma).$$

2.3 Supermodularity Property

Let $\frac{\partial}{\partial b_+}, \frac{\partial}{\partial b_-}$ denote the partial derivatives from the right, or left respectively. We say that the payoff function has the supermodularity property in the strong form at (v_i, b) if

$$\frac{\partial}{\partial b_{+(-)}} \frac{\partial}{\partial v_i} u(v_i, b, \sigma) > 0. \tag{14}$$

It has the supermodularity in the weak form at (v_i, b) if

$$\frac{\partial}{\partial b_{+(-)}} \frac{\partial}{\partial v_i} u(v_i, b, \sigma) \geq 0. \tag{15}$$

When there is no speculator, and no resale allowed, we have

$$\frac{\partial}{\partial v_i} u_0(v_i, b, \phi) = F^{N-1}(\phi(b)).$$

Notice that $\frac{\partial}{\partial v_i} u_0(v_i, b, \phi)$ is the winning probability of a regular buyer with value v_i bidding b, hence

$$\frac{\partial}{\partial b}\frac{\partial}{\partial v_i} u(v_i, b, \sigma) = \frac{\partial}{\partial b} F^{N-1}(\phi(b)) > 0,$$

and we have the supermodularity property in the strong form.

We can now prove the supermodularity property from Lemma 5. The proof is similar to the case of the first-price auction in Cheng and Tan (2014).

Theorem 1 *If the regular buyers use the same increasing strategies, then the supermodularity property in the strong form (14) holds when $H(b) > 0$, $\beta > \phi(b) \geq v_i > \rho$. Otherwise, the weak form (15) holds.*

In the auction without speculator or resale, the supermodularity property always holds in the strong form. The above supermodularity property is a little weaker, but is sufficient for the first-order condition $\frac{\partial}{\partial b} u(v_i, b, \sigma)|_{b=b(v_i)} = 0$ to yield the optimality property for the strategy $b(v_i)$.

3 Equilibrium with No Speculator

In this section, we consider first the case when there is no speculator, but resale is allowed. We will deal with the case of no bid revelation, and the regular buyers use pure bidding strategies. The hybrid auctions have a simple equilibrium strategy.

Theorem 2 *Let the first-stage auction be the $\theta-$hybrid auction and there is no speculator, and no resale. Let $\rho < \beta$ be the reserve price and $N > 1$. Then the equilibrium is unique and is given by*

$$b^*_\theta(v) = v - \int_\rho^v F^{\frac{N-1}{\theta}}(x|v)dx = \int_\rho^v x dF^{\frac{N-1}{\theta}}(x|v). \tag{16}$$

The equilibrium strategy $b^*_\theta(v)$ is strictly decreasing in θ at all $v > 0$.

Proof. The first-order condition of an optimal bid is

$$\frac{(N-1)f(\phi(b))}{F(\phi(b))}\phi'(b) = \frac{\theta}{\phi(b) - b}, \tag{17}$$

or

$$\frac{d}{db}\ln F^{\frac{N-1}{\theta}}(\phi(b)) = \frac{1}{\phi(b) - b},$$

and we get the unique solution (16) under the boundary condition $\phi(\rho) = \rho$. ■

It is convenient to write the first-order condition (17) in variable v, written as

$$b'(v) = \frac{(N-1)f(v)}{\theta F(v)}(v - b(v)) \tag{18}$$

This is also the first-order condition in an interval in which the speculator is not active. The equilibrium strategy depends on both ρ, θ, and will be denoted by $b^*_\rho(.), b^*_\theta(.)$, or depending on which variables need to be emphasized.

Under the symmetry assumption, we can easily show that the first-order condition without the speculator but allowing resale is the same as that of the model with no resale. Combine this with the supermodularity property, and we know that equilibrium must be unique in a model without a speculator but allowing resale.

Theorem 3 *If there is no speculator, the symmetric equilibrium with resale is unique and is the same as the one without resale given by Theorem (2).*

Proof. When resale is allowed, but there is no speculator (or $H(0) = 1$), we can write the payoff of a regular buyer i with value v bidding $b, \phi(b) < v$, as

$$u_i(v, b, \phi) = u_{1i}(v, b, \phi) + \pi_{li0},$$

and we have the first-order condition

$$\frac{\partial}{\partial b} u_i(v, b, \phi)|_{b=b(v)} = \frac{\partial}{\partial b} u_{1i}(v, b, \phi)|_{b=b(v)} = 0. \tag{19}$$

When $\phi(b) \geq v$, we have

$$u(v, b, \phi) = u_{1i}(v, b, \phi) + \int_{r(v,\phi(b))}^{\phi(b)} (J(x, \phi(b)) - v) dF^{N-1}(x|\phi(b)),$$

and we have the same first-order condition (19). Therefore if we allow resale with no speculator, the first-order condition of an equilibrium is the same as that of a model with no speculator and no resale. From the supermodularity property in the Appendix, the equilibrium with resale is the same as the equilibrium without resale. ■

Note that the equilibrium bidding strategy of a hybrid auction is decreasing in θ. When $\theta \to 0$, we have $b^*_\theta(v) \to v$. Thus we have truthful bidding in second-price auctions. Although the buyers bid higher when θ is lower, the expected payment after winning remains the same. This is the result of the revenue equivalence theorem. Therefore the revenue is independent of θ.

4 Equilibrium Condition of the Speculator

Here we also deal with the case of no bid revelation, and the regular buyers use pure bidding strategies. This framework gives us a better understanding of how regular buyers bid with the presence of a actively bidding speculator.

Let $\phi(.)$ be the inverse equilibrium strategy of the regular buyers. After winning the auction bidding b, the speculator sells to N regular buyers with the value distribution $F(.|\phi(b))$ in the resale market. The speculator profit, given the regular buyer's inverse bidding strategy $\phi(.)$, can be written as

$$u_s(b, \phi) = F^N(\phi(b))B(\phi(b)) - x_s(b, \phi).$$

Equilibrium profit for the speculator bidding b in the support of the $H(.)$ must be a constant independent of b. It has been shown that when the first-stage auction is a first-price auction, the speculator makes zero profit in equilibrium. The same proof for the first-price auction also applies to the $\theta-$hybrid auction, $\theta > 0$. See the proof in Cheng and Tan (2014). We state it as a lemma for easy reference.

Lemma 1 *In any symmetric equilibrium, the speculator makes zero expected profit, and $H(b) > 0$ for all non-fictitous bid b.*

When $\theta > 0$, $v \geq \rho, v > 0$, define

$$B_\theta(v) = B(\rho)F^{\frac{N}{\theta}}(\rho|v) + \int_\rho^v B^t(x)dF^{\frac{N}{\theta}}(x|v) = B^t(v) - (B^t(\rho) - B(\rho))F^{\frac{N}{\theta}}(\rho|v) - \int_\rho^v F^{\frac{N}{\theta}}(x|v)dB^t(x).$$

We have $B_\theta(v) < B^t(v)$, for all $v > 0$. The function $B_\theta(v)$ is strictly decreasing in θ at any $v > 0$. When $\rho = 0$, we let $B_\theta(0) = 0$, then $B_\theta(v)$ is a continuous function on $[0, \infty)$ for all $\theta > 0$, and $B_\theta(v) = B(v)$. The following lemma gives the derivative of the function $B_\theta(v)$. Note that $B_\theta(.), \eta_\theta(.)$ depend on ρ as well.

Lemma 2 *We have*

$$B'_\theta(v) = \frac{Nf(v)}{\theta F(v)}(B^t(v) - B_\theta(v)) > 0 \text{ at all } v > 0. \tag{20}$$

Proof. Since

$$B_\theta(v) = \frac{1}{F^{\frac{N}{\theta}}(v)}\left[B(\rho)F^{\frac{N}{\theta}}(\rho) + \int_\rho^v B^t(x)dF^{\frac{N}{\theta}}(x)\right],$$

we have

$$\begin{aligned} B_\theta(v) &= \frac{Nf(v)}{\theta F(v)}B^t(v) - \frac{Nf(v)}{\theta F^{\frac{N}{\theta}+1}(v)}\left[B(\rho)F^{\frac{N}{\theta}}(\rho) - \int_\rho^v F^{\frac{N}{\theta}}(x)dB^t(x)\right] \\ &= \frac{Nf(v)}{\theta F(v)}\left(B^t(v) - B_\theta(v)\right). \end{aligned} \tag{21}$$

■

The main usefulness of the function $B_\theta(.)$ is due to the following property. Let $\eta_\theta(.)$ be the inverse function of $B_\theta(.)$.

Lemma 3 *If the regular buyers use the bidding strategy $B_\theta(.)$, then a speculator bidding in the range of $B_\theta(.)$ makes zero expected profit.*

Proof. The expected profit of the speculator bidding b after winning is

$$B(\eta_\theta(b)) - x_s(b, \eta_\theta) = B(\eta_\theta(b)) - \theta b - \frac{1-\theta}{F^N(\eta_\theta(b))}\left[B(\rho)F^N(\rho) + \int_\rho^b ydF^N(\eta_\theta(y))\right] \tag{22}$$

Change to the variable $v = \eta_\theta(b)$, we have the expression of the right-hand side of (22) in v :

$$K(v) = B(v) - \theta B_\theta(v) - \frac{1-\theta}{F^N(v)}\left[B(\rho)F^N(\rho) + \int_\rho^v B_\theta(x)dF^N(x)\right]. \tag{23}$$

We want to show that the expression in (22) is equal to 0. This is equivalent to showing $K(v) = 0$ for all $v \geq \rho$. Taking the derivative of (23) with respect to v, we get

$$\begin{aligned} K'(v) &= B'(v) - \theta B'_\theta(v) - \frac{1-\theta}{F^N(v)}NF^{N-1}(v)f(v)B_\theta(v) + \frac{1-\theta}{F^{N+1}(v)}Nf(v)\left[\rho F^N(\rho) + \int_\rho^v B_\theta(x)dF^N(x)\right] \\ &= \frac{Nf(v)}{F(v)}[B_\theta(v) - B(v)] - (1-\theta)\frac{Nf(v)}{F(v)}B_\theta(v) + \frac{Nf(v)}{F(v)}[B(v) - \theta B_\theta(v)] \\ &= \frac{Nf(v)}{F(v)}[B_\theta(v) - B(v)] - \frac{Nf(v)}{F(v)}[B_\theta(v) - B(v)] = 0. \end{aligned}$$

Since $K(\rho) = 0$, we have $K(v) = 0$ for all v. The proof is complete. ■

The following example illustrates the function $B_\theta(v)$ for the case $N = 1, F(v) = v, \rho = 0$.

Example 1 *There is one speculator, one regular buyer with the use value distribution $F(v) = v$. Let $\rho = 0$. We have*

$$B_\theta(v) = \frac{1}{v^{1/\theta}}\int_0^v B^t(x)dx^{1/\theta},$$

where $B^t(v) = r(v) = \frac{v}{2}$. Hence

$$B_\theta(v) = \frac{v}{2(1+\theta)}.$$

5 Revenue Ranking

In Myerson (1981), one main idea is the relationship between the equilibrium payoff of a buyer and the equilibrium winning probabilities. We reinterpret the winning probabilities as the holding probabilities at the end of the second stage in the auction with resale, and establish the new expressions between the payoffs and the winning probabilities.

For symmetric strategies, the holding probabilities can be easily calculated when the regular buyers use pure strategies, and only the winning bid is announced. This is what we do in this section. When all the bids are announced, and the regular buyers adopt mixed strategies to avoid the revelation of new information after losing, then the calculations here are valid only if the updated belief is given by $G(.,b_1) = F(.|\phi(b_1))$ after observing the losing bid b_1 or at least the same beyond $x \geq r(\phi(b_1))$ so that there is no difference in the resale revenue. If we can prove that in equilibrium, this is correct, then we will have only one auctioneer revenue which is independent of the bid revelation policy. The conjecture is that it is indeed an equilibrium property, as intuitively, it is best for the buyers to use mixed bidding to hide information without hurting the incentive properties, and that means that either no new information information is revealed in equilibrium, or that whatever revealed information makes no difference in the equilibrium outcome. For example, if we announce only the lower losing bids (not the highest losing bid), then that information makes no difference in the equilibrium outcome.

Let $Q_{i1}(v_i, b, \sigma)$ be the probability of winning in the first-stage auction, but failing to resell, and $Q_{i2}(v_i, b, \sigma)$ be the probability of losing in the first-stage auction, but winning back during resale. Define the holding probability as the sum $Q_{i1}(v_i, b, \sigma) + Q_{i2}(v_i, b, \sigma)$. The following is the Myerson Lemma in our model. It has been proved in Cheng and Tan (2014) for the case of first-price auctions. The proof for the hybrid auctions is similar.

Lemma 4 *We have*

$$\frac{\partial}{\partial v_i} u_i(v_i, b, \sigma) = Q_{i1}(v_i, b, \sigma) + Q_{i2}(v_i, b, \sigma).$$

From the above lemma, we have the following immediate corollary.

Lemma 5 *In equilibrium, we have*

$$\frac{\partial}{\partial v_i} u_i(v_i, b, \sigma)|_{b=b(v_i)} = H(\tilde{y}(v_i))F^{N-1}(v_i).$$

Proof. By definition, we have

$$Q_{i1}(v_i, b, \sigma)|_{b=b(v_i)} = H(b)F^{N-1}(v_i). \tag{24}$$

The probability of losing to a speculator in the first stage, then buying it back is

$$Q_{i2}(v_i, b, \sigma)|_{b=b(v_i)} = F^{N-1}(v_i)(H(\tilde{y}(v_i)) - H(b)) \tag{25}$$

The sum of (24) and (25) gives us the result by Lemma 4. ■

In this model of speculative resale, we will establish the revenue formula similar to that of Myerson (1981) with some natural adjustments needed. Although regular buyers make payments to both the auctioneer and the speculator, all revenue ends up in the hands of the auctioneer as the speculator makes zero profit in equilibrium. From Lemma 5, and the envelop theorem, we have the following formula of the equilibrium payoff of a regular buyer. Let $\sigma = (H,\ b)$ be the equilibrium strategy profile, and $U(v) = u(v, b(v), \sigma)$ be the equilibrium payoff of a regular buyer with use value v. Recall the definition of the function $\tilde{y}(.)$ in (**??**).

Lemma 6 *The equilibrium payoff of a regular buyer is given by*

$$U(v) = \int_\rho^v H(\tilde{y}(x))F^{N-1}(x)dx.$$

The following revenue formula is derived from Lemma 6.

Theorem 4 *Let $\rho \geq 0$ be the reserve price. We have the following equilibrium revenue formula*

$$R(\theta) = \int_{\rho}^{\beta} H_\theta(\tilde{y}(x)) J(x,\beta) dF^N(x). \tag{26}$$

Proof. Let $t(x) = H(\tilde{y}(x))$ in the following computations. By Lemma 6, we must have the following expected contribution of revenue from a regular buyer with value v

$$F^{N-1}(v)t(v)v - \int_{\rho}^{v} F^{N-1}(x)t(x)dx.$$

Hence the expected revenue of the auction with resale is given by

$$\begin{aligned}
& N\int_{\rho}^{\beta} \left(F^{N-1}(v)t(v)v - \int_{\rho}^{v} F^{N-1}(x)t(x)dx \right) dF(v) \\
&= \int_{\rho}^{\beta} t(v)vdF^N(v) - N\int_{\rho}^{\beta} \left(\int_{\rho}^{v} F^{N-1}(x)t(x)dx \right) dF(v) \\
&= \int_{\rho}^{\beta} t(v)vdF^N(v) - N\int_{\rho}^{\beta} t(x)F^{N-1}(x) \left(\int_{x}^{\beta} dF(v) \right) dx \\
&= \int_{\rho}^{\beta} t(v)vdF^N(v) - N\int_{\rho}^{\beta} t(x)(1-F(x))F^{N-1}(x)dx \\
&= \int_{\rho}^{\beta} t(x)xdF^N(x) - \int_{\rho}^{\beta} t(x)(\frac{1-F(x)}{f(x)})dF^N(x) = \int_{\rho}^{\beta} t(x)J(x,\beta)dF^N(x).
\end{aligned}$$

This proves the theorem. ■

We now apply Theorem 4 to obtain the revenue ranking result of the paper. First we consider the case $N=1$. In this case, the speculator is fully active in equilibrium for all $\theta > 0$. We will show that the revenue is independent of θ, so that there is revenue equivalence for all hybrid auctions. If we adopt the equilibrium $r(v)$ strategy for the second-price auction, revenue equivalence applies to all $\theta \geq 0$. Revenue equivalence holds only for the case $N=1$. When $N>1$, the revenue is strictly increasing in θ. This is our main result.

Theorem 5 *If $N=1$, we have revenue equivalence for all $\theta-$hybrid auctions, $\theta > 0$. If $N>1$, the revenue is strictly increasing in θ.*

Proof. When $N=1$, the speculator is fully active in equilibrium. We have

$$\tilde{H}_\theta(v) = F(v),$$

and

$$H_\theta(\tilde{y}(v)) = H_\theta(b(r^{-1}(v))) = \tilde{H}_\theta(r^{-1}(v)) = F(r^{-1}(v)).$$

when $v \leq r(\beta)$, and $H_\theta(\tilde{y}(v)) = 1$ when $v > r(\beta)$.Hence

$$R(\theta) = \int_{\rho}^{r(\beta)} F(r^{-1}(x))J(x,\beta)dF(x) + \int_{r(\beta)}^{\beta} J(x,\beta)dF(x)$$

when $\rho \leq r(\beta)$, and

$$R(\theta) = \int_{\rho}^{\beta} J(x,\beta)dF(x)$$

when $\rho > r(\beta)$. The revenue is therefore independent of θ. If $N > 1$, the revenue is given by

$$R(\theta) = \int_{\rho}^{\beta} H_\theta(\tilde{y}(x))J(x,\beta)dF^N(x) = \int_{\rho}^{r(\beta)} \tilde{H}_\theta(r^{-1}(x))J(x,\beta)dF^N(x) + \int_{r(\beta)}^{\beta} J(x,\beta)dF^N(x).$$

Note that the holding probability of a regular buyer is smaller when θ is higher, as a regular buyer bids lower. The speculator becomes more active in bidding, and wins with higher probability. This means that the speculator holds the object more often, and the holding probability of a regular buyer is lower. Since the function $\tilde{H}_\theta(v)$ is strictly decreasing in θ, and $J(x,\beta) < 0$ in the first term on the right, we know $R(\theta)$ is strictly increasing in θ. ∎

Corollary 6 *If $N > 1$, the first-price auction yields strictly higher revenue than the second-price auction*

The following example illustrates the revenue equivalence for the uniform distribution case when $N = 1$.

Example 2 *There is one speculator, one regular buyer with the use value distribution $F(v) = v$. Let $\rho = 0$. The equilibrium bidding strategy of a regular buyer is given by*

$$B_\theta(v) = \frac{v}{2\theta + 2}.$$

We have

$$\tilde{H}_\theta(v) = H_\theta(B_\theta(v)) = v.$$

Therefore $\tilde{H}_\theta(v)$ is independent of θ. The revenue is given by

$$\int_0^{\frac{1}{2}} 2x(2x-1)dx + \int_{\frac{1}{2}}^{0} (2x-1)dx = \frac{1}{6}.$$

Note that the optimal bid of a regular buyer is decreasing in θ. However the expected payment of a regular buyer with value v is always $B(v)$, and so is the expected payment of the speculator. We therefore have the revenue equivalence for all θ.

The revenue equivalence is no longer true when $N > 1$. This is because the regular buyers with use value $v < r(\beta)$ have higher holding probability when θ is lower. Since the marginal revenue $J(x,\beta)$ is negative, this leads to a lower revenue for the original seller according to the revenue formula. Another good intuition is that as θ becomes lower, the speculator is less active in equilibrium. Thus regular buyers have higher holding probabilities, and higher equilibrium payoffs, hence they contribute less revenue to the original seller in equilibrium. The speculator has zero payoff, and she affects the revenue by making it more difficult for the regular buyers to win the object. The following example illustrates the ranking result.

Example 3 *There are two regular buyers, and one speculator. Let the regular buyer's use value distribution be*

$$F(v) = 0.5(1+v^2) \text{ over } [0,1].$$

Let $\rho = 0$. First we compute the equilibrium without the speculator. We have

$$b^*_\rho(v) = v - \frac{2^{1/\theta}}{(1+v^2)^{1/\theta}} \int_0^v (\frac{1+x^2}{2})^{1/\theta} dx = \int_0^v x dF^{1/\theta}(x|v).$$

When $\theta = 1$, the strategy is

$$b^*_\rho(v) = \frac{2}{3}\frac{v^3}{v^2+1},$$

and this is the equilibrium bidding strategy of the regular buyers in the first-price auction without the speculator. The revenue is

$$\int_0^1 \frac{2}{3}\frac{v^3}{v^2+1}d(\frac{1+v^2}{2})^2 = \int_0^1 \frac{4}{3}\frac{v^4}{v^2+1}(\frac{1+v^2}{2})dv = \frac{2}{15} = 0.133\,333$$

This is also the revenue of the truthful equilibrium in the second-price auction with a speculator. The virtual value is

$$J(x,\beta) = x - \frac{1-0.5(1+x^2)}{x} = \frac{1}{2x}(3x^2-1).$$

The optimal reserve price for revenue during resale is given by $r(v) = \frac{v}{\sqrt{3}}$. We can compute the revenue of the first-price auction with a speculator by the formula

$$\int_0^{\frac{1}{\sqrt{3}}} \tilde{H}(\sqrt{3}x)J(x,1)dF^2(x) + \int_{\frac{1}{\sqrt{3}}}^1 J(x,1)dF^2(x) = 0.140\,539\,7.$$

The revenue is larger than that of the second-price auction with a speculator.

6 Equilibrium Analysis

It is possible to have an explicit solution of the equilibrium bidding strategy profile and $H_\theta(.)$. Here we show how the equilibrium is analyzed. The main ideas for our analysis is very similar to that of the first-price auctions with resale in Cheng and Tan (2014). We show that the equilibrium profit of the speculator is 0. This determines the equilibrium strategy $b_\theta(v)$ of the regular buyers over the active intervals. For inactive intervals, the first-order condition is identical to the one without the speculator, and is determined from the boundary condition. The speculator bid distribution is determined by the first-order condition of the regular buyer's optimal bid. This gives rise to a simple solution of the bid distribution $H_\theta(b)$. Therefore the equilibrium is uniquely determined when the active or inactive intervals are uniquely determined. We then show that both $b_\theta(v), H_\theta(v)$ are strictly decreasing in θ for $v > 0$.

We say that the speculator is fully active in equilibrium if the support of $H(.)$ is the same as that of the equilibrium bid distribution of a regular buyer. In equilibrium, the speculator typically bids only in certain intervals with positive probability. To define an interval in which the speculator is active, we say that a bid interval $[a_1, a_2]$ is an active interval if $H'(.) > 0$ in the interval a.e. (everywhere, i.e. except a closed set of measure 0. A maximal active interval is a closed active interval which is not a proper subset of another closed active interval. A maximal active interval is simply a maximal connected component of the support of $H(.)$. When $[a_1, a_2]$ is a maximal active interval, we say that a_1 is the beginning of an active interval, and a_2 is the end of an active interval. An interval $[a_1, a_2]$ is inactive if it contains no active intervals. This is the same as saying $H(.)$ is constant over the interval. Since most of our functions are defined in the space of use values, It is convenient to call an interval $[z_1, z_2]$ of use values an active interval if $[B_\theta(z_1), B_\theta(z_1)]$ is an active interval of bids. Since we know $B_\theta(.)$ must be the equilibrium bidding strategy over an active interval $[z_1, z_2]$, an active value interval is simply the value-types of a regular buyer who bids inside an active bid interval in equilibrium. A maximal value interval can be similarly defined, so are the beginning and end of an active value interval, and the inactive intervals.

6.1 Equilibrium Conditions for the Regular Buyers

Let $N > 1, \rho > 0$. If the speculator is active in equilibrium, let $a(\rho) = \min\{b : b \geq \rho$, and b is in the support of $H(.)\}$ be the minimum non-fictious bid $a(\rho)$. The following lemma says we must have $\alpha(\rho) > \rho$, and regular buyers with value between ρ and $a(\rho)$ bid as if there is no speculator.

Lemma 7 *Let $\theta > 0, \rho > 0, N > 1$, and assume that the speculator is active in equilibrium Then we have $a(\rho) > \rho$.*

In principle a regular buyer with use value $v \leq \rho$ may be able to purchase from the winning speculator during resale if the optimal reserve price $r(\phi(y))$ is sufficiently low. In this case, a regular buyer with use value $v \leq \rho$ may have a positive winning probability. However, we will show that this is not possible in equilibrium so that such regular buyers have zero winning probability. This result is actually needed for our revenue formula in section 4.

Lemma 8 *In equilibrium, we have: (i) A regular buyer with use value $v < \rho$ bids below ρ, and has zero holding probability; (ii) A regular buyer with use value $v \leq \rho$ has zero payoff.*

The equilibrium condition for a regular buyer in inactive intervals is very similar to the equilibrium condition with no speculator. It can be easily seen that in inactive intervals, (18) is also the first-order condition of equilibrium. Therefore if the boundary condition is determined, then (18) uniquely determines the bidding behavior of the regular buyers in inactive intervals. When we solve (18) with the initial condition $b(z) = b_0$ at $z \geq \rho$, the solution is given by

$$b_0 F^{\frac{N-1}{\theta}}(z|v) + \int_z^v x dF^{\frac{N-1}{\theta}}(x|v),$$

for $v \in [z, \beta]$. We often take z to be either ρ, the beginning or the end of an active interval, and $b_0 = b(z)$. In this case, we use the notation $C_z(.)$ to denote the solution (27). When $b_0 = b(z) = B_\theta(z)$, we have

$$C_z(v) = B_\theta(z) F^{\frac{N-1}{\theta}}(z|v) + \int_z^v x dF^{\frac{N-1}{\theta}}(x|v). \tag{27}$$

In a symmetric increasing equilibrium only the speculator will sell the object after winning it in the first-stage auction. In determining whether the speculator is active or not, the two functions $b^*_\rho(.), C_z(.)$ can be regarded as "cost functions" for the speculator because it is the cost needed to win the object to be able to sell to N regular buyers with value distribution $F(.|v)$. Note that $C_z(v)$ is a strictly decreasing function of θ, as $F(z|v) > 1, F(x|v) < 1$.

Another function closely related to $B^t(v)$ is now introduced to simplify the first-order condition of equilibrium. Let

$$B^c(v) = v - N \int_{r(v)}^v F^{N-1}(x|v) dx.$$

Note that, by definition, we have

$$v - B^c(v) = N(v - B^t(v)). \tag{28}$$

When $N = 1, B^c(v) = B^t(v) = r(v)$.

The following gives the derivative of a regular buyer's payoff with respect to b.

Lemma 9 *We have the following derivative in active intervals for a regular buyer i with value v for $b > \rho$:*

$$\frac{\partial u(v, b, \sigma)}{\partial b}|_{v=\phi(b)} = H'(b) F^{N-1}(\phi(b)) B^t(\phi(b)) - \frac{\partial}{\partial b} x(b, \sigma_{-i}) + v H(b) \frac{\partial F^{N-1}(\phi(b))}{\partial b}|_{v=\phi(b)}. \tag{29}$$

Proof. For the case 1) payoff expressions of section 2.1, we can take the derivative of (8) with respect to b,and we have

$$\frac{\partial}{\partial b} \pi_{li}(v, b, \sigma)|_{v=\phi(b)} = -H'(b)(v - B^t(\phi(b))) F^{N-1}(\phi(b)). \tag{30}$$

Since $J(\phi(b),\phi(b)) = v$, $J(r(v,\phi(b)),\phi(b)) = v$, $r(\phi(b),\phi(b)) = \phi(b)$, from (7), with v_i now denoted by v. we have

$$\frac{\partial u_{wi}(v,b,\sigma)}{\partial b}|_{v=\phi(b)} = H'(b)\int_{r(v,\phi(b))}^{\phi(b)} (J(x,\phi(b)) - v)dF^{N-1}(x)|_{\phi(b)} = 0.$$

Hence taking the derivative with respect to b of

$$u(v,b,\sigma) = H(b)F^{N-1}(\phi(b))v - x(b,\sigma_{-i}) + u_{wi}(v,b,\sigma) + \pi_{li}(v,b,\sigma)$$

at $b = b(v)$, we get (29). For the case 2) with $\phi(b) < v$, the term π_{li0} is either 0 or given by (9). Taking the derivative with respect to b, we have

$$\frac{\partial}{\partial b}\pi_{li0} = \frac{\partial}{\partial b}\int_b^{b(v)} (v - \phi(y))H(y)dF^{N-1}(\phi(y))|_{v=\phi(b)} = 0.$$

From (12), we have $\frac{\partial}{\partial b}\pi_{lis}$ equal to (30). Thus the formula (29) holds in the this case as well. ■

For v in an active interval, let

$$\mathcal{L}(v) = \frac{B^c(v) - B_\theta(v)}{B^t(v) - B_\theta(v)}; \tag{31}$$

and for v in an inactive interval, let $\mathcal{L}(v) = 0$. Let

$$L(b) = \mathcal{L}(\eta_\theta(b)).$$

When $N = 1$, we have $B^c(v) = B^t(v)$, hence $\mathcal{L}(v) = 1$. In general $\mathcal{L}(v) \leq 1$, and may take negative values. When we want to express the dependence on θ, we use the notations $\mathcal{L}_\theta(v), L_\theta(v)$.

Lemma 10 *For $N > 1$, in an active interval, the function $\mathcal{L}_\theta(v)$ is a strictly increasing function of θ for any $v > 0$.*

Proof. By definition, we know $B^t(v) - B^c(v) > 0$ for $v > 0$. From Lemma **??**, we also know $B^t(v) - B_\theta(v)$ for $v > 0$. Note that

$$\mathcal{L}_\theta(v) = \frac{B^c(v) - B_\theta(v)}{B^t(v) - B_\theta(v)} < 1,$$

and $B_\theta(v)$ is decreasing in θ, hence

$$\frac{B^c(v) - B_\theta(v)}{B^t(v) - B_\theta(v)} = 1 - \frac{B^t(v) - B^c(v)}{B^t(v) - B_\theta(v)}$$

is increasing in θ. ■

Let

$$\tilde{H}(v) = H(B_\theta(v))$$

be the bid distribution of the speculator in variable v. For hybrid auctions, we further simplify the first-order condition of a regular buyer as follows.

Lemma 11 *For the $\theta-$hybrid auction, the first-order condition of a regular buyer in an active interval is*

$$\frac{H'(b)}{H(b)} = L_\theta(b)\frac{d}{db}\ln F(\eta_\theta(b)), \tag{32}$$

or

$$\frac{\tilde{H}'(v)}{\tilde{H}(v)} = \mathcal{L}_\theta(v)\frac{d}{dv}(\ln F(v)) \tag{33}$$

Proof. In active intervals, by Lemma 9, we rewrite the first-order condition by a change of variable to $v = \phi(b_i)$. We have

$$\frac{\tilde{H}'(v)}{\tilde{H}(v)} F^{N-1}(v) B^t(v) = \frac{\partial}{\partial v} x_i(B_\theta(v), \sigma_{-i}) - v\tilde{H}(v) \frac{d}{dv} F^{N-1}(v). \tag{34}$$

Rewrite the payment rule in the variable v, we have

$$x_i(B_\theta(v), \sigma_{-i}) = \theta B_\theta(v) F^{N-1}(v)\tilde{H}(v) + (1-\theta)\int_0^v B_\theta(x) d(F^{N-1}(x)\tilde{H}(x)),$$

hence we have

$$\frac{\partial}{\partial v} x_i(B_\theta(v), \sigma_{-i}) = \theta B'_\theta(v) F^{N-1}(v)\tilde{H}(v) + B_\theta(v)\tilde{H}(v)\frac{d}{dv}F^{N-1}(v) + B_\theta(v)F^{N-1}(v)\tilde{H}'(v). \tag{35}$$

From (34) and (35), we have

$$\begin{aligned}
\frac{\tilde{H}'(v)}{\tilde{H}(v)} F^{N-1}(v)\left(B^t(v) - B_\theta(v)\right) &= \theta B'_\theta(v) F^{N-1}(v) - (v - B_\theta(v))\frac{d}{dv}F^{N-1}(v) \\
&= F^{N-1}(v)\left(\theta B'_\theta(v) - (v - B_\theta(v))\frac{(N-1)f(v)}{F(v)}\right)
\end{aligned}$$

From Lemma 2, we have

$$\begin{aligned}
\frac{\tilde{H}'(v)}{\tilde{H}(v)}\left(B^t(v) - B_\theta(v)\right) &= \frac{Nf(v)}{F(v)}(B^t(v) - B_\theta(v)) - (v - B_\theta(v))\frac{(N-1)f(v)}{F(v)} \\
&= \frac{f(v)}{F(v)}\left(N(B^t(v) - B_\theta(v)) - (N-1)(v - B_\theta(v))\right) \\
&= \frac{f(v)}{F(v)}\left(v - B_\theta(v) - N(v - B^t(v))\right) \\
&= \frac{f(v)}{F(v)}\left(B^c(v) - B_\theta(v)\right),
\end{aligned}$$

thus we have (33). We obtain (32) by a simple change of variable. ■

The following summarizes the first-order conditions of a regular buyer in active and inactive intervals.

Lemma 12 *Let $H(.), b(.)$ be an equilibrium strategy profile, $\phi(.) = b^{-1}(.)$. In inactive intervals, we have*

$$\frac{\partial u_0(v,b,\phi)}{\partial b}(v,b,\phi)|_{v=\phi(b)} = 0, b > \rho. \tag{36}$$

In active intervals, we have $\phi(b) = \eta_\theta(b)$,

$$\frac{H'(b)}{H(b)} = L_\theta(b)\frac{d}{db}\ln F(\eta_\theta(b)). \tag{37}$$

The following is an immediate implication of Lemma 12.

Lemma 13 *In an active open interval I of the speculator, we must have $L_\theta(b) > 0$, a.e. or equivalently $B^c(v) > B_\theta(v)$ in the active interval $\eta_\theta(I)$ a.e.*

The following result states that the active and inactive intervals are uniquely determined, so that there is only one equilibrium which has the symmetry property.

Theorem 7 *(i) The collection of maximal active and inactive intervals are uniquely determined. (ii) There can only be one equilibrium which satisfies the symmetry property.*

Proof of Theorem 7:

Let $b(.)$ be the equilibrium bidding strategy of a regular buyer. When $N = 1$, the speculator is fully active, hence we must have $b(.) = B_\theta(.)$. We now assume $N > 1$. By Lemma 19, the active and inactive intervals are uniquely determined. It is sufficient to show that $b(.)$ is uniquely determined in all active or inactive intervals. In active intervals, we must have $b(.) = B_\theta(.)$, hence it is unique in active intervals. Let $[z_1, z_2]$ be any maximal inactive interval. We have $b(z_1) = B(z_1)$. let $C_{z_1}(.)$ be defined by the initial condition $C_{z_1}(z_1) = B_\theta(z_1)$, and we must have $b(.) = C_{z_1}(.)$ in $[z_1, z_2]$. Hence we have shown that $b(.)$ is uniquely determined in inactive intervals as well. Thus the bidding strategy of a regular buyer is uniquely determined. Now we want to show that $H(.)$ is also uniquely determined. Since the active intervals are uniquely determined, the maximum bid b_s^* of the speculator is uniquely determined. Hence we have the boundary condition $H(b_s^*) = 1$ for $H(.)$. The function $H(.)$ satisfies the differential equation stated in Lemma 12 in any active interval. The differential equation is the same for any equilibrium. The function $H(.)$ stays constant on inactive intervals. Hence $H(.)$ must be uniquely determined, as long as the active and inactive intervals are uniquely determined. The proof is complete.

6.2 Equilibrium Strategy

We are now ready to state the unique symmetric equilibrium of the model. The case $N = 1$ is a special interest.

Theorem 8 *Let $N = 1, \theta > 0$. If $\rho = 0$, the speculator is fully active in equilibrium. The equilibrium strategy $b(.)$ of the regular buyer is given by*

$$b(v) = B_\theta(v), \tag{38}$$

The speculator's equilibrium bid distribution is given by

$$\bar{H}(b) = F(\eta_\theta(y)), b \in [0, B_\theta(\beta)]. \tag{39}$$

If $\rho > 0$, we have (38) for $v \geq \eta_\theta(\rho)$, $b(v) = \rho$ for all $v \in [\rho, \eta_\theta(\rho))]$, and $\bar{H}(.)$ is given by (39) for $b \geq \rho$, $\bar{H}(0) = \bar{H}(\rho)$.

Proof. When $N = 1, L_\theta(v) = 1$. The first-order condition of a regular bidder, by Lemma 12, is

$$\frac{H^{'}(b)}{H(b)} = \frac{d}{db} \ln H(b) = \frac{d}{db} \ln F(\eta_\theta(b)), \tag{40}$$

with the boundary condition $H(\eta_\theta(\beta)) = 1$. Hence we get $\bar{H}(.)$ given in the lemma. ■

The following gives a simple characterization of the equilibrium bidding strategy profile in our model.

Theorem 9 *Let $N > 1$. In an active interval, the equilibrium strategy of the regular buyers in a $\theta-$hybrid auction, $\theta > 0$, is given by*

$$b(v) = B_\theta(v).$$

In a maximal inactive interval $[z, z']$, it is given by

$$b(v) = C_z(v), \tag{41}$$

where $C_z(.) = b^*_\rho(.)$ when $z = \rho$, and is the cost function with the initial condition $C_z(z) = B_\theta(z)$ when $z > \rho$. The speculator bid distribution is given by

$$\bar{H}(b) = \exp\left(-\int_b^{\bar{b}_s} L(y)d\left(\ln F(\eta_\theta(b))\right)\right), \tag{42}$$

in $[\rho, \bar{b}_s]$, where $\bar{b}_s$ is the maximum equilibrium bid of the speculator. Moreover the equilibrium has the following first-order stochastic dominance property

$$\bar{H}(b) > F(\eta_\theta(b)) \text{ or } F(\phi(b)) \text{ for all } b < \bar{b}_s. \tag{43}$$

Proof. The bidding strategy of a regular buyer in an active or inactive interval is clear and (42) is the only solution of the first-order condition in Lemma 12 with the boundary condition $H(\bar{b}_s) = 1$. For the regular buyers, the first-order condition of equilibrium is satisfied because of the way $\bar{H}(.)$ is defined. The supermodularity property insures that $b(.)$ is an optimal strategy for a regular buyer in active and inactive intervals. This proves that the given strategy profile is an equilibrium strategy. To show (43), consider a maximal active interval $[a, b] = [B(z), B(w)]$, we have

$$\ln \bar{H}(b) - \ln \bar{H}(a) = \int_a^b L(b)d\left(\ln F(\eta(b))\right) = \int_z^w \mathcal{L}(v)d\left(\ln F(v)\right)$$

$$< \int_z^w d\left(\ln F(v)\right) = \ln F(\eta(b)) - \ln F(\eta(a)).$$

Over a maximal inactive bid interval $[a', b']$, we also have

$$\ln \bar{H}(a') - \ln \bar{H}(b') < \ln F(\eta(a')) - \ln F(\eta(b')),$$

as the left-hand side is always 0. Hence for all $b < \bar{b}_s$, we must have

$$\ln \bar{H}(\bar{b}_s) - \ln \bar{H}(b) < \ln F(\eta(\bar{b}_s) - \ln F(\eta(b)) < -\ln F(\eta(b)),$$

and we have

$$\ln \bar{H}(b) > \ln F(\eta(b)),$$

or

$$\bar{H}(b) > F(\eta(b)).$$

Rewrite in the variable v,

$$\bar{H}(B_\theta(v)) > F(v) \text{ for all } v < \eta(\bar{b}_s).$$

Note that we have $B_\theta(v) = b(v)$ over active intervals, and $\bar{H}(.)$ is constant over inactive intervals. We have $\bar{H}(B_\theta(v)) = \bar{H}(b(v))$, and this translates into

$$\bar{H}(B_\theta(v)) > F(v) \text{ for all } v < \eta(\bar{b}_s),$$

which is equivalent to

$$\bar{H}(b) > F(\eta_\theta(b)) \text{ for } b < \bar{b}_s,$$

which is also valid for $b < \bar{b}$, as $\bar{H}(b) = 1$ when $b \geq \bar{b}_s$. The proof is complete. ■

The following gives the comparative static result of the equilibrium when θ varies.

Theorem 10 *Let $N > 1$. The equilibrium strategy $\bar{H}_\theta(v), b_\theta(v)$ is strictly decreasing in θ.*

Proof. In an active interval, we have shown that $B_\theta(.)$ is the bidding strategy of the regular buyers, and it is strictly decreasing in θ. In an inactive interval $[\rho, z]$, the equilibrium bidding strategy is given by $b^*_\rho(v)$ which is also strictly decreasing in θ. In a maximal inactive interval $[z, z'], z > \rho$, we have $C_z(z) = B_\theta(z)$. Since $B_\theta(z)$ is strictly decreasing in θ, $C_z(.)$ is also strictly decreasing in θ. Moreover we show that if $\theta' > \theta$, then an active interval of the $\theta-$equilibrium must be an active interval of a $\theta'-$equilibrium. This is because the endpoint is determined by the crossing of $B_\theta(.)$ by $B^c(.)$ from above. As θ increases, $B_\theta(.)$ is smaller, while $B^c(.)$ remains unchanged. This implies that the endpoint of an active interval moves higher with a higher θ. Moreover, the revenue function $B(v)$ does not change when θ varies, but the cost function $C(v)$ is strictly decreasing in θ. This implies that the beginning point of an inactive interval becomes lower when θ increases. Thus $b_\theta(.)$ must be strictly increasing in θ at all $v > \rho$. Moreover, we have

$$\bar{H}_\theta(v) = \exp\left(-\int_v^\beta \mathcal{L}_\theta(x) d\ln F(x)\right),$$

Lemma 10 says that $\mathcal{L}_\theta(x)$ is strictly increasing in θ. Hence $\bar{H}_\theta(v)$ is strictly decreasing in θ. ■

Example 4 *Let $N = 1$. There is one speculator, one regular buyer with the use value distribution $F(v) = v$. Let $\rho = 0$. The equilibrium bidding strategy of a regular buyer is given by*

$$b_\theta(v) = \frac{v}{2\theta + 2}.$$

The highest equilibrium bid of a regular buyer is $\frac{1}{2\theta+2}$. We have the first-order condition

$$\frac{H'_\theta(b)}{H_\theta(b)} = \frac{1}{b}, \tag{44}$$

with the boundary condition $H(\frac{1}{2\theta+2}) = 1$. Hence we have

$$H_\theta(b) = (2\theta + 2)b,$$

and

$$\tilde{H}_\theta(v) = H_\theta(b_\theta(v)) = v.$$

Example 5 *Let $N = 2$. There are two regular buyers, and one speculator. Let the regular buyer's use value distribution be*

$$F(v) = 0.5(1 + v^2) \text{ over } [0, 1].$$

Let $\rho = 0$. First we compute the equilibrium without the speculator. We have

$$b^*_\rho(v) = v - \frac{2^{1/\theta}}{(1 + v^2)^{1/\theta}} \int_0^v (\frac{1 + x^2}{2})^{1/\theta} dx = \int_0^v x dF^{1/\theta}(x|v).$$

When $\theta = 1$, the strategy is

$$b^*_\rho(v) = \frac{2}{3} \frac{v^3}{v^2 + 1},$$

and this is the equilibrium bidding strategy of the regular buyers in the first-price auction without the speculator.

$$B(v) = \frac{4v^3}{(1 + v^2)^2} \left[\frac{1}{3\sqrt{3}} + \frac{1}{15}(2 + \frac{1}{3\sqrt{3}})v^2\right],$$

$$B^t(v) = \frac{v}{1+v^2}\left[\frac{1}{\sqrt{3}} + \frac{1}{3}\left(2 + \frac{1}{3\sqrt{3}}\right)v^2\right],$$

$$B^c(v) = \frac{v}{1+v^2}\left[\frac{2}{\sqrt{3}} - 1 + \frac{1}{3}(1 + \frac{2}{3\sqrt{3}})v^2\right].$$

From Theorem , we have

$$B_\theta(v) = \left(\frac{2}{1+v^2}\right)^{\frac{2}{\theta}} \int_0^v B^t(x) d(\frac{1+x^2}{2})^{\frac{2}{\theta}},$$

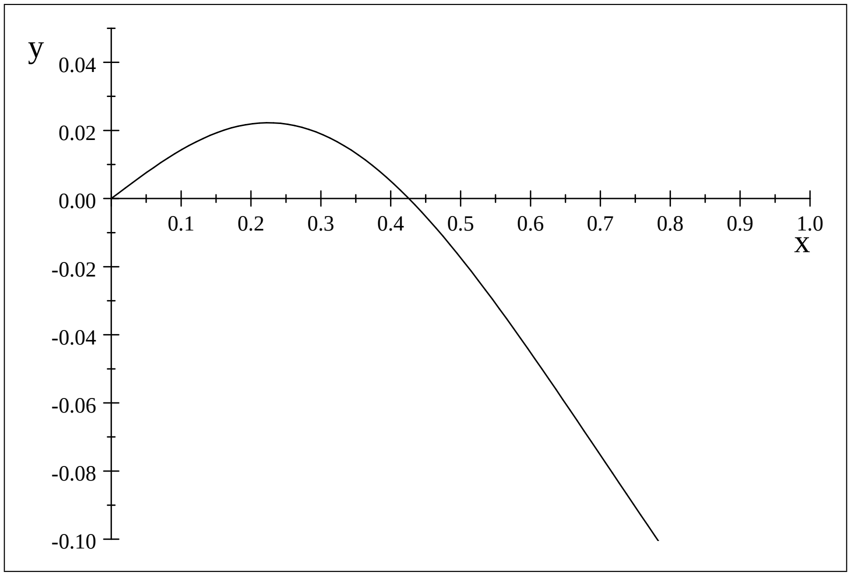
for $\theta \in [0, \bar{b}_s]$*, with* $\bar{b}_s$ *determined by the positive intersection of* $B^c(v) = B_\theta(v)$*. When* $\theta = 1$*, we have* $\bar{b}_s = 0.812\,063\,1$.

When θ is smaller, the active interval shrinks. To determine the active interval for $\theta = 0.5$, we have

$$B_{0.5}(v) = \left(\frac{2}{1+v^2}\right)^4 \int_0^v B^t(x) d(\frac{1+x^2}{2})^4 = \left(\frac{2}{1+v^2}\right)^4 \int_0^v 4(\frac{1+x^2}{2})^3 x B^t(x) dx,$$

Solving $B^c(v) - B_\theta(v) = 0$, we get $v = 0.425\,904$, hence the active interval is $[0, 0.425904]$.

The graph of $B^c(v) - B_\theta(v)$ is given by



7 Appendix

Proofs of Section 2

Proof of Lemma 4:
We have

$$\begin{aligned} Q_{i1}(v_i, b, \sigma) &= H(b)F^{N-1}(r(v_i, \phi(b))) \text{ when } v_i \leq \phi(b), \\ &= H(b)F^{N-1}(\phi(b)) \text{ when } v_i > \phi(b). \end{aligned} \tag{45}$$

When $v_i \leq \phi(b)$,and $b \leq \tilde{y}(v_i)$, $Q_{i2}(v_i, b, \sigma)$ is the same as the probability of buying from the speculator, and we have

$$Q_{i2}(v_i, b, \sigma) = F^{N-1}(v_i)(H(\tilde{y}(v_i) - H(b)). \tag{46}$$

When $v_i \leq \phi(b)$,and $b > \tilde{y}(v_i)$, we have

$$Q_{i2}(v_i, b, \sigma) = 0.$$

When $v_i > \phi(b)$, the regular buyer wins during resale from the speculator when (i) all rival regular buyers have use value below $\phi(y)$ and the speculator wins with a bid $y \in (b, b(v_i))$, or (ii) all rival regular buyers have values below v_i and the speculator wins with a bid $y \in [b(v_i), \tilde{y}(v_i))$. Hence the probability of winning from a speculator is

$$\int_b^{b(v_i)} F^{N-1}(\phi(y))dH(y) + F^{N-1}(v_i)(H(\tilde{y}(v_i)) - H(b(v_i))). \tag{47}$$

The probability of buying from other regular buyers is either 0 or given by

$$\int_b^{b(v_i)} H(y)dF^{N-1}(\phi(y)). \tag{48}$$

When the probability of buying from the regular buyers is zero, $Q_{i2}(v_i, b, \sigma)$ is given by (47), otherwise $Q_{i2}(v_i, b, \sigma)$ is given by the sum of (47) and (48). From case 1) of the payoff expressions (7),(3) we have

$$\frac{\partial}{\partial v_i}(\pi_{i1}(v_i, b, \sigma) + \pi_{wi}(v_i, b, \sigma)) = H(b)F^{N-1}(r(\phi(b)) = Q_{i1}(v_i, b, \sigma).$$

From (8), we have

$$\frac{\partial}{\partial v_i}\pi_{li}(v_i, b, \sigma) = F^{N-1}(v_i)(H(\tilde{y}(v_i) - H(b)) = Q_{i2}(v_i, b, \sigma).$$

Hence the lemma is proved. For case 2) with no resale from the winning regular buyers, from (12), we have

$$\frac{\partial}{\partial v_i}\pi_{li}(v_i, b, \sigma) = \int_b^{b(v_i)} F^{N-1}(\phi(y))dH(y) + F^{N-1}(v_i)(H(\tilde{y}(v_i)) - H(b(v_i))) = Q_{i2}(v_i, b, \sigma).$$

By (47), in this case, we have

$$\frac{\partial}{\partial v_i}\pi_{li}(v_i, b, \sigma) = Q_{i2}(v_i, b, \sigma). \tag{49}$$

When there is resale from the winning regular buyers, from (9), we have

$$\frac{\partial}{\partial v_i}\pi_{li0}(v_i, b, \sigma) = \int_b^{b(v_i)} H(y)dF^{N-1}(\phi(y)).$$

By (47), and (48), we also have (49). The lemma is proved.

Proof of Theorem 1:

We use the simpler derivative notations instead of the one-sided derivatives. Consider the case 1), $\phi(b) \geq v_i$. From (46),(45), we have

$$\frac{\partial}{\partial b}Q_{i1}(v_i, b, \sigma) = H(b)\frac{\partial}{\partial b}F^{N-1}(r(v_i, \phi(b))) + H'(b)F^{N-1}(r(v_i, \phi(b))),$$

and

$$\frac{\partial}{\partial b}Q_{i2}(v_i, b, \sigma) = -H'(b)F^{N-1}(v_i).$$

Hence Lemma 4 implies that

$$\frac{\partial}{\partial b}\frac{\partial}{\partial v_i}u(v_i, b, \sigma) \geq H(b)\frac{\partial}{\partial b}F^{N-1}(r(v_i, \phi(b))) > 0.$$

For case 2), when there is positive payoffs buying from the speculator and the regular buyers, we have

$$\frac{\partial}{\partial b}Q_{i2}(v_i, b, \sigma) = -H(b)\frac{\partial}{\partial b}F^{N-1}(\phi(b)) - H'(b)F^{N-1}(\phi(b)).$$

From the second line in (45), we have

$$\frac{\partial}{\partial b}Q_{i1}(v_i, b, \sigma) = H(b)\frac{\partial}{\partial b}F^{N-1}(\phi(b)) + H'(b)F^{N-1}(\phi(b)),$$

hence Lemma 4 implies that

$$\frac{\partial}{\partial b}\frac{\partial}{\partial v_i}u(v_i, b, \sigma) = 0.$$

In all other cases, we have

$$\frac{\partial}{\partial b}\frac{\partial}{\partial v_i}u(v_i, b, \sigma) > 0.$$

Proofs of Section 6:

Proof of Lemma 7:

Let $b(v)$ be the equilibrium bidding function of a regular buyer, and $H_r(.)$ be the bid distribution of $b(.)$. If $a(\rho) = \rho$, by Lemma 1, there exists $\bar{v} > \rho$ such that $b(\bar{v}) = B_\theta(\bar{v}) = \rho$. If $H_r(.)$ has an atom at ρ, then a regular buyer with value $\bar{v}$ can bid slightly higher than ρ, and the existence of the atom implies that this is a profitable deviation. Similarly, if $H(.)$ has an atom at ρ, a regular buyer bidding ρ can increase the payoff deviating slightly above ρ. Assume that neither of $H_r(.)$ or $H(.)$ has an atom at ρ. We will get another contradiction. The existence of no atom means that a regular buyer with value v below $\bar{v}$ bids below ρ. Hence he loses the first-stage auction, and has zero payoff in the first stage. He has a payoff during resale. If he bids ρ instead, he has the first-stage payoff $H(\rho)F^{N-1}(\rho)(v - \rho) > 0$ while maintaining the same payoff during resale. Hence the deviation is profitable. This is another contradiction. This proves that $a(\rho) = \rho$ cannot be true, and we must have $a(\rho) > \rho$.

Proof of Lemma 8: (i) If the speculator is not active in equilibrium, the equilibrium is the same the model without the speculator. The lemma is obvious. We can now assume that the speculator is active in equilibrium. It is clear that a regular buyer with use value $v < \rho$ has zero winning probability in the first stage, as he would not bid above ρ. Any winning in the first-stage leads to negative profit. We first show that his winning probability is 0 also during resale, if the winner of the first-stage auction is a speculator. If $N = 1$, when the speculator wins the first-stage auction with a bid $b \geq \rho$, she sets a reserve price $r(\phi(b)) > B(\phi(b))$. Since the speculator must have non-negative profit, we also have $B(\phi(b)) \geq \rho$, hence $r(\phi(b)) > \rho$. Thus a regular buyer with value $v < \rho$ cannot win the object in the second-stage. This gives the proof for the case $N = 1$.

Now consider the case $N > 1$. Let $b_0 = a(\rho), v = v(\rho)$ in Lemma 7. By Lemma 7, we have $b_0 = b^*_\rho(v)$. Let $r(\phi(b))$ be the reserve price set by the speculator after winning the first-stage auction with the active bid b. If $r(\phi(b)) < \rho$, we have

$$b^*_\rho(v) = v - \int_\rho^v F^{N-1}(x|v)dx > v - \int_{r(\phi(b))}^v F^{N-1}(x|v)dx = B^t(\phi(b)) > B(\phi(b)).$$

The speculator profit must be non-negative, hence we have

$$B(\phi(b)) \geq b \geq b_0,$$

hence we have $b^*_\rho(v) > b_0$ contradicting the equality $b_0 = b^*_\rho(v)$ earlier. The contradiction means that the reserve price $r(\phi(b))$ set by the speculator must be at least ρ. This proves that the regular buyer with value $v < \rho$ cannot win in the second-stage auction, and the proof is complete. (ii) If $\rho \geq 0$, it is clear that a regular buyer with value $v = 0$ has zero payoff in equilibrium. Hence we can just consider the case $\rho > 0, v > 0$. We know that the holding probability of a regular buyer with value $v < \rho$ is zero, hence the payoff must be zero. If $v = \rho$, he has positive winning probability, but whenever he wins in the first stage or the resale stage, the cost is v, hence the payoff is also zero.

Lemma 14 *When $N > 1$, the function $C_z(.)$ has the following properties: (i) $C_z(z) = B_\theta(z)$, (ii) $C_z(.)$ is strictly increasing in $[z, \beta]$, and (iii) the derivative is given by*

$$C'_z(v) = \frac{(N-1)f(v)}{\theta F(v)}(v - C_z(v)).$$

for $v \geq z$.

The following is a useful lemma on the derivative of the function $B_\theta(.) - C_z(v)$.

Lemma 15 *We have*

$$B'_\theta(v) - C'_z(v) = \frac{Nf(v)}{\theta F(v)}\left[\frac{1}{N}B^c(v) + \frac{N-1}{N}C_z(v) - B_\theta(v)\right]. \tag{50}$$

Proof. From Lemma 14 and (20), we have

$$\theta\left(B'_\theta(v) - C'_z(v)\right) = \frac{Nf(v)}{F(v)}[B^t(v) - B_\theta(v)] - \frac{(N-1)f(v)}{F(v)}[v - C_z(v)]$$

$$= \frac{f(v)}{F(v)}[Nv - \frac{N}{F^{N-1}(v)}\int_{r(v)}^{v} F^{N-1}(x)dx - NB_\theta(v) - (N-1)(v - C_z(v))]$$

$$= \frac{f(v)}{F(v)}\left[B^c(v) - NB_\theta(v) + (N-1)C_z(v)\right] = \frac{Nf(v)}{F(v)}\left[\frac{1}{N}B^c(v) + \frac{N-1}{N}C_z(v) - B_\theta(v)\right].$$

■

The following says that at the point where $B^c(.)$ crosses $B_\theta(.)$, we also have the crossing of $C_z(.)$ and $B_\theta(.)$ in the other direction.

Lemma 16 *Let $N > 1$, and $C_z(.)$ be the endogenous cost function defined in (z, v) with the boundary condition $C_z(z) = B_\theta(z)$. Then the following three statements hold (with the understanding that the inequalities hold in (z, v) a.e.):*

$$\begin{aligned}
&\text{If } B^c(.) \; > \; B_\theta(.)(.) \text{ in } (z,v), \text{ then we have } B_\theta(.)(.) > C_z(.) \text{ in } (z,v),\\
&\text{If } B^c(.) \; < \; B_\theta(.)(.) \text{ in } (z,v), \text{ then we have } B_\theta(.)(.) < C_z(.) \text{ in } (z,v),\\
&\text{If } B^c(.) \; = \; B_\theta(.)(.) \text{ in } (z,v), \text{ then we have } B_\theta(.)(.) = C_z(.) \text{ in } (z,v).
\end{aligned}$$

Proof of Lemma 16: Assume that $B^c(.) > B_\theta(.)$ in (z, z'). We want to show that $B_\theta(.) > C_z(.)$ in (z, v). If it is not true, then we can find an interval $(x, x+\varepsilon) \subset [z, v]$ such that $B_\theta(x) = C_z(x)$ and $B_\theta(.) \leq C_z(.)$ in $(x, x+\varepsilon)$. By Lemma 15, we must have $B'_\theta(x) > C'_z(x)$ in $(x, x+\varepsilon)$, which implies $B_\theta(.) > C_z(.)$ in $(x, x+\varepsilon)$, a contradiction. The contradiction finishes the proof. Similarly, if $B^c(.) < B_\theta(.)$ in (z, v), the same arguments show that we must have $B_\theta(.) < C_z(.)$ in (z, v). If $B^c(.) = B_\theta(.)$ in (z, v), then for the same reason, $B_\theta(.)$ cannot be above or below $C_z(.)$, and we must have $B_\theta(.) = C_z(.)$ in (z, v).

Since the equilibrium strategy of the regular buyer and the speculator is uniquely determined in active as well as inactive intervals, and the speculator bid distribution is uniquely determined by the (33), we know that there is a unique symmetric equilibrium. It can be characterized as in the case of first-price auctions once we know that active and inactive intervals.

From the zero profit condition, we know that if b is in the support of the $H(.)$, $\eta_\theta(.)$ is the inverse equilibrium bidding strategy of the regular buyers.

Lemma 17 *Let $N > 1$. The speculator is inactive in equilibrium if and only if $B_\theta(v) \leq b^*_\rho(v)$ for all $v \in [\rho, \beta]$. In an equilibrium in which the speculator is inactive, the equilibrium is the same as if there is no speculator and is given by (**??**).*

Proof. Assume that $B_\theta(v) - b^*_\rho(v) \leq 0$ for all $v \in [\rho, \beta]$. Let $\varphi(.)$ be the inverse equilibrium bidding strategy of the regular bidders, and $\rho > 0$. If the speculator is active in equilibrium, by Lemma 7, there exists an active interval $(a(\rho), b')$ such that the speculator is inactive in $(\rho, a(\rho))$, and $\varphi(b) = \eta_\theta(b)$ on the interval $(a(\rho), b')$. Since $B_\theta(.) \leq b^*_\rho(.)$ in the interval $(v(\rho), \eta(b'))$, Lemma 16 implies that $B^c(.) \leq B_\theta(.)$ in the interval $(v(\rho), \eta(b'))$. This contradicts Lemma 13. If $\rho = 0$, and the active interval starts at 0, we can apply Lemma 16 to this active interval, and get the same contradiction $B^c(.) \leq B_\theta(.)$. The contradiction implies that the speculator cannot be active in equilibrium. We obtain the same equilibrium $b^*_\rho(.)$ in the model without resale as shown in Theorem 3. If $\pi_\rho(b) > 0$ for some $b \in [\rho, b^*_\rho(\beta)]$, and the speculator is inactive in an equilibrium, then regular bidders will bid $b^*_\rho(.)$ in equilibrium, but in this case the speculator can make a strictly positive profit by bidding b, contradicting the zero profit condition. Hence the theorem is proved. ■

Given the equilibrium strategy $b(.)$, and any value interval $[z, z']$, we can define the endogenous cost function $C_z(.)$ with the initial condition $C_z(z) = b(z)$. If $[z, z']$ is an inactive interval, we know that the first-order condition (18) must be satisfied, and both $C_z(.), b(.)$ have the same initial condition, therefore we have $C_z(.) = b(.)$ in $[z, z']$. The speculator makes maximum profit 0 in equilibrium, and we must have $B_\theta(.) \leq b(.) = C_z(.)$. The condition $B_\theta(.) \leq C_z(.)$ is also sufficient for being an inactive interval as stated in the following. The proof is the same as that of Theorem 17.

Lemma 18 *The value interval $[z, z']$ is an inactive interval if and only if $B_\theta(.) \leq C_z(.)$ in $[z, z']$.*

The difference between Lemma 17 and Lemma 18 is that the condition in Lemma 17 is based on the primitives of the model, while the latter uses the value $b(z)$ which needs to be determined in equilibrium. Note that a corollary of the characterization of inactive equilibrium is that when the speculator is inactive in one equilibrium, she must be inactive in another equilibrium, as the condition $B_\theta(v) - b^*_\rho(v)$ is independent of the equilibrium profile. Hence the equilibrium is unique when $H(0) = 1$ in an equilibrium. We define inactive intervals through $H(.)$. Now we want to show that if an interval is inactive in one equilibrium, it must be inactive in another equilibrium, if there is another one.

Lemma 19 *Let $N > 1$. If an interval is inactive in one equilibrium, it must be inactive in another equilibrium. Similarly, if an interval is active in one equilibrium, it must be active in another equilibrium.*

Proof. It is sufficient to prove this for a maximal inactive interval. Let $[z, z']$ be a maximal inactive

interval in the equilibrium $H(.), b(.)$. Lemma 18 implies $B_\theta(.) \leq C_z(.)$ in $[z, v]$, and Lemma 16 implies that $B^c(.) \leq B_\theta(.)$ in $[z, v]$. Assume that there is another equilibrium $\bar{H}(.), \bar{b}(.)$ such that $[z, z']$ is an active interval. There is a maximal active interval (z_0, z'') of $\bar{H}(.)$ that overlaps with $[z, z']$ in some open interval (x, x'). By continuity, the speculator profit must be 0 at z_0, hence $B_\theta(z_0) = \bar{b}(z_0)$. Define $\bar{C}_{z_0}(.)$ with the initial condition $\bar{C}_{z_0}(z_0) = B_\theta(z_0)$. By Lemma 13, we have $B^c(.) > B_\theta(.)$. This is a contradiction. The contradiction proves that $[z, v]$ must be inactive in another equilibrium. Since the active intervals are complementary to inactive intervals, we also conclude that the active intervals are uniquely determined regardless of the $H(.)$ used to define it. ■

The following says that the end of an active interval occurs at the place where $B^c(.)$ crosses $B_\theta(.)$ from above.

Lemma 20 *If $[z_1, z_2]$ is a maximal active interval, then $B^c(z_2) = B_\theta(z_2)$ if $z_2 < \beta$.*

Proof of Lemma 20: Since z_2 is the endpoint of an active interval, either there exists an inactive interval $(z_2, z_2 + \varepsilon)$ or there is a sequence of maximal inactive intervals $[x_n, x_n']$ with $x_n' \to z_2, x_n \geq z_2$. If $B^c(z_2) > B_\theta(z_2)$, define the cost function $C_{z_2}(.)$ with the initial condition $C_{z_2}(z_2) = B_\theta(z_2)$, then we have $B_\theta'(z_2) - C_{z_2}^{'}(z_2) > 0$ by Lemma 15. Hence $B_\theta(.) > C_{z_2}(.) > 0$ in a neighborhood $(z_2, z_2 + \varepsilon)$. This is a contradiction.

When we combine Lemma 20 with Lemma 15, we know that $B_\theta(.)$ cannot cross $C_z(.)$ unless $B^c(.)$ is above $B_\theta(.)$. Once $B_\theta(.)$ crosses $C_z(.)$, Lemma 16 then tells us $B_\theta(.)$ will stay above $C_z(.)$ leading to an active interval, until $B^c(.)$ ends the active interval at the place it crosses $B_\theta(.)$ from above. This is essentially the way a maximal active interval is determined.

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