Implementation of Efficient Investments in Mechanism Design

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Abstract

A general mechanism design model with transferable utility and unobservable investments is considered. First, when agents make investments only before participating in the mechanism, it is shown that efficient investments are not implementable for any allocatively efficient mechanisms. We then allow agents to make investments before and after the mechanism. In this environment, for constrained-efficient mechanisms, efficient investments are subgame perfect implementable if and only if the mechanism is commitment-proof, which is a weaker requirement than strategy-proofness. In the provision of public goods, our result implies that budget balancing does not preclude the subgame perfect implementation of efficient investments under efficient mechanisms.

1 Introduction

The theory of mechanism design has identified what social choice rules are implementable when agents play certain equilibria under mechanisms. The standard assumption in the literature is that the participants' types (or the distribution of types) are exogenously given. In many applications, however, there are opportunities to change their own types outside the market. For example, in a procurement auction, firms make efforts to reduce the cost of production in preparation for bidding, and will make further efforts if they win the right to

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produce the project (Tan, 1992; Piccione and Tan, 1996; Arozamena and Cantillon, 2004). When the license to use scarce resources such as spectrums is auctioned off, firms make investments in technologies to make use of the resource before or after participating the auction. These investments increase the value of winning the auction at the market clearing stage. Moreover, the incentives for investments can also be affected by the structure of the mechanism, which implies that investment decisions are endogenous.

The difficulty of the problem stems from the fact that investment decisions are not observable to the mechanism designer in many cases. If they were observable to him, investments can be specified as a part of the outcome of social choice rules and the standard implementation theory applies. However, investment behaviors are usually difficult to observe; they are multi dimensional and they involve the expenditure of time and effort as well as the expenditure of money. In the example of a procurement auction, in addition to the monetary investments for physical capital, other forms of investments such as efforts to search for new technologies or reorganization of the firm's system can increase the value of the project while they are difficult to describe. This unobservability of investments has also been argued in the literature of hold-up problems (Klein, Crawford, and Alchian, 1978; Williamson, 1979, 1983).

In the literature, there are several notable papers which analyzed the incentives for pre-mechanism actions and examined the existence of ex ante efficient equilibria. Hatfield, Kojima and Kominers (2015) provided a characterization result in a general transferable utility environment with an ex ante investment stage; for allocatively efficient mechanisms, strategy-proofness is equivalent to inducement of efficient investments (which implies the existence of investment efficient equilibria). This result completes the findings of Rogerson (1992) who offered specific mechanisms to induce efficient investments. In the context of information acquisition (Milgrom, 1981; Obara, 2008), Bergemann and Välimäki (2002) also indicate the link between ex ante efficiency and strategy-proofness; the VCG mechanism ensures ex ante efficiency under private values.

Although these works ensure the existence of ex ante efficient equilibria under certain mechanisms, it has rarely been examined under what mechanisms *every* equilibrium is ex ante efficient. The possibility of inefficient equilibria is not a trivial problem even under strategy-proof mechanisms.[1] Consider an example where firms are competing for a government project. The ex ante investment stage allows a firm with high unit cost of investment to commit to production. If the firm is the only one which makes an ex ante investment,

[1] Hatfield, Kojima and Kominers (2015) note this problem in their Example 4.

at the market clearing stage, the value of the project for the firm can be higher than others because the cost of pre-mechanism investment is already sunk. Therefore, there is an equilibrium in which a firm with costly investments commits to production ex ante and deters other firms from making investments. As a result, an auction rule induces socially inefficient investments by the firms even though the auction itself is allocatively efficient.

The objective of this paper is to show when we can implement efficient investments. We consider a general public choice setting with transferable utility. The valuation functions of agents at the market clearing stage are endogenously determined; agents explicitly choose their own valuations with the cost of investments. We will examine the possibility of the implementation of full efficiency, which requires that given an allocatively efficient mechanism, any equilibrium of the investment game maximize the total utility of agents net of the cost of investments. First, we will analyze the environments where agents make investments only before participating in the mechanism. However, the result is not permissible; under such environments, any allocatively efficient mechanism fails to implement efficient investments in subgame perfect equilibrium (Theorem 1). This theorem implies that an inefficient equilibrium is inevitable because the ex ante investment stage can create a commitment power to a certain agent.

To alleviate this problem, we analyze environments where there is an additional investment stage after the mechanism is run. Some of the readers might think that agents only make investments in preparation for the mechanism, but investments after the market clearing stage are often possible in several applications. In the context of bidding for government contracts, firms invest in cost reduction once they are selected by the government to perform the task (McAfee and McMillan, 1986; Laffont and Tirole, 1987). The ex post investment stage allows agents to substitute investments between the two stages if the technology of investments does not dramatically change over time. Our main theorem characterizes the mechanisms which implement efficient investments under this environment; for constrained-efficient mechanisms, commitment-proofness is sufficient and necessary for SPE implementing efficient investments (Theorem 2). Commitment-proofness requires no agent be able to benefit from becoming another type by committing to throw away some of her wealth contingent on the outcome of the mechanism. I argue that this theorem is permissible because commitment-proofness is weaker than well-known strategy-proofness. Moreover, since this is a characterization result, we can also detect a mechanism which has an inefficient investment equilibrium by examining whether it implements a commitment-proof social choice function.

To be precise, our theorem characterizes the set of social choice functions and does not explicitly argue mechanisms. The reason why we do this is that any social choice function is SPE implementable using an extensive form mechanism under transferable utility environments (Moore and Repullo, 1988). Therefore, allowing for extensive form mechanisms, we can simply analyze the properties of social choice functions rather than the details of mechanisms. If we slightly weaken the concept of implementation to virtual implementation, any social choice function is virtually implementable by a static mechanism (Abreu and Matsushima, 1992; Maskin, 1999). Thus, our result also implies that a static mechanism can virtually implement efficient investments if it virtually implements a commitment-proof social choice function.

We will demonstrate how we obtain positive results in our environments using an example of a procurement auction (Example 3). Especially, we compare the equilibria of investments when there is an additional investment stage and when there is not for the second-price and first-price auctions. Finally, we consider the provision of public goods where budget balancing is required. In this environment, it is shown that there exists a commitment-proof, efficient and budget balancing social choice function (Proposition 1), which implies that the SPE implementation of full efficiency is still possible.

Regarding the two distinct features in our paper that (i) post-mechanism investments are allowed and (ii) unique implementation of investment efficiency is shown, Piccione and Tan (1996) share closest interest with ours in the literature. They analyze a procurement auction where firms make R&D investments prior to the auction and the firm which wins the procurement contract exerts an additional effort to reduce costs. One of the main results of their paper is that the full-information solution (in which investments and alternative are efficient) can be uniquely implemented by first-price and second-price auctions when the R&D technology exhibits decreasing returns to scale. This elegant theorem resembles ours, but neither their result nor our theorem is a special case of the other because of several different modeling choices. First, their possibility result is only for specific mechanisms while we provide a characterization result. Moreover, they assume a common R&D technology and do not allow for ex ante heterogeneity of cost functions as we do in our model. However, the relationship between ex ante R&D investment technology and ex post cost reduction effort is more general in their model than ours. Thus, our result can be seen as complementary to Piccione and Tan (1996)'s earlier work.

The rest of the paper is organized as follows. Section 2 introduces our model and defines relevant concepts. The impossibility and possibility results of unique implementation are presented in Section 3. Several applications are introduced in Section 4. Section 5 concludes.

2 Model

There is a finite set of agents I and a finite set of alternatives Ω. A valuation function of agent $i \in I$ is $v^i : \Omega \to \mathbb{R}$. The valuation function is endogenously determined by each agent's investment decision. The set of possible valuation functions is $V^i \subseteq \mathbb{R}^\Omega$. Assume that V^i is a compact set. Denote the profile of the sets of valuations by $V \equiv \times_{i\in I} V^i$. We assume complete information among agents, but investments and the costs of investments are not observable to the mechanism designer. Therefore, a mechanism chooses an alternative and transfers, but not agents' investment behaviors. We will argue the relationship between social choice rules and mechanisms later in this section.

Each agent makes an investment decision to determine her valuation over alternatives. The investment is modeled as an explicit choice of a valuation function with the flow cost of investment determined by a cost function $c^i : V^i \to \mathbb{R}_+$. The cost of investment is assumed to be non-negative, but this is just a normalization. Without loss of generality, we can also assume that there is $v^i \in V^i$ such that $c^i(v^i) = 0$. We assume that there are two investment stages; before and after participating in the mechanism. We regard each of the investment stages as a simultaneous move game by all agents. Assume that the investment is irreversible; if agent i chooses $v^i \in V^i$ before the mechanism, she can only choose a valuation function from the set $\{\bar{v}^i \in V^i | c^i(\bar{v}^i) \geq c^i(v^i)\}$ in the second investment stage. To clarify, the timeline of the investment game induced by a mechanism is:

1. Each agent makes a prior investment by choosing a valuation function $v^i \in V^i$.

2. Agents participate in a mechanism. After the mechanism is run, each agent can make an additional investment, i.e., she chooses a valuation function from $\{\bar{v}^i \in V^i | c^i(\bar{v}^i) \geq c^i(v^i)\}$.

The ex ante utility function of an agent has the following three components: the valuations she chooses in the first and the second investment stages, the cost of investment and the discount factor. Let $\beta \in (0, 1]$ be a discount factor which discounts the utility realized in the second stage. For an alternative $\omega \in \Omega$, a transfer vector $t \equiv (t^i)_{i\in I} \in \mathbb{R}^I$ and an investment schedule $(v^i, \bar{v}^i) \in (V^i)^2$ where v^i is the valuation function chosen before the mechanism and $\bar{v}^i$ is the final valuation function, the ex ante utility of agent i is

$$-c^i(v^i) + \beta\Big[\bar{v}^i(\omega) - t^i - \big(c^i(\bar{v}^i) - c^i(v^i)\big)\Big].$$

When agents face the mechanism in the second stage, the cost of investment made in the first stage is already sunk. Moreover, the valuation of an alternative $\omega \in \Omega$ should be evaluated by the maximum value over possible valuation functions which can be taken in the second stage net of the cost of additional investments. Therefore, we can define the valuations of agents at the time of the mechanism as follows using the notation b^{c^i,v^i} for any $c^i : V^i \to \mathbb{R}_+$ and the prior investment $v^i \in V^i$.

Definition 1. *The valuation function $b^{c^i,v^i} : \Omega \to \mathbb{R}$ at the time of the mechanism given $c^i : V^i \to \mathbb{R}_+$ and $v^i \in V^i$ is defined by*

$$b^{c^i,v^i}(\omega) = \max_{\bar{v}^i \in \{\tilde{v}^i \in V^i | c^i(\tilde{v}^i) \geq c^i(v^i)\}} \left\{ \bar{v}^i(\omega) - c^i(\bar{v}^i) \right\} + c^i(v^i)$$

for each $\omega \in \Omega$. Let $b^{c,v} \equiv (b^{c^i,v^i})_{i \in I}$.

A social choice function is defined in the standard way. A social choice function is $h \equiv (h_\omega, h_t)$ where $h_\omega : \mathbb{R}^{\Omega \times I} \to \Omega$ is an allocation rule and $h_t : \mathbb{R}^{\Omega \times I} \to \mathbb{R}^I$ is a transfer rule. The transfer rule for each agent is denoted by $h_t^i : \mathbb{R}^{\Omega \times I} \to \mathbb{R}$. Note that a social choice function is defined only for a tuple (I, Ω) because the domain $\mathbb{R}^{\Omega \times I}$ is not restricted by the choice of V.

We are interested in whether efficient investments and outcomes can be implemented in subgame perfect equilibria (SPE) by an investment game induced by a mechanism. Here we allow for extensive form mechanisms. Since it is known that any social choice function is SPE implementable using the extensive form mechanism of Moore and Repullo (1988) under transferable utility environments, we can define an investment game solely by the social choice function to be implemented, rather than the details of the mechanism. Therefore, the formal definition of SPE implementability of efficient investments involves a social choice function instead of a mechanism itself. For this reason, we do not explicitly define mechanisms and rather use "social choice functions" and "mechanisms" interchangeably in this paper, although these two concepts are usually distinguished. Moreover, when we say that efficient investments are SPE implementable by a social choice function, it implicitly assumes that we make use of the Moore-Repullo extensive form mechanism which requires a large penalty to SPE implement the social choice function itself.[2] As argued in the introduction, however, if we only require virtual implementation (Abreu and Matsushima, 1992), we do not even need an extensive form mechanism and a large penalty in the mechanism.

[2] A large penalty will not be paid by any agent on the equilibrium path.

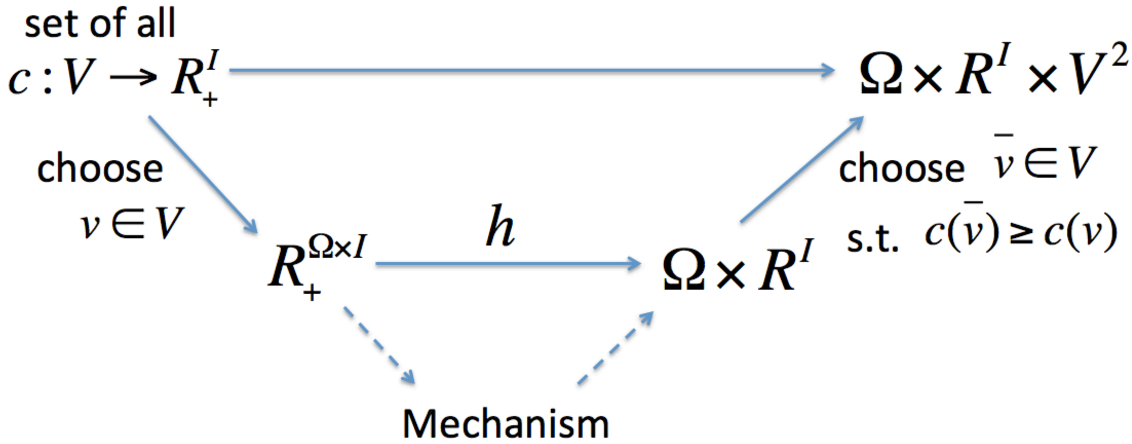

Figure 1. The structure of a social choice function and the investment game.

To introduce the SPE implementability of efficient investments, first define the set of subgame perfect equilibria of the investment game given a social choice function.

Definition 2. *For any $V \subseteq \mathbb{R}^{\Omega\times I}$ and a profile of cost functions $c : V \to \mathbb{R}^I_+$, an investment profile $(v^*, \bar{v}^*) \in V^2$ is a subgame perfect equilibrium of the investment game given a social choice function $h : \mathbb{R}^{\Omega\times I} \to \Omega \times \mathbb{R}^I$ if for each $i \in I$, there exists $\mu^i : V^i \to V^i$ such that*

1. $\bar{v}^{i*} = \mu^i(v^{i*}) \in \arg\max_{\bar{v}^i \in \{\tilde{v}^i \in V^i | c^i(\tilde{v}^i) \geq c^i(v^i)\}} \Big\{ \bar{v}^i(h_\omega(b^{c,v^*})) - c^i(\bar{v}^i) \Big\}$,
2. $v^{i*} \in \arg\max_{v^i \in V^i} \Big\{ -c^i(v^i) + \beta \Big[\mu^i(v^i)(h_\omega(b^{c,v})) - h^i_t(b^{c,v}) - \big(c^i(\mu^i(v^i)) - c^i(v^i) \big) \Big] \Big\}$.

Let $SPE(h, V, c) \subseteq V^2$ denote the set of all subgame perfect equilibria of the investment game given a social choice function h.

A social choice function h (which is not necessarily efficient) SPE implements efficient investments if for any $V \subseteq \mathbb{R}^{\Omega\times I}$ and $c : V \to \mathbb{R}^I_+$, the set of all subgame perfect equilibria of the investment game given h and c coincides with the set of investment schedules which maximize the total utility of agents net of cost of investments given h.

Definition 3. *A social choice function $h : \mathbb{R}^{\Omega\times I} \to \Omega \times \mathbb{R}^I$ SPE implements efficient investments if for any $V \subseteq \mathbb{R}^{\Omega\times I}$ and a profile of cost functions $c : V \to \mathbb{R}^I_+$,*

$$SPE(h, V, c) = \arg \max_{(v,\bar{v}) \in \{(p,q) \in V^2 | c^i(q^i) \geq c^i(p^i) \forall i \in I\}} \sum_{i \in I} \Big\{ -c^i(v^i) + \beta \Big[\bar{v}^i(h_\omega(b^{c,v})) - \big(c^i(\bar{v}^i) - c^i(v^i) \big) \Big] \Big\}.$$

The main theorem of this paper identifies the set of (efficient) social choice functions which SPE implement efficient investments. For this purpose, we define strategy-proofness and commitment-proofness of social choice functions. The definition of strategy-proofness is standard: a social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ is strategy-proof if for any $i \in I$, $b \in \mathbb{R}^{\Omega \times I}$ and $\tilde{b}^i \in \mathbb{R}^\Omega$,

$$b^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) \leq b^i(h_\omega(b)) - h_t^i(b)$$

holds. We introduce a new concept commitment-proofness, which is a weaker requirement than strategy-proofness.

Definition 4. *A social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ is commitment-proof if for any $i \in I$, $b \in \mathbb{R}^{\Omega \times I}$, $\tilde{b}^i \in \mathbb{R}^\Omega$ and $x \geq 0$ such that $\tilde{b}^i(\omega) \leq b^i(\omega) + x$ for all $\omega \in \Omega$,*

$$\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) - x \leq b^i(h_\omega(b)) - h_t^i(b). \tag{1}$$

Here, x can be interpreted as the cost of commitment. Consider a situation where each agent can make a contract with a third party that she pays x to the third party in advance and x or less will be paid back to her depending on the realization of $\omega \in \Omega$ (the payback can be negative). Note that this assumption is far from demanding because the third party wouldn't lose anything from this contract. When agent i's original valuation function is b^i, her valuation function given this contract will be $\tilde{b}^i$ such that $\tilde{b}^i(\omega) \leq b^i(\omega) + x$ for all $\omega \in \Omega$. Commitment-proofness of a social choice function h says that no agent would benefit from such a commitment under h. Showing that commitment-proofness is implied by strategy-proofness is straightforward: for any $i \in I$, $b \in \mathbb{R}^{\Omega \times I}$, $\tilde{b}^i \in \mathbb{R}^\Omega$ and $x \geq 0$ such that $\tilde{b}^i(\omega) \leq b^i(\omega) + x$ for all $\omega \in \Omega$,

$$\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) - x \leq b^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) \leq b^i(h_\omega(b)) - h_t^i(b),$$

where the first inequality follows from the definition of $\tilde{b}^i$, and the second inequality holds from the strategy-proofness of h.

Commitment-proofness is not a restrictive concept because two of the most common auction rules, the first-price and the second-price auction (as social choice functions), satisfy this. The second-price auction is commitment-proof because it is strategy-proof. The first-price auction is commitment-proof although it is not strategy-proof. This is because the payoff net of the transfer is always zero in the first-price auction when the agent reports her true valuation and equation (1) holds for any $b \in \mathbb{R}^{\Omega \times I}$, $\tilde{b}^i \in \mathbb{R}^\Omega$ and $x \geq 0$.

The definition of (allocative) efficiency is standard. A social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ is efficient if for any $b \in \mathbb{R}^{\Omega \times I}$, the allocation rule satisfies

$$h_\omega(b) = \arg\max_{\omega \in \Omega} \sum_{i \in I} b^i(\omega).$$

Constrained-efficiency is a weaker efficiency criterion which gurantees efficiency within a certain subset of alternatives.

Definition 5. *A social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ is constrained-efficient for $\Omega' \subseteq \Omega$ with $\Omega' \neq \emptyset$ if for any $b \in \mathbb{R}^{\Omega \times I}$, the allocation rule satisfies*

$$h_\omega(b) = \arg\max_{\omega \in \Omega'} \sum_{i \in I} b^i(\omega).$$

3 Implementation of Efficient Investments

3.1 Impossibility Theorem

In the literature, it is often assumed that investments are made only before the mechanism. To see the consequence of this assumption, we will first examine whether it is possible to implement efficient investments without the post-mechanism investments. For this purpose, we will redefine the implementability of efficient investments for this environment accordingly.

When post-mechanism investments are not allowed, the investment game induced by a social choice function is a one-shot game which takes place before the mechanism. Thus, the equilibrium concept we employ in the investment game reduces to Nash equilibrium in this case.

Definition 6. *For any $V \subseteq \mathbb{R}^{\Omega \times I}$ and a profile of cost functions $c : V \to \mathbb{R}^I_+$, an investment profile $v^* \in V$ is a Nash equilibrium of the ex ante investment game given a social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ if for each $i \in I$,*

$$v^{i*} \in \arg\max_{v^i \in V^i} \Big\{ - c^i(v^i) + \beta \Big[v^i(h_\omega(v)) - h^i_t(v) \Big] \Big\}$$

holds. Let $NE(h, V, c) \subseteq V$ denote the set of all Nash equilibria of the ex ante investment game given a social choice function h.

Implementability of efficient investments is defined by the set of Nash equilibria of the ex ante investment game. In this environment, investment efficiency requires that the total utility of agents be maximized given that the ex post choices of valuations must coincide with the ones chosen before the mechanism.

Definition 7. *A social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ Nash implements efficient ex ante investments if for any $V \subseteq \mathbb{R}^{\Omega \times I}$ and a profile of cost functions $c : V \to \mathbb{R}^I_+$,*

$$NE(h, V, c) = \arg\max_{v \in V} \sum_{i \in I} \Big\{ - c^i(v^i) + \beta v^i(h_\omega(v)) \Big\}.$$

The question is whether there is any social choice function which implements efficient investments. The result is negative when we require allocative efficiency; for any efficient social choice function, there is $V \subseteq \mathbb{R}^{\Omega\times I}$ and cost functions $c : V \to \mathbb{R}^I_+$ for which the set of Nash equilibria of the investment game does not coincide with the set of efficient investments.

Theorem 1. *Suppose $|I| \geq 2$ and $|\Omega| \geq 2$. There is no efficient social choice function $h : \mathbb{R}^{\Omega\times I} \to \Omega \times \mathbb{R}^I$ which Nash implements efficient ex ante investments.*

Proof: Consider any I and Ω with $|I| \geq 2$ and $|\Omega| \geq 2$. Consider any arbitrary efficient social choice function $h : \mathbb{R}^{\Omega\times I} \to \Omega \times \mathbb{R}^I$. We will examine two cases where h is not strategy-proof and h is strategy-proof. In the former case, we will construct a profile of the sets of valuations and cost functions under which there is an inefficient Nash equilibrium of the investment game. In the latter case, we will show that a simple auction has multiple equilibria in the investment game and one of them is less efficient than the other.

[h is not strategy-proof] Since the social choice function h is not strategy-proof, there are $i \in I$, $v \in \mathbb{R}^{\Omega\times I}$ and $\tilde{v}^i \in \mathbb{R}^{\Omega\times I}$ such that

$$v^i(h_\omega(\tilde{v}^i, v^{-i})) - h^i_t(\tilde{v}^i, v^{-i}) > v^i(h_\omega(v)) - h^i_t(v). \tag{2}$$

Consider the sets of valuations $V \subseteq \mathbb{R}^{\Omega\times I}$ such that

$$\begin{aligned} &V^i = \{v^i, \tilde{v}^i\} \text{ and} \\ &V^j = \{v^j\} \text{ for all } j \in I \setminus \{i\}. \end{aligned}$$

Consider a profile of cost functions $c : V \to \mathbb{R}^I_+$ such that

$$\begin{aligned} c^i(v^i) &= \max\Big\{0, \beta\Big[v^i(h_\omega(v)) - h^i_t(v) - \big(\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h^i_t(\tilde{v}^i, v^{-i})\big)\Big]\Big\} \\ c^i(\tilde{v}^i) &= \max\Big\{0, \beta\Big[\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h^i_t(\tilde{v}^i, v^{-i}) - \big(v^i(h_\omega(v)) - h^i_t(v)\big)\Big]\Big\} \\ c^j(v^j) &= 0 \quad \text{for all } j \in I \setminus \{i\}. \end{aligned}$$

Note that

$$c^i(v^i) - c^i(\tilde{v}^i) = \beta\Big[v^i(h_\omega(v)) - h^i_t(v) - \big(\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h^i_t(\tilde{v}^i, v^{-i})\big)\Big]$$

always holds. Here, the only choice of valuations for each $j \in I \setminus \{i\}$ is v^j. Thus, we only need to analyze the choice of agent i's valuation for the Nash equilibrium and investment efficiency.

First, consider i's incentive for choosing between v^i and $\tilde{v}^i$. The total utility from choosing v^i when the valuations of other agents are v^{-i} is

$$-c^i(v^i) + \beta\Big[v^i(h_\omega(v)) - h_t^i(v)\Big],$$

and that from choosing $\tilde{v}^i$ is

$$-c^i(\tilde{v}^i) + \beta\Big[\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h_t^i(\tilde{v}^i, v^{-i})\Big].$$

The difference is

$$\begin{aligned}
& -c^i(v^i) + \beta\Big[v^i(h_\omega(v)) - h_t^i(v)\Big] - \Big\{ - c^i(\tilde{v}^i) + \beta\Big[\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h_t^i(\tilde{v}^i, v^{-i})\Big]\Big\} \\
= \;& \beta\Big[v^i(h_\omega(v)) - h_t^i(v) - \big(\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h_t^i(\tilde{v}^i, v^{-i})\big)\Big] - \Big(c^i(v^i) - c^i(\tilde{v}^i)\Big) \\
= \;& 0.
\end{aligned}$$

Therefore, v^i and $\tilde{v}^i$ are indifferent for agent i, and both v and $(\tilde{v}^i, v^{-i})$ are the Nash equilibria of the investment game.

Next, compare the social welfare between v and $(\tilde{v}^i, v^{-i})$. For v, the social welfare is

$$\sum_{j\in I}\Big\{ - c^j(v^j) + \beta v^j(h_\omega(v))\Big\} = -c^i(v^i) + \beta \sum_{j\in I} v^j(h_\omega(v)),$$

and the social welfare for $(\tilde{v}^i, v^{-i})$ is

$$-c^i(\tilde{v}^i) + \beta\Big[\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) + \sum_{j\in I\setminus\{i\}} v^j(h_\omega(\tilde{v}^i, v^{-i}))\Big].$$

The difference of these two is:

$$\begin{aligned}
& -c^i(v^i) + \beta \sum_{j\in I} v^j(h_\omega(v)) + c^i(\tilde{v}^i) - \beta\Big[\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) + \sum_{j\in I\setminus\{i\}} v^j(h_\omega(\tilde{v}^i, v^{-i}))\Big] && (3)\\
\geq \;& \beta\Big[\sum_{j\in I} v^j(h_\omega(\tilde{v}^i, v^{-i})) - \tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - \sum_{j\in I\setminus\{i\}} v^j(h_\omega(\tilde{v}^i, v^{-i}))\Big] - (c^i(v^i) - c^i(\tilde{v}^i)) && (4)\\
= \;& \beta\Big[v^i(h_\omega(\tilde{v}^i, v^{-i})) - \tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i}))\Big] - (c^i(v^i) - c^i(\tilde{v}^i)) && (5)\\
> \;& \beta\Big[v^i(h_\omega(v)) - h_t^i(v) + h_t^i(\tilde{v}^i, v^{-i}) - \tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i}))\Big] - (c^i(v^i) - c^i(\tilde{v}^i)) && (6)\\
= \;& 0, && (7)
\end{aligned}$$

in which the inequality in (4) follows from the efficiency of h; the inequality in (6) follows from equation (2). Therefore, $(\tilde{v}^i, v^{-i})$ is not an efficient investment profile. Thus, by $(\tilde{v}^i, v^{-i}) \in NE(h, V, c)$, h does not Nash implement efficient ex ante investments.

[h is strategy-proof] We will consider a slight modification of Example 4 of Hatfield, Kojima and Kominers (2015); a second-price auction for one item, where two agents bid for a single good. Suppose $\{i, j\} \subseteq I$ and $\{\omega^i, \omega^j\} \subseteq \Omega$. Since $|I|$ and $|\Omega|$ may be more than two, we choose the sets of valuation functions in the following way:

$$\begin{aligned} V^i &= \{a\mathbb{1}_{\{\omega=\omega^i\}} : a \in [0, 10]\}, \\ V^j &= \{a\mathbb{1}_{\{\omega=\omega^j\}} : a \in [0, 10]\}, \\ V^k &= \{0\} \text{ for any } k \in I \setminus \{i, j\}. \end{aligned}$$

Here ω^i and ω^j each represent the alternatives where i and j obtains the item respectively. Consider the following cost functions:

$$\begin{aligned} c^i(a\mathbb{1}_{\{\omega=\omega^i\}}) &= \beta a^2, \\ c^j(a\mathbb{1}_{\{\omega=\omega^j\}}) &= \frac{3}{2}\beta a^2, \\ c^k(0) &= 0 \text{ for any } k \in I \setminus \{i, j\}. \end{aligned}$$

Since the utility of agents other than i and j is always zero, focus on the valuation choices of agents i and j.

First, it is clear that $(\frac{1}{2}\mathbb{1}_{\{\omega=\omega^i\}}, 0)$ is the unique investment profile of i and j which maximizes the social welfare. Then consider another investment profile $(0, \frac{1}{3}\mathbb{1}_{\{\omega=\omega^j\}})$, and show that it is a Nash equilibrium of the investment game. Given $\bar{v}^j \equiv \frac{1}{3}\mathbb{1}_{\{\omega=\omega^j\}}$,

$$\begin{aligned} & \arg\max_{v^i \in V^i} \Big\{ - c^i(v^i) + \beta v^i(h_\omega(v^i, \bar{v}^j)) + \beta \bar{v}^j(h_\omega(v^i, \bar{v}^j)) \Big\} \\ = \; & \arg\max_{v^i \in V^i} \Big\{ - \frac{1}{\beta} c^i(v^i) + v^i(h_\omega(v^i, \bar{v}^j)) + \bar{v}^j(h_\omega(v^i, \bar{v}^j)) \Big\} \\ = \; & 0 \end{aligned}$$

holds. Since h is efficient and strategy-proof, $h^i_t(\cdot, \bar{v}^j)$ is written as a Groves function (Green and Laffont, 1977):

$$h^i_t(v^i, \bar{v}^j) = g(\bar{v}^j) - \bar{v}^j(h_\omega(v^i, \bar{v}^j)).$$

Hence,

$$\begin{aligned} & \arg\max_{v^i \in V^i} \Big\{ - \bar{c}^i(v^i) + v^i(h_\omega(v^i, \bar{v}^j)) - h^i_t(v^i, \bar{v}^j) \Big\} \\ = \; & \arg\max_{v^i \in V^i} \Big\{ - \bar{c}^i(v^i) + v^i(h_\omega(v^i, \bar{v}^j)) - g(\bar{v}^j) + \bar{v}^j(h_\omega(v^i, \bar{v}^j)) \Big\} \\ = \; & \arg\max_{v^i \in V^i} \Big\{ - \bar{c}^i(v^i) + v^i(h_\omega(v^i, \bar{v}^j)) + \bar{v}^j(h_\omega(v^i, \bar{v}^j)) \Big\} \end{aligned}$$

should hold for any cost function $\bar{c}^i : V^i \to \mathbb{R}_+$. Thus, we have

$$\begin{aligned}
& \arg\max_{v^i \in V^i} \Big\{ -c^i(v^i) + \beta v^i(h_\omega(v^i, \bar{v}^j)) - \beta h^i_t(v^i, \bar{v}^j) \Big\} \\
= \;& \arg\max_{v^i \in V^i} \Big\{ -\frac{1}{\beta} c^i(v^i) + v^i(h_\omega(v^i, \bar{v}^j)) - h^i_t(v^i, \bar{v}^j) \Big\} \\
= \;& \arg\max_{v^i \in V^i} \Big\{ -\frac{1}{\beta} c^i(v^i) + v^i(h_\omega(v^i, \bar{v}^j)) + \bar{v}^j(h_\omega(v^i, \bar{v}^j)) \Big\} \\
= \;& 0.
\end{aligned}$$

This means that $(0, \frac{1}{3}\mathbb{1}_{\{\omega=\omega^j\}})$ is a NE investment profile. However, this does not achieve investment efficiency given h because it is less efficient than $(\frac{1}{2}\mathbb{1}_{\{\omega=\omega^i\}}, 0)$. Therefore, $NE(h, V, c)$ differs from the set of efficient investments, which means that h fails to Nash implement efficient ex ante investments. □

The first half of the proof follows Theorem 1 and 2 of Hatfield, Kojima and Kominers (2015) which show that given a profile of the sets of valuations, the social choice function is strategy-proof for agent i if and only if the ex ante choice of valuation that maximizes i's utility always maximizes social welfare. The construction of cost functions is slightly different from theirs because the cost of investment in our model is non-negative whereas it is not assumed in their paper.

3.2 Possibility Theorem

Now we turn our attention to the original model in which the additional investments are possible after the mechanism. This extension allows us to obtain a more permissible result; a large class of social choice functions SPE implement efficient investments, and we can characterize such a set. For the purpose of the main theorem, the following lemma is proved.

Lemma 1. *For any $i \in I$, $V^i \subseteq \mathbb{R}^{\Omega \times I}$ and $c^i : V^i \to \mathbb{R}_+$,*

$$c^i(v^i) \geq \max_{\omega \in \Omega} \Big\{ b^{c^i, v^i}(\omega) - b^{c^i, v^{0i}}(\omega) \Big\}$$

holds for any $v^i \in V^i$ and $v^{0i} \in V^i$ such that $c^i(v^{0i}) = 0$.

Proof: From the definition of the valuation at the time of the mechanism,

$$\begin{aligned}
b^{c^i, v^{0i}}(\omega) &= \max_{\bar{v}^i \in V^i} \Big\{ \bar{v}^i(\omega) - c^i(\bar{v}^i) \Big\} \\
&\geq \max_{\bar{v}^i \in \{\tilde{v}^i \in V^i | c^i(\tilde{v}^i) \geq c^i(v^i)\}} \Big\{ \bar{v}^i(\omega) - c^i(\bar{v}^i) \Big\} \\
&= b^{c^i, v^i}(\omega) - c^i(v^i)
\end{aligned}$$

holds for any $\omega \in \Omega$. Thus, we have $c^i(v^i) \geq \max_{\omega \in \Omega} \left\{ b^{c^i, v^i}(\omega) - b^{c^i, v^{0i}}(\omega) \right\}$. $\square$

The next theorem shows that for constrained-efficient social choice functions, commitment-proofness is sufficient and necessary for SPE implementing efficient investments.

Theorem 2. *Consider any I, Ω and a social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ which is constrained-efficient for some $\Omega' \subseteq \Omega$ with $\Omega' \neq \emptyset$. The social choice function h SPE implements efficient investments if and only if it is commitment-proof.*

Proof: We will fist show that when h is constrained-efficient for some $\Omega' \subseteq \Omega$ with $\Omega' \neq \emptyset$, any efficient valuations $(v, \bar{v}) \in V^2$ is such that $c^i(v^i) = 0$ and $\bar{v}^i$ is optimal given the outcomes of h for each $i \in I$. Next, the sufficiency of commitment-proofness for SPE implementation is proved by showing that such profiles of valuations are the only SPE of the investment game induced by h. Finally, we will show the necessity of commitment proofness of h by constructing a profile of the sets of valuations and associated cost functions under which the set of SPE of the investment game differs from the set of efficient investments.

[Efficient Investments] First, take any $V \subseteq \mathbb{R}^{\Omega \times I}$ and $c : V \to \mathbb{R}^I_+$, and fix them. For any $i \in I$, let V^{0i} be the set of all valuation functions in V^i whose costs are zero, i.e., $V^{0i} \equiv \{v^i \in V^i | c^i(v^i) = 0\}$. Let $V^0 \equiv \times_{i \in I} V^{0i}$. Moreover, for each $i \in I$, let $\mu^i : V \twoheadrightarrow V^i$ be the optimal choice correspondence of ex post valuations under h for valuation functions $v \in V$ chosen before the mechanism:

$$\mu^i(v) \equiv \arg \max_{\bar{v}^i \in \{\tilde{v}^i \in V^i | c^i(\tilde{v}^i) \geq c^i(v^i)\}} \left\{ \bar{v}^i(h_\omega(b^{c,v})) - c^i(\bar{v}^i) \right\}.$$

We will show that the set of efficient investment schedules is

$$\{(p, q) \in V^2 | p \in V^0 \text{ and } q^i \in \mu^i(p) \text{ for all } i \in I\}.$$

Take any profile of valuations $(v^0, \bar{v}^*)$ from this set:

$$(v^0, \bar{v}^*) \in \{(p, q) \in V^2 | p \in V^0 \text{ and } q^i \in \mu^i(p) \text{ for all } i \in I\}.$$

Also take another profile of valuations $(v, \bar{v})$ with a costly prior investment:

$$(v, \bar{v}) \in \{(p, q) \in V^2 | p \notin V^0 \text{ and } c^i(q^i) \geq c^i(p^i) \text{ for all } i \in I\}.$$

It is shown that the social welfare given h under $(v, \bar{v})$ is strictly less than that under $(v^0, \bar{v}^*)$:

$$
\begin{aligned}
& \sum_{i \in I} \Big\{ -c^i(v^i) + \beta \Big[\bar{v}^i(h_\omega(b^{c,v})) - \big(c^i(\bar{v}^i) - c^i(v^i) \big) \Big] \Big\} && (8) \\
= & \sum_{i \in I} \Big\{ \beta \Big[\bar{v}^i(h_\omega(b^{c,v})) - c^i(\bar{v}^i) \Big] - (1-\beta) c^i(v^i) \Big\} && (9) \\
< & \sum_{i \in I} \beta \Big[\bar{v}^i(h_\omega(b^{c,v})) - c^i(\bar{v}^i) \Big] && (10) \\
\le & \sum_{i \in I} \beta b^{c^i, v^{0i}}(h_\omega(b^{c,v})) && (11) \\
\le & \sum_{i \in I} \beta b^{c^i, v^{0i}}(h_\omega(b^{c,v^0})) && (12) \\
= & \sum_{i \in I} \beta \Big[\bar{v}^{*i}(h_\omega(b^{c,v^0})) - c^i(\bar{v}^{*i}) \Big], && (13)
\end{aligned}
$$

in which the last equation (13) is the social welfare given h under $(v^0, \bar{v}^*)$. The inequality in (10) holds because $c^i(v^i) > 0$ and $\beta < 1$; the inequality in (11) follows from the definition of $b^{c^i, v^{0i}}$; the inequality in (12) follows from the fact that h is constrained-efficient; the equality of (13) follows from the definition of $b^{c^i, v^{0i}}$ and $\bar{v}^{*i}$. Moreover, for any prior investments $v^0 \in V^0$, the valuations b^{c,v^0} at the time of the mechanism is the same, and hence, the outcome of the social choice function should also be the same. Therefore, any profile of valuations in $\{(p, q) \in V^2 | p \in V^0 \text{ and } q^i \in \mu^i(p) \text{ for all } i \in I\}$ maximizes the social welfare given h. Thus, the set of efficient investment schedules is characterized by

$$\{(p, q) \in V^2 | p \in V^0 \text{ and } q^i \in \mu^i(p) \text{ for all } i \in I\}.$$

[Sufficiency of commitment-proofness] We still fix arbitrary $V \subseteq \mathbb{R}^{\Omega \times I}$ and $c : V \to \mathbb{R}^I_+$. We will show that the set of SPE of the investment game is exactly $\{(p, q) \in V^2 | p \in V^0 \text{ and } q^i \in \mu^i(p) \text{ for all } i \in I\}$. Take any agent $i \in I$ and $v^{-i} \in V^{-i}$, and consider i's incentive for investments when the valuation functions of other agents at the time of the mechanism are fixed to $b^{-i} \equiv b^{c^{-i}, v^{-i}}$.

Take any investment schedule $(v^{0i}, \bar{v}^{*i})$ of agent i from the following set:

$$(v^{0i}, \bar{v}^{*i}) \in \{(p^i, q^i) \in (V^i)^2 | p^i \in V^{0i} \text{ and } q^i \in \mu^i(p^i, v^{-i})\}.$$

Also take another investment schedule $(v^i, \bar{v}^i)$ with a costly prior investment:

$$(v^i, \bar{v}^i) \in \{(p^i, q^i) \in (V^i)^2 | p^i \notin V^{0i} \text{ and } c^i(q^i) \ge c^i(p^i)\}.$$

We will show that $(v^{0i}, \bar{v}^{*i})$ is strictly better than $(v^i, \bar{v}^i)$ for agent i. To see this, the ex ante utility from $(v^i, \bar{v}^i)$ is written as:

$$
\begin{aligned}
& -c^i(v^i) + \beta\Big[\bar{v}^i(h_\omega(b^{c^i,v^i}, b^{-i})) - h_t^i(b^{c^i,v^i}, b^{-i}) - \big(c^i(\bar{v}^i) - c^i(v^i)\big)\Big] && (14)\\
= \;& \beta\Big[\bar{v}^i(h_\omega(b^{c^i,v^i}, b^{-i})) - h_t^i(b^{c^i,v^i}, b^{-i}) - c^i(\bar{v}^i)\Big] - (1-\beta)c^i(v^i) && (15)\\
< \;& \beta\Big[\bar{v}^i(h_\omega(b^{c^i,v^i}, b^{-i})) - h_t^i(b^{c^i,v^i}, b^{-i}) - c^i(\bar{v}^i)\Big] && (16)\\
\leq \;& \beta\Big[b^{c^i,v^i}(h_\omega(b^{c^i,v^i}, b^{-i})) - h_t^i(b^{c^i,v^i}, b^{-i}) - c^i(v^i)\Big] && (17)\\
\leq \;& \beta\Big[b^{c^i,v^i}(h_\omega(b^{c^i,v^i}, b^{-i})) - h_t^i(b^{c^i,v^i}, b^{-i}) - \max\Big\{0, \max_{\omega\in\Omega}\big\{b^{c^i,v^i}(\omega) - b^{c^i,v^{0i}}(\omega)\big\}\Big\}\Big] && (18)\\
\leq \;& \beta\Big[b^{c^i,v^{0i}}(h_\omega(b^{c^i,v^{0i}}, b^{-i})) - h_t^i(b^{c^i,v^{0i}}, b^{-i})\Big] && (19)\\
= \;& \beta\Big[\bar{v}^{*i}(h_\omega(b^{c^i,v^{0i}}, b^{-i})) - h_t^i(b^{c^i,v^{0i}}, b^{-i}) - c^i(\bar{v}^{*i})\Big], && (20)
\end{aligned}
$$

in which the last equation (20) is the ex ante utility from $(v^{0i}, \bar{v}^{*i})$. The inequality in (16) holds because $c^i(v^i) > 0$ and $\beta < 1$; the inequality in (17) follows from the definition of b^{c^i,v^i}; the inequality in (18) follows from Lemma 1; the inequality in (19) follow from the fact that h is commitment-proof; and the equality in (20) follows from the definition of $b^{c^i,v^{0i}}$ and $\bar{v}^{*i}$. Moreover, any investment schedule in $\{(p^i, q^i) \in (V^i)^2 | p^i \in V^{0i}$ and $q^i \in \mu^i(p^i, v^{-i})\}$ gives exactly the same utility as $(v^{0i}, \bar{v}^{*i})$ does. Therefore, $\{(p^i, q^i) \in (V^i)^2 | p^i \in V^{0i}$ and $q^i \in \mu^i(p^i, v^{-i})\}$ is the set of best responses of agent i to any $b^{-i} \in \mathbb{R}^{\Omega\times(|I|-1)}$.

For any prior investment $v^j \in V^{0j}$, agent j should have the same valuation at the time of the mechanism. Therefore, the utility agent i obtains from choosing an investment schedule from $\{(p^i, q^i) \in (V^i)^2 | p^i \in V^{0i}$ and $q^i \in \mu^i(p^i, v^{-i})\}$ is the same as long as v^j is taken from V^{0j} for any $j \in I\backslash\{i\}$. From this argument, the set of SPE of the investment game is characterized by any combination of valuations from $\{(p^i, q^i) \in (V^i)^2 | p^i \in V^{0i}$ and $q^i \in \mu^i(p^i, v^{-i})\}_{i\in I}$. Therefore, $SPE(h, V, c) = \{(p, q) \in V^2 | p \in V^0$ and $q^i \in \mu^i(p)$ for all $i \in I\}$ and it coincides with the set of efficient investment schedules.

[Necessity of commitment-proofness] Consider a social choice function h which is constrained-efficient for some $\Omega' \subseteq \Omega$ with $\Omega' \neq \emptyset$ but is not commitment-proof. Since h is constrained-efficient, for any $V \subseteq \mathbb{R}^{\Omega\times I}$ and $c : V \to \mathbb{R}^I_+$, the set of efficient investment schedules is

$$\{(p, q) \in V^2 | p \in V^0 \text{ and } q^i \in \mu^i(p) \text{ for all } i \in I\}.$$

by the argument above. We will show that for some $V \subseteq \mathbb{R}^{\Omega\times I}$ and $c : V \to \mathbb{R}^I_+$, there is a profile of investment schedules in $\{(p, q) \in V^2 | p \in V^0$ and $q^i \in \mu^i(p)$ for all $i \in I\}$ which is not a SPE of the investment game.

First, since h is not commitment-proof, there are $i \in I$, $b \in \mathbb{R}^{\Omega \times I}$ and $\tilde{b}^i \in \mathbb{R}^{\Omega}$ such that

$$\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h^i_t(\tilde{b}^i, b^{-i}) - \Big(b^i(h_\omega(b)) - h^i_t(b)\Big) > \max\Big\{0, \max_{\omega \in \Omega}\big\{\tilde{b}^i(\omega) - b^i(\omega)\big\}\Big\}. \quad (21)$$

Consider the following profile of the set of valuations:

$$V^i = \{b^i, \tilde{b}^i\},$$
$$V^j = \{b^j\} \text{ for all } j \in I \setminus \{i\}.$$

Consider a profile of cost functions $c : V \to \mathbb{R}^I_+$ such that

$$c^i(\tilde{b}^i) = \begin{cases} \max_{\omega \in \Omega}\Big\{\tilde{b}^i(\omega) - b^i(\omega)\Big\} & \text{if } \max_{\omega \in \Omega}\Big\{\tilde{b}^i(\omega) - b^i(\omega)\Big\} > 0 \\ \delta & \text{otherwise} \end{cases}$$
$$c^j(b^j) = 0 \text{ for all } j \in I$$

where $\delta > 0$. Any agent $j \in I \setminus \{i\}$ always chooses $b^j \in V^j$ in the investment game because there is only one choice in V^j. Agent i has two choices b^i and $\tilde{b}^i$. When she chooses $\tilde{b}^i$ prior to the mechanism, since $c^i(\tilde{b}^i) > c^i(b^i)$, the valuation at the time of the mechanism is

$$b^{c^i, \tilde{b}^i}(\omega) = \Big\{\tilde{b}^i(\omega) - c^i(\tilde{b}^i)\Big\} + c^i(\tilde{b}^i) = \tilde{b}^i(\omega)$$

for each $\omega \in \Omega$. On the other hand, when she chooses b^i prior to the mechanism, the valuation at the time of the mechanism is

$$b^{c^i, b^i}(\omega) = \max_{\bar{v}^i \in \{b^i, \tilde{b}^i\}} \Big\{\bar{v}^i(\omega) - c^i(\bar{v}^i)\Big\} = b^i(\omega)$$

for each $\omega \in \Omega$ since we have $b^i(\omega) \geq \tilde{b}^i(\omega) - c^i(\tilde{b}^i)$ by the construction of $c^i(\tilde{b}^i)$. Thus, we need to compare the following two investment schedules, $(\tilde{b}^i, \tilde{b}^i)$ and (b^i, b^i).

Consider agent i's incentive for investments given the valuations b^{-i} of other agents. The total utility of agent i when choosing an investment schedule $(\tilde{b}^i, \tilde{b}^i)$ is

$$-c^i(\tilde{b}^i) + \beta\Big[\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h^i_t(\tilde{b}^i, b^{-i})\Big]$$

and when choosing an investment schedule (b^i, b^i), it is

$$\beta\Big[b^i(h_\omega(b)) - h^i_t(b)\Big].$$

The difference of these two is calculated as:

$$\begin{aligned}
& -c^i(\tilde{b}^i) + \beta\Big[\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h^i_t(\tilde{b}^i, b^{-i})\Big] - \beta\Big[b^i(h_\omega(b)) - h^i_t(b)\Big] \\
= \;& -(1-\beta)c^i(\tilde{b}^i) + \beta\Big[\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h^i_t(\tilde{b}^i, b^{-i}) - c^i(\tilde{b}^i)\Big] - \beta\Big[b^i(h_\omega(b)) - h^i_t(b)\Big] \\
= \;& -(1-\beta)c^i(\tilde{b}^i) \\
& +\beta\Big[\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h^i_t(\tilde{b}^i, b^{-i}) - \Big(b^i(h_\omega(b)) - h^i_t(b)\Big) - \max\Big\{\delta, \max_{\omega\in\Omega}\big\{\tilde{b}^i(\omega) - b^i(\omega)\big\}\Big\}\Big] \\
> \;& 0,
\end{aligned}$$

in which $c^i(\tilde{b}^i) = \max\Big\{\delta, \max_{\omega\in\Omega}\big\{\tilde{b}^i(\omega) - b^i(\omega)\big\}\Big\}$ holds for sufficiently small $\delta > 0$, and the final inequality holds from equation (21) when we take β sufficiently close to 1 and $\delta > 0$ sufficiently small. Therefore, (b, b) is not a SPE in this investment game. However for this V and c, since h is constrained-efficient, (b, b) is a profile of efficient inevstment schedules because $c^j(b^j) = 0$ for all $j \in I$ and $\arg\max_{\bar{v}^j\in V^j}\Big\{\bar{v}^j(\omega) - c^j(\bar{v}^j)\Big\} = b^j(\omega)$ for any $\omega \in \Omega$. Thus, h fails to SPE implement efficient investments. $\square$

As the proof shows, when the social choice function h is constrained-efficient, the efficient investment profile is such that every agent makes a costly investment only after the mechanism. In order for agents not to have an incentive to make costly investments before the mechanism, commitment-proof of h is sufficient and necessary; no agent would get better off by investing in some costly valuation and manipulating the outcome of the mechanism under a commitment-proof social choice function.

The sufficiency of commitment-proofness no longer holds if the social choice function is not constrained-efficient for any $\Omega' \subseteq \Omega$. The following example demonstrates that a strategy-proof social choice function fails to implement efficient investments when it is not constrained-efficient.

Observation 1. *Suppose $|I| \geq 2$ and $|\Omega| \geq 2$. There is a strategy-proof social choice function $h : \mathbb{R}^{\Omega\times I} \to \Omega \times \mathbb{R}^I$ which is not constrained-efficient for any $\Omega' \subseteq \Omega$ and does not SPE implement efficient investments for some $\beta \in (0, 1)$.*

Example 1. Let $\{i, j\} \subseteq I$ and $\{\omega_1, \omega_2\} \subseteq \Omega$. Consider a social choice function $h : \mathbb{R}^{\Omega\times I} \to \Omega \times \mathbb{R}^I$ such that for any $b \in \mathbb{R}^{\Omega\times I}$,

$$h_\omega(b) \in \arg\max_{\omega\in\Omega} h_\omega(b^i),$$

$$h_t(b) = 0.$$

This means that the best alternative for agent i is always chosen and no transfer is made under h. This h is strategy-proof because i does not have an incentive to manipulate and j's report does not affect the outcome. It is clear that h is not constrained-efficient because j's valuation is not taken into account. Consider the following sets of valuations:

$$\begin{aligned}
&V^i = \{b^i, \tilde{b}^i\},\\
&V^j = \{b^j\},\\
&V^k = \{0\} \text{ for any } k \in I \setminus \{i, j\}
\end{aligned}$$

where

$$\begin{aligned}
&b^i(\omega_1) = 5,\ b^i(\omega_2) = 4,\ b^i(\omega) = 0 \text{ for any } \omega \in \Omega \setminus \{\omega_1, \omega_2\}\\
&\tilde{b}^i(\omega_1) = 5,\ \tilde{b}^i(\omega_2) = 6,\ \tilde{b}^i(\omega) = 0 \text{ for any } \omega \in \Omega \setminus \{\omega_1, \omega_2\}\\
&b^j(\omega_1) = 0,\ b^j(\omega_2) = 2,\ b^j(\omega) = 0 \text{ for any } \omega \in \Omega \setminus \{\omega_1, \omega_2\}.
\end{aligned}$$

Also consider the following cost functions:

$$\begin{aligned}
&c^i(b^i) = 0,\ c^i(\tilde{b}^i) = 2,\\
&c^j(b^j) = 0,\\
&c^k(0) = 0 \text{ for any } k \in I \setminus \{i, j\}.
\end{aligned}$$

Since the only choice of valuation is 0 for any $k \in I \setminus \{i, j\}$, we can ignore these agents. For j, the only choice of valuation is b^j. Since $c^i(\tilde{b}^i) > c^i(b^i)$, if i chooses $\tilde{b}^i$ before the mechanism, then the only valuation she can choose after the mechanism is $\tilde{b}^i$. If i chooses b^i before the mechanism, since $b^i(\omega) \geq \tilde{b}^i(\omega) - c^i(\tilde{b}^i)$ holds for any $\omega \in \Omega$, her optimal valuation after the mechanism is also b^i. Thus, compare two invetment schedules (b^i, b^i) and $(\tilde{b}^i, \tilde{b}^i)$ of agent i for the equilibrium and the investment efficiency.

First, it is shown that $(\tilde{b}^i, \tilde{b}^i)$ gives higher social welfare than (b^i, b^i) for sufficiently large $\beta \in (0, 1)$. Given j's valuation b^j, the social welfare when i takes $(\tilde{b}^i, \tilde{b}^i)$ is

$$-2 + \beta(6 + 2) = 8\beta - 2.$$

The social welfare when i takes (b^i, b^i) is

$$0 + \beta(5 + 0) = 5\beta.$$

Since the former is larger for $\beta > \frac{2}{3}$, $(\tilde{b}^i, \tilde{b}^i)$ is an efficient investment schedules for agent i and (b^i, b^i) is not.

Next, consider the incentive of agent i. Given j's valuation b^j, compare the total utility of i when she chooses $(\tilde{b}^i, \tilde{b}^i)$ and (b^i, b^i). When i chooses $(\tilde{b}^i, \tilde{b}^i)$, her utility is $6\beta - 2$ whereas it is 5β when i chooses (b^i, b^i). From $6\beta - 2 < 5\beta$ for any $\beta \in (0, 1)$, agent i chooses (b^i, b^i). Thus, (b^i, b^i) is a SPE investment schedule but is not efficient. Therefore, h does not SPE implement efficient investments. □

Time-discounting also plays an important role in Theorem 2 although β can be arbitrarily close to one. When it is exactly one, the sufficiency part of Theorem 2 does not hold anymore. The next observation shows that investment efficiency given a VCG social choice function, which is efficient and strategy-proof, may not be SPE implemented if $\beta = 1$. For any I and Ω, a VCG social choice function $h^{VCG} : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ is defined as follows: for any $b \in \mathbb{R}^{\Omega \times I}$,

$$h_\omega^{VCG}(b) \in \arg\max_{\omega \in \Omega} \sum_{i \in I} b^i(\omega),$$

$$h_t^{VCG} = \max_{\omega \in \Omega} \sum_{j \in I \setminus \{i\}} b^j(\omega) - \sum_{j \in I \setminus \{i\}} b^j(h_\omega^{VCG}(b)).$$

Observation 2. *Suppose $|I| \geq 2$, $|\Omega| \geq 2$ and assume $\beta = 1$. A VCG social choice function $h^{VCG} : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ does not SPE implement efficient investments.*

Example 2. Let $\{i, j\} \subseteq I$ and $\{\omega_1, \omega_2\} \subseteq \Omega$. Consider the following sets of valuations:

$$V^i = \{b^i, \tilde{b}^i\},$$
$$V^j = \{b^j, \tilde{b}^j\},$$
$$V^k = \{0\} \text{ for any } k \in I \setminus \{i, j\}$$

where

$$b^i(\omega_1) = b^j(\omega_1) = 5,\ b^i(\omega_2) = b^j(\omega_2) = 4,\ b^i(\omega) = b^j(\omega) = 0 \text{ for any } \omega \in \Omega \setminus \{\omega_1, \omega_2\}$$
$$\tilde{b}^i(\omega_1) = \tilde{b}^j(\omega_1) = 5,\ \tilde{b}^i(\omega_2) = \tilde{b}^j(\omega_2) = 6,\ \tilde{b}^i(\omega) = \tilde{b}^j(\omega) = 0 \text{ for any } \omega \in \Omega \setminus \{\omega_1, \omega_2\}.$$

Consider the cost functions as follows:

$$c^i(b^i) = c^j(b^j) = 0,$$
$$c^i(\tilde{b}^i) = c^j(\tilde{b}^j) = 2,$$
$$c^k(0) = 0 \text{ for any } k \in I \setminus \{i, j\}.$$

Since the only choice of valuation is 0 for any $k \in I \setminus \{i, j\}$, we can ignore these agents. Given a VCG social choice function h^{VCG}, the most efficient investment schedules of agent i and j is (b, b) because it achieves the maximum social welfare $\beta(5 + 5) = \beta 10 = 10$.

Next, consider another profile of investment schedules $(\tilde{b}, \tilde{b})$ of agent i and j. When the other agent (say j) chooses $\tilde{b}^j$ before the mechanism, her valuation at the time of the mechanism is $\tilde{b}^j = (0, 6)$. Since h^{VCG} is efficient, it chooses ω_2 whichever of b^i and $\tilde{b}^i$ agent i chooses. The utility when she chooses (b^i, b^i) or $(b^i, \tilde{b}^i)$ is 4β, and when she chooses $(\tilde{b}^i, \tilde{b}^i)$, it is $6\beta - 2$. Since $\beta = 1$, these choices are indifferent and therefore, $(\tilde{b}, \tilde{b})$ constitutes a SPE of the investment game. But $(\tilde{b}, \tilde{b})$ is not efficient investments, and thus h^{VCG} does not SPE implement efficient investments. $\square$

4 Applications

In this section, we present several applications of our results. First, we will explain the main results (Theorem 1 and 2) using an example of a procurement auction. Next, we consider the provision of public goods where budget balancing is required.

4.1 Procurement Auction

Consider a procurement auction in which firms are competing for a single government project. We still assume complete information among firms. In a standard (efficient) procurement auction, firms bid production costs and the government buys the project from the firm with the lowest bid. Here we put a slightly more structure in the model.

Assume that the original production cost is $\alpha > 0$ for any firm. Each firm makes investments to reduce the production cost of the project. Suppose firm i's cost reduction is $a^i \in [0, \alpha]$. Then the actual value of the project (without including the cost of investments) is negative: $b^i = a^i - \alpha \leq 0$. For example, consider the second-price auction. If firm i's value b^i is the highest and the second highest value is b^j by firm j, then firm i produces the project and the payment from the government to firm i is $-b^j$. Thus, the utility of firm i from the mechanism is

$$b^i - b^j = a^i - \alpha - (a^j - \alpha) = a^i - a^j.$$

This is equivalent to running a second-price auction for the amount of cost reduction, not the production cost itself.

From this observation, we will define the set of valuation functions for firm i as $V^i = \{a\mathbb{1}_{\{\omega=i\}} : a \in [0, \alpha]\}$, in which each valuation function represents the cost reduction. This makes the example look simple because the valuations are non-negative. We consider (ex ante) social choice functions and mechanisms given these sets of valuation functions.

The next example demonstrates the implications of Theorem 1 and 2; under the first and second-price auction as social choice functions, (i) there is an inefficient investment equilibrium with only an ex ante investment stage, and (ii) the unique SPE achieves investment efficiency when they can also make investments after the mechanism.

Example 3. Let $I = \{i, j\}$, $\Omega \cong I$, $V^i = \{a\mathbb{1}_{\{\omega=i\}} : a \in [0, 10]\}$ and $V^j = \{a\mathbb{1}_{\{\omega=j\}} : a \in [0, 10]\}$. Their cost functions for investments are:

$$c^i(a\mathbb{1}_{\{\omega=i\}}) = \frac{1}{6}a^2,$$
$$c^j(a\mathbb{1}_{\{\omega=j\}}) = \frac{1}{4}a^2.$$

For firm i, the optimal investment when it only invests ex ante and produces the project is

$$\max_{a\in[0,10]} \Big\{ -\frac{1}{6}a^2 + \beta a\Big\} = 3\beta.$$

For firm i, the optimal investment when it only invests ex post and produces the project is

$$\max_{a\in[0,10]} \beta\Big\{ -\frac{1}{6}a^2 + a\Big\} = 3.$$

For firm j, the optimal investment when it only invests ex ante and produces the project is

$$\max_{a\in[0,10]} \Big\{ -\frac{1}{4}a^2 + \beta a\Big\} = 2\beta.$$

For firm j, the optimal investment when it only invests ex ante and produces the project is

$$\max_{a\in[0,10]} \beta\Big\{ -\frac{1}{4}a^2 + a\Big\} = 2.$$

<u>Second-price Auction</u>

[1] Investments only before the mechanism:

Consider an ex ante investment profile $(0, 2\beta\mathbb{1}_{\{\omega=j\}})$. This is a Nash equilibrium because firm i does not have an incentive to win the auction as $\beta^2(\frac{3}{2} - 2) < 0$ and it is also optimal for firm j. However, this is not an efficient investment profile because $(3\beta\mathbb{1}_{\{\omega=i\}}, 0)$ is more efficient.

[2] Investments before and after the mechanism:

Consider firm i's incentive. Since the second-price auction is strategy-proof, for any firm j's valuation b^{c^j,v^j} at the time of the mechanism, it is optimal for firm i to have a valuation function $\frac{3}{2}\mathbb{1}_{\{\omega=i\}}$ at the mechanism. For any $\beta \in (0, 1)$, it is strictly optimal not to make an ex ante investment because $b^{c^i,0} = \frac{3}{2}\mathbb{1}_{\{\omega=i\}}$. The argument is the same for firm j. Therefore, the unique SPE is $((0, \frac{3}{2}\mathbb{1}_{\{\omega=i\}}), (0, 0))$, which achieves investment efficiency.

First-price Auction

[1] Investments only before the mechanism:

If the social choice function is the first-price auction, the benefit from the mechanism is always zero. Thus, it is always optimal for any firm to make no ex ante investments. However, this is obviously not an efficient investment profile.

[2] Investments before and after the mechanism:

As the same reason as above, it is always optimal for any firm to make no ex ante investments. Therefore, the valuation at the mechanism is $b^{c^i,0} = \frac{3}{2}\mathbb{1}_{\{\omega=i\}}$ for firm i and $b^{c^j,0} = \mathbb{1}_{\{\omega=i\}}$ for firm j. The unique SPE is $((0, \frac{3}{2}\mathbb{1}_{\{\omega=i\}}), (0, 0))$ and investment efficiency is achieved. $\square$

4.2 Provision of Public Goods

Consider the problem of providing public goods by agents. The provision of public goods is a choice of an alternative $\omega \in \Omega$, and making transfers $(t^i)_{i \in I}$ is allowed. The only difference from the original model is that we require budget balancing for social choice functions, i.e., the sum of the transfers must be equal to zero.

Definition 8. *A social choice function h satisfies budget balancing if*

$$\sum_{i \in I} h_t^i(b) = 0$$

for any $b \in \mathbb{R}^{\Omega \times I}$.

Budget balancing is considered to be a part of allocative efficiency because the transfer collected by the mechanism designer is regarded as the loss of welfare in this model.

In this environment, it is known that there is no social choice function that is strategy-proof, efficient and budget balancing (Walker, 1980). Therefore, when there is only an ex ante investment stage, it is impossible to even ensure the existence of efficient investment equilibria (Hatfield, Kojima and Kominers, 2015). However, we can show that commitment-proof is compatible with other properties; there is a social choice function which is commitment-proof, efficient and budget balancing. This implies that allowing for the ex post investments, budget balancing does not preclude the SPE implementation of full efficiency.

Proposition 1. *For any I, Ω and an efficient allocation rule $h_\omega : \mathbb{R}^{\Omega\times I} \to \Omega$, there exists a transfer rule $h_t : \mathbb{R}^{\Omega\times I} \to \mathbb{R}^I$ with which $h = (h_\omega, h_t)$ is commitment-proof and budget balancing.*

Proof: For any efficient allocation rule h_ω, consider the following transfer rule h_t which divides the maximized value equally among all agents:

$$h_t^i(b) = b^i(h_\omega(b)) - \frac{1}{n}\sum_{i\in I} b^i(h_\omega(b)).$$

It is clear that h is budget balancing. It suffices to show that h is commitment-proof. Consider any $i \in I$, $b \in \mathbb{R}^{\Omega\times I}$, $\tilde{b}^i \in \mathbb{R}^\Omega$ and $x \geq 0$ such that $\tilde{b}^i(\omega) \leq b^i(\omega) + x$ for all $\omega \in \Omega$. We will show:

$$\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) - x \leq b^i(h_\omega(b)) - h_t^i(b).$$

Since $x \geq \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\}$ holds,

$$\begin{aligned}
& \text{(RHS) - (LHS)} \\
\geq\ & \left[b^i(h_\omega(b)) - h_t^i(b)\right] - \left[\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i})\right] + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
=\ & \frac{1}{n}\sum_{i\in I} b^i(h_\omega(b)) - \frac{1}{n}\left\{\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) + \sum_{j\in I\setminus\{i\}} b^j(h_\omega(\tilde{b}^i, b^{-i}))\right\} + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
=\ & -\frac{1}{n}\left\{\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) + \sum_{j\in I\setminus\{i\}} b^j(h_\omega(\tilde{b}^i, b^{-i})) - \sum_{i\in I} b^i(h_\omega(b))\right\} + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
=\ & -\frac{1}{n}\left\{\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - b^i(h_\omega(\tilde{b}^i, b^{-i})) + \sum_{i\in I} b^i(h_\omega(\tilde{b}^i, b^{-i})) - \sum_{i\in I} b^i(h_\omega(b))\right\} \\
& + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
\geq\ & -\frac{1}{n}\max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
=\ & \frac{n-1}{n}\max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
\geq\ & 0.
\end{aligned}$$

The second inequality holds from the efficiency of h. Therefore, this h is commitment-proof and the proof is done. □

Corollary 1. *For any I and Ω, there exists an efficient and budget balancing social choice function $h : \mathbb{R}^{\Omega\times I} \to \Omega \times \mathbb{R}^I$ which SPE implements efficient investments.*

5 Concluding Remarks

Our main result shows that allowing for ex post investments, commitment-proofness is equivalent to SPE implementing efficient investments for allocatively efficient mechanisms. This has two following implications. First, whenever it is possible, the mechanism should be run sufficiently before actual production or consumption is carried out. This allows agents to substitute investments between before and after the mechanism. Otherwise, by Theorem 1, we cannot eliminate the possibility of inefficient equilibria. Second, commitment-proofness of the mechanism is essential. This ensures that no agent has an incentive to commit to a certain type by making prior investments. Moreover, this is not a restrictive concept since not just strategy-proof mechanisms but the first-price auction (as a social choice function) also satisfies this.

In this paper, we have strong assumptions on the information and the technology of investments. First, complete information is assumed throughout so that there is no uncertainty about their initial types and the consequence of investments. Moreover, the investment technology does not change over time. These make the analysis simple because agents can defer investments after the mechanism unless there is an incentive for commitment. In future research, these assumptions can be relaxed. Agents may invest in the distribution of their own types ex ante, and further investments are conducted given the realization of types. This incomplete information structure will be a generalization of Piccione and Tan (1996) and other papers on information acquisition (Bergemann and Välimäki, 2002; Obara 2008). Under the incomplete information setting, we hope to obtain conditions on mechanisms or cost structures which make unique implementation of investment efficiency possible.

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