Serial Dictatorship with Infinitely Many Agents*

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Abstract. We extend an existing impossibility theorem to the environment where there are infinitely many agents. When the number of agents is infinite, it is impossible to identify a dictator whose preference dictates the outcome and it is proved that there is an "invisible" dictator (Fishburn (1970) and Kirman and Sondermann (1972)). We extend the result of invisible dictators in a domain of weak preference profiles and formulate a serial dictatorship by using a hierarchy of ultrafilters. An immediate consequence of this characterization is the existence of an individual dictator in the case of a finite number of agents and it gives a concise proof of an existing impossibility theorem with serial dictatorship. At the same time, the same characterization shows that effectively serial dictatorship persists also in the infinite case.
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1 Introduction

In this paper, we establish a serial dictatorship with infinitely many voters and more than three alternatives. When the number of agents is infinite, it is impossible to identify a dictator whose preference dictates the outcome and it is proved that there is an "invisible" dictator (Fishburn (1970) and Kirman and Sondermann (1972)). We extend the result of invisible dictators in a domain of weak preference profiles and formulate a serial dictatorship by using a hierarchy of ultrafilters. An immediate consequence of this characterization is the existence of an individual dictator in the case of a finite number of agents and it gives a concise proof of an existing impossibility theorem with serial dictatorship. At the same time, the same characterization shows that effectively serial dictatorship persists also in the infinite case.

Within the framework of social choice, recently Man and Takayama (2013) have proposed the independence and stability axioms together with unanimity and showed that many well-known impossibility theorems follow

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their main theorem as corollaries when the number of voters is finite. We extend the analysis by allowing the number of voters to be infinite, and on the other hand we extend the analysis in Kirman and Sondermann (1972) by allowing voters to have weak preferences. Our main theorem shows that a social choice correspondence satisfying the three axioms is characterized by a serial dictatorship.

This paper adds to the literature by showing a structure of serial dictatorship when the number of voters can be infinite. Our analysis is not confined to the number of voters being infinite. Our theorem also includes the case of finitely many voters. In this sense, we extend the previous research of dictatorship to a more general environment. It combines two strands of Kirman and Sondermann (1972) and Man and Takayama (2013) in social choice theory. The organization of the paper is as follows. Section 2 presents the model. Section 3 presents the main theorem. Section 4 proves the existence of invisible serial dictators.

2 The Model

Let $\mathcal{X}$ be the set of potential alternatives. We assume that $|\mathcal{X}| \geq 3$ and it is finite. Let $\mathcal{N}$ be the set of potential agents. We assume that $\mathcal{N}$ is infinite. Let $\mathcal{R}$ be the entire space of weak preferences over $\mathcal{X}$. Let $\succsim_i \in \mathcal{R}$ be agent i's preference, and, for each $N \subset \mathcal{N}$, let $\succsim \in \mathcal{R}^N$ be a preference profile of all the agents in N. For each $x, y \in \mathcal{X}$, we say that $x \succ_i y$ if x is *strictly preferred* to y, i.e., $x \succsim_i y$ but not $y \succsim_i x$, and say that $x \sim_i y$ if x is *indifferent* to y, i.e., $x \succsim_i y$ and $y \succsim_i x$.

An *economy* is a list of $\{X, N, \succsim\} \in 2^{\mathcal{X}} \times 2^{\mathcal{N}} \times \mathcal{R}^N$. Given population N, A solution ϕ_N is a correspondence from economies to alternatives such that

$$\begin{aligned} \phi_N : 2^{\mathcal{X}} \backslash \{\varnothing\} \times \mathcal{R}^N &\rightrightarrows \mathcal{X}, \\ \text{s.t. } \phi_N(X, \succsim) &\subset X, \\ \phi_N(X, \succsim) &\neq \varnothing. \end{aligned}$$

We omit the subscript of a solution ϕ throughout the paper when we take the entire population $\mathcal{N}$ for ϕ. Next we define three properties which we want solutions to satisfy.

Definition 2.1. ***Strong Unanimity*** *(St. Unanimity, hereafter)*
For each $X \in 2^{\mathcal{X}} \backslash \{\varnothing\}$, $\succsim \in \mathcal{R}^N$, and $x \in X$, if, for each $i \in N$ and $y \in X$, $x \succsim_i y$, and there exists $j \in N$ s.t. $x \succ_j y$, then $\phi_N(X, \succsim) = \{x\}$.

We say that $\succsim$ and $\succsim' \in \mathcal{R}^N$ *agree* on X if $\succsim$ and $\succsim'$ are the same on X. Then we say that $\succsim =_X \succsim'$.

Definition 2.2. ***Independence of Irrelevant Alternatives*** *(IA-Independence)*
For each $X \in 2^{\mathcal{X}} \backslash \{\varnothing\}$ and each $\succsim, \succsim' \in \mathcal{R}^N$, if $\succsim =_X \succsim'$, then $\phi_N(X, \succsim) = \phi_N(X, \succsim')$.

Definition 2.3. ***Stability with Losing Alternatives*** *(LA-Stability)*
For each $X, Y \in 2^{\mathcal{X}} \backslash \{\varnothing\}$ and each $\succsim \in \mathcal{R}^N$, if $X \subset Y$ and $\phi_N(Y, \succsim) \cap X \neq \varnothing$, then $\phi(X, \succsim) = \phi(Y, \succsim) \cap X$.

3 Dictatorial ultrafilter

In this section, we show that dictatorship holds when solutions satisfy the three properties. However such dictatorship is not necessarily governed by a single dictator, but sometimes by a "invisible" dictator (Kirman and Sondermann (1972)). First we show that each solution ϕ_N can be represented as if they are chosen by a social preference over X. For each $R \in \mathcal{R}$, let $Top(X, R) \equiv \{x \in X \mid \text{for each } y \in X,\ x\ R\ y\}$.

Lemma 3.1. *If a solution ϕ_N satisfies St. Unaminity, IA-Independence and LA-Stability, then, for each $\succsim \in \mathcal{R}^N$, there exists $R \in \mathcal{R}$ such that, for each $X \subset \mathcal{X}$ with $|X| \geq 2$, $\phi_N(X, \succsim) = Top(X, R)$.*

Proof. Let $\succsim \in \mathcal{R}^N$. We define a binary relationship R on X so that $x\ R\ y$ if $x \in \phi_N(\{x,y\}, \succsim)$. Then, since $\phi_N(\{x,y\}, \succsim) \neq \varnothing$, R is complete.

Next we show transitivity. Suppose that $x, y, z \in \mathcal{X}$ satisfy that $x\ R\ y$ and $y\ R\ z$. Then $x \in \phi_N(\{x,y\}, \succsim)$ and $y \in \phi_N(\{y,z\}, \succsim)$. If $x \notin \phi_N(\{x,z\}, \succsim)$, then $x \notin \phi_N(\{x,y,z\}, \succsim)$ by LA-Stability. It means that $y \notin \phi_N(\{x,y,z\}, \succsim)$. Otherwise $\phi_N(\{x,y\}, \succsim) = \{y\}$ by LA-Stability. This leads to that $\phi_N(\{x,y,z\}, \succsim) = \{z\}$. But, by LA-Stability, this implies that $y \notin \phi_N(\{y,z\}, \succsim)$. It is a contradiction. Therefore $x \in \phi_N(\{x,z\}, \succsim)$, i.e., $x\ R\ z$. Now R is a complete transitive relationship, that is, a weak preference.

Finally we show that ϕ_N is the top set of R. Let $X \subset \mathcal{X}$ with $|X| \geq 2$. First, suppose that there exists $x \in \phi_N(X, \succsim)$ but $x \notin Top(X, R)$. Then there exists $y \in X$ such that $y \in \phi_N(\{x,y\}, \succsim)$ but $x \notin \phi_N(\{x,y\}, \succsim)$. However, since $\{x,y\} \subset X$, $x \in \phi_N(\{x,z\}, \succsim)$ by LA-Stability. It is a contradiction. Thus $\phi_N(X, \succsim) \subset Top(X, R)$. Next suppose that $x \in Top(X, R)$. Then, for each $y \in X$, $x\ R\ y$. It means that for each $y \in X$, $x \in \phi_N(\{x,y\}, \succsim)$. By LA-stability, it implies that $x \in \phi_N(X, \succsim)$. As a result, $Top(X, R) \subset \phi_N(X, \succsim)$. $\square$

Here, we define an *ultrafilter* $\mathscr{F}$ on population N.

Definition 3.2. *A family of sets $\mathscr{F}$ is an ultrafilter of $N \subset \mathcal{N}$ if*

(1) $\varnothing \notin \mathscr{F}$.

(2) If $S \in \mathscr{F}$, and $S' \supset S$, then $S' \in \mathscr{F}$.

(3) If $S, S' \in \mathscr{F}$, then $S \cap S' \in \mathscr{F}$.

(4) If $S \in N$, then either $S \in \mathscr{F}$ or $N \backslash S \in \mathscr{F}$.

For each $\succsim\ \mathcal{R}^N$, we denote the weak preference established in Lemma 3.1 as $R^N(\succsim)$. We say that $x\ P^N(\succsim)\ y$ if x is strictly preferred to y under $R^N(\succsim)$, and say that $x\ I^N(\succsim)\ y$ if x is indifferent to y. Now $R^N(\cdot)$ is a mapping from $\mathcal{R}^N$ to $\mathcal{R}$, so we can consider $R^N(\cdot)$ as a *social welfare function*. Therefore we can apply Arrow's impossibility theorem, especially the one extended to infinitely many agents by Kirman and Sondermann (1972).

For each $U \subset N$ and each $x, y \in \mathcal{X}$, we denote $x \succsim_U y$ if, for each $i \in U$, $x \succsim_i y$.

Theorem 3.3. *If a solution ϕ_N satisfies St.Unanimity, IA-Independence, and LA-Stability, then there exists an ultrafilter $\mathcal{U}$ on N s.t., for each $\succsim \in \mathcal{R}^N$, if there exists $U \in \mathcal{U}$ such that, for each $x, y \in \mathcal{X}$, $x \succ_U y$, then $x \; P^N(\succsim) \; y$.*

This idea originally comes from Kirman and Sondermann (1972). However they assumed that preferences are strict, so we have to extend their result so that we can incorporate weak preferences. We show this theorem through the lemmas below. Before going to the proofs, we define some families of sets of agents. In the following arguments, we assume that ϕ_N satisfies St.Unanimity, IA-Independence, and LA-Stability. And, for each $U \subset N$, let $U' \equiv N \backslash U$, and, for each $\tilde{U} \subset U'$, let $U'' \equiv U' \backslash \tilde{U}$.

The first family is;

$$\mathcal{U} \equiv \{U \subset N \mid \forall x, y \in \mathcal{X}, \forall \succsim \in \mathcal{R}^N, \; x \succ_U y \; \wedge \; y \succsim_{U'} x \Rightarrow x \; P^N(\succsim) \; y.\},$$

We can interpret each member of $\mathcal{U}$ as a "coalition of dictators". Let the other two families of sets be:

$$\begin{aligned}
\mathcal{U}' &\equiv \{U \subset N \mid \exists x, y \in \mathcal{X} \text{ with } x \neq y, \forall \succsim \in \mathcal{R}^N, \; x \succ_U y \; \wedge \; y \succsim_{U'} x \Rightarrow x \; P^N(\succsim) \; y\}, \\
\mathcal{U}'' &\equiv \Big\{U \subset N \mid \exists x, y \in \mathcal{X} \text{ with } x \neq y, \\
&\qquad \forall \tilde{U} \subset U', \; \exists \succsim \in \mathcal{R}^N, \; x \succ_U y \; \wedge \; y \sim_{\tilde{U}} x \; \wedge \; y \succ_{U''} x \; \wedge \; x \; P^N(\succsim) \; y \Big\}.
\end{aligned}$$

All we have to show is that $\mathcal{U}$ is an ultrafilter on N. For this purpose, we show that $\mathcal{U} = \mathcal{U}''$ and $\mathcal{U}''$ is an ultrafilter.

We show that these three families are the same. First note that $\mathcal{U} \subset \mathcal{U}' \subset \mathcal{U}''$ by definition. We show the inverse inclusion by the following two lemmas.

Lemma 3.4. *$\mathcal{U}'' \subset \mathcal{U}'$.*

Proof. Let $U \in \mathcal{U}''$ and $U' \equiv N \backslash U$. Let $\succsim \in \mathcal{R}^N$. Suppose that $x \succ_U y$ and $y \succsim_{U'} x$. Now let $\tilde{U} \equiv \{i \in U' \mid x \sim_i y\}$, and let $U'' \equiv U' \backslash \tilde{U}$. Then, since $U \in \mathcal{U}''$, there exists $x, y \in \mathcal{X}$ and $\succsim^* \in \mathcal{R}^N$ such that $x \succ^*_U y$, $y \sim^*_{\tilde{U}} x$, $y \succ^*_{U''} x$, and $x P^N(\succsim^*) y$. It is clear that $\succsim^* =_{\{x,y\}} \succsim$. Thus, by IA-independence, $\phi_N(\{x,y\}, \succsim) = \phi_N(\{x,y\}, \succsim^*)$. Since $x \; P^N(\succsim^*) \; y$, $\phi_N(\{x,y\}, \succsim) = \phi_N(\{x,y\}, \succsim^*) = \{x\}$. It means that $x \; P^N(\succsim) \; y$. Thus $U \in \mathcal{U}'$. $\square$

Lemma 3.5. *$\mathcal{U}' \subset \mathcal{U}$.*

Proof. Let $U \in \mathcal{U}'$, and let $U' \equiv N \backslash U$. Then there exists $x, y \in \mathcal{X}$ with $x \neq y$ such that

$$\forall \succsim \in \mathcal{R}^N, \; x \succ_U y \; \wedge \; y \succsim_{U'} x \Rightarrow x \; P^N(\succsim) \; y.$$

Let $z \in \mathcal{X} \backslash \{x, y\}$ and $\succsim \in \mathcal{R}^N$. Suppose that $z \succ_U y$ and $y \succsim_{U'} z$. Now we want to show that $z \; P^N(\succsim) \; y$. Let $\succsim' \in \mathcal{R}^N$ such that $z \succ'_U x \succ'_U y$, $y \succsim'_{U'} z \succ'_{U'} x$, and, for each $i \in U'$, $\succsim_i =_{\{y,z\}} \succsim'_i$. Then, by St. Unanimity, $\phi_N(\{x, z\}, \succsim') = \{z\}$. It means that $z \; P^N(\succsim') \; x$. Now $x \succ'_U y$ and $y \succ'_{U'} x$. Since $U \in \mathcal{U}'$, we have $x\ P^N(\succsim')\ y$. Since $P^N(\succsim') \in \mathcal{R}^N$, we have $z\ P^N(\succsim')\ y$ by transitivity. Since $\succsim =_{\{y,z\}} \succsim'$, we have that $\phi_N(\{y,z\},\succsim) = \phi_N(\{y,z\},\succsim')$. Therefore $z\ P^N(\succsim)\ y$.

Let $w \in \mathcal{X} \backslash \{x,y,z\}$. Now, for y,z, it holds that, for each $\succsim \in \mathcal{R}^N$, if $z \succ_U y$ and $y \succsim_{U'} z$, then $z\ P^N(\succsim)\ y$. Thus, by applying the same argument as above, we have that, for each $\succsim \in \mathcal{R}^N$, if $z \succ_U w$ and $w \succsim_{U'} z$, then $z\ P^N(\succsim)\ w$. It implies that $U \in \mathcal{U}$. $\square$

Corollary 3.6. $\mathcal{U} = \mathcal{U}' = \mathcal{U}''$

Finally we show that $\mathcal{U}$ is an ultrafilter.

Proposition 3.7. *$\mathcal{U}$ is an ultrafilter.*

Proof. By Corollary 3.6, we have show that $\mathcal{U}''$ is an ultrafilter. First, we show that $\varnothing \notin \mathcal{U}''$. If not so, by takeing $U = \varnothing$ and $\tilde{U} \neq N$, there exist $x,y \in \mathcal{X}$ and $\succsim \in \mathcal{R}^N$ such that $y \sim_{\tilde{U}} x$, $y \succ_{N\backslash\tilde{U}} x$, and $x\ P^N(\succsim)\ y$. It contradicts St. Unanimity.

Next we show that $\mathcal{U}''$ is closed under finite intersection. Let $W_1, W_2 \in \mathcal{U}''$. We separate N into four disjoint subsets:

$$
\begin{aligned}
&V_1 \equiv W_1 \cap W_2,\\
&V_2 \equiv W_1 \backslash V_1,\\
&V_3 \equiv W_2 \backslash V_1,\\
&V_4 \equiv N \backslash (W_1 \cup W_2).
\end{aligned}
$$

Let $\{a,b,c\} \subset \mathcal{X}$ and $\tilde{V}_1 \subset N\backslash V_1$. Then we can find $\succsim^* \in \mathcal{R}^N$ such that

$$
\begin{aligned}
&c \succ^*_{V_1} a \succ^*_{V_1} b,\ a \succ^*_{V_2} b \succsim^*_{V_2} c,\\
&b \succsim^*_{V_3} c \succ^*_{V_3} a,\ b \succsim^*_{V_4} a \succsim^*_{V_4} c,\\
&b \sim^*_{\tilde{V}_1} c, \text{ and } b \succ^*_{N\backslash(V\cup\tilde{V}_1)} c.
\end{aligned}
$$

Note that $a \succ^*_{W_1} b$, and $b \succsim^*_{N\backslash W_1} a$. By Corollary 3.6, we have $W_1 \in \mathcal{U}'' = \mathcal{U}$. Therefore $a\ P^N(\succsim^*)\ b$. By the same way, $W_2 \in \mathcal{U}$, $c \succ^*_{W_2} a$, and $a \succsim^*_{N\backslash W_2} c$. It implies that $c\ P^N(\succsim^*)\ a$. By transitivity, we have $c\ P^N(\succsim^*)\ b$. Now we also have that $c \succ^*_{V_1} b$, $b \sim^*_{\tilde{V}_1} c$, and $b \succ^*_{N\backslash(V\cup\tilde{V}_1)} c$. Thus, by the definition of $\mathcal{U}''$, $V_1 = W_1 \cap W_2 \in \mathcal{U}''$.

Next we show that, for each $V \subset N$, $V \in \mathcal{U}''$ or $N\backslash V \in \mathcal{U}''$. If $V \in \mathcal{U}''$, then it is immediately satisfied. So we suppose that $V \notin \mathcal{U}''$. Then, for each $\{a,b,c\} \subset \mathcal{X}$, there exists $\tilde{V} \subset N\backslash V$ such that, for each $\succsim \in \mathcal{R}^N$, if $b \succ_V a$, $b \sim_{\tilde{V}} a$, and $a \succ_{N\backslash(V\cup\tilde{V})} b$, then $a\ R^N(\succsim)\ b$. Now let $\hat{V} \subset V$. Then we can find $\tilde{\succsim} \in \mathcal{R}^N$ such that

$$
\begin{aligned}
&b \mathrel{\tilde{\succ}}_{\hat{V}} c \mathrel{\tilde{\sim}}_{\hat{V}} a,\ b \mathrel{\tilde{\succ}}_{V\backslash\hat{V}} c \mathrel{\tilde{\succ}}_{V\backslash\hat{V}} a,\\
&a \mathrel{\tilde{\sim}}_{\tilde{V}} b \mathrel{\tilde{\succ}}_{\tilde{V}} c, \text{ and } a \mathrel{\tilde{\succ}}_{N\backslash(V\cup\tilde{V})} b \mathrel{\tilde{\succ}}_{N\backslash(V\cup\tilde{V})} c.
\end{aligned}
$$

It is clear that, by St. Unanimity, $b\ P^N(\tilde{\succsim})\ c$. Note that $b \tilde{\succ}_V a$, $a \tilde{\sim}_{\hat{V}} b$, and $a \tilde{\succ}_{N\backslash(V\cup\hat{V})} b$. By the hypothesis, we have that $a\ R^N(\tilde{\succsim})\ b$. Thus $a\ P^N(\tilde{\succsim})\ c$. At the same time, we have that $a \tilde{\succ}_{N\backslash V} b$, $c \tilde{\sim}_{\hat{V}} a$, and $c \tilde{\succ}_{V\backslash\hat{V}} a$. Since $\hat{V}$ can be an arbitrary subset of V, by the definition of $\mathcal{U}''$, we have that $N\backslash V \in \mathcal{U}''$.

Finally we show that, for each $U \in \mathcal{U}''$, if $W \supset U$, then $W \in \mathcal{U}''$. Suppose not. Then there exist $W, U \subset N$ with $W \supset U$ such that $U \in \mathcal{U}''$ but $W \notin \mathcal{U}''$. Then, by the above argument, $N\backslash W \in \mathcal{U}''$. Since $\mathcal{U}''$ is closed under intersection, we have that $U \cap (N\backslash W) \in \mathcal{U}''$. However $U \cap (N\backslash W) = \varnothing$. It is a contradiction. $\square$

Given a population N, we denote the decisive ultrafilter above as $\mathcal{U}^N$.

4 Characterization with serially dictatorial coalitions

Based on the dictatorial ultrafilter, in this section, we show that ϕ is chosen as if groups of agents serially choose their desirable alternatives. First we slightly extend our definition $Top(X, \succsim)$:

Definition 4.1. *For each $U \subset \mathcal{N}$, $X \subset \mathcal{X}$, and $\succsim \in \mathcal{R}^N$, let*

$$Top_{(U)}(X, \succsim) \equiv \{x \in X \mid \forall i \in U,\ \forall y \in X,\ x \succsim_i y\}.$$

Lemma 4.2. *Let $N \subset \mathcal{N}$. Suppose that ϕ_N satisfies St.Unanimity, IA-Independence, and LA-Stability. Then, for each $\succsim \in \mathcal{R}$, there exists $U^* \in \mathcal{U}^N$ such that (i) for each $i, i' \in U^*$ and $j \in N\backslash U^*$, it holds that $\succsim_i = \succsim_{i'}$ and $\succsim_i \neq \succsim_j$, and (ii) for each $X \subset \mathcal{X}$ with $X \neq \varnothing$, it holds that $\phi_N(X, \succsim) \subset Top_{(U^*)}(X, \succsim)$.*

Proof. Let $N \subset \mathcal{N}$ and $\succsim \in \mathcal{R}$. Since $|\mathcal{X}| < \infty$, we can separate N into finite numbers of equivalent classes with respect to the agents' preferences over $\mathcal{X}$. Let $\{U_k\}_{k\in\{1,\cdots,K\}}$ be such a partition, i.e., for each $k, l \in \{1, \cdots, K\}$, $i, i' \in U_k$, and $j \in U_l$, we have that $\succsim_i = \succsim_{i'}$ and $\succsim_i \neq \succsim_j$. Now $\mathcal{U}^N$ is ultrafilter. Thus there exists a unique $k^* \in \{1, \cdots, K\}$ such that $U_{k^*} \in \mathcal{U}^N$. [1]

Next we show that, for each $X \subset \mathcal{X}$, $\phi_N(X, \succsim) \subset Top_{(U_{k^*})}(X, \succsim)$. When $|X| = 1$, it is clear. So we assume that $|X| \geq 2$. Suppose that there exists $X \subset \mathcal{X}$ such that $\phi_N(X, \succsim) \not\subset Top_{(U_{k^*})}(X, \succsim)$. Then, there exists $x \in \phi_N(X, \succsim)$ such that $x \notin Top_{(U_{k^*})}(X, \succsim)$. Then there exists $y \in Top_{(U_{k^*})}(X, \succsim)$ such that $y \succ_{U_{k^*}} x$. It means that $U_{k^*} \subset \{i \in N \mid y \succ_i x\}$. Now $U_{k^*} \in \mathcal{U}^N$. Thus, $\{i \in N \mid y \succ_i x\} \in \mathcal{U}^N$. This implies that $y\ P^N(\succsim)\ x$. However $x \in \phi_N(X, \succsim)$, and so, by LA-Stability, we have that $x \in \phi_N(\{x, y\}, \succsim)$. This implies that $xR(\succsim)y$. It is a contradiction. Thus $\phi_N(X, \succsim) \subset Top_{(U_{k^*})}(X, \succsim)$. $\square$

Now the following two lemmas show that when a solution satisfies St.Unanimity, IA-Independence, and LA-Stability, there exists a serially dictatorial groups of agents with some tie-breaking rule.

[1] For detailed arguments on mathematical properties of ultrafilters, see Aliprantis and Border (2006) etc.

Let $\{U_k\}_{k\in\{1,\cdots,K\}}$ be a finite partition on $\mathcal{N}$. We define $\{T^*_k(X,\succsim)\}_{k\in\{0,\cdots,K\}}$ recursively;

$$\begin{aligned}
&T^*_0(X,\succsim)\equiv X,\\
&T^*_1(X,\succsim)\equiv Top_{(U_1)}(X,\succsim),\\
\text{for } \forall\, k\in\{2,\cdots,K\},\quad &T^*_k(X,\succsim)\equiv Top_{(U_k)}(T^*_{k-1}(X,\succsim),\succsim).
\end{aligned}$$

Lemma 4.3. *For each $X\subset\mathcal{X}$ with $X\neq\varnothing$ and $\succsim\in\mathcal{R}^{\mathcal{N}}$, there exists a finite partition $\mathscr{P}\equiv\{U_k\}_{k\in\{1,\cdots,K\}}$ on $\mathcal{N}$ such that $\phi(X,\succsim)\subset T^*_K(X,\succsim)$.*

Proof. Let $X\subset\mathcal{X}$ with $X\neq\varnothing$ and $\succsim\in\mathcal{R}^{\mathcal{N}}$. Let $\mathscr{P}$ be a finite partition on $\mathcal{N}$ generated by the equivalent classes regarding agents' preferences over $\mathcal{X}$. Let $|\mathscr{P}|\equiv K$. As shown in the proof of Lemma 4.2, there exists a unique $U\in\mathscr{P}$ such that $U\in\mathcal{U}^{\mathcal{N}}$ and $\phi(X,\succsim)\subset Top_{(U)}(X,\succsim)$. Name this set U_1 and this ultrafilter $\mathcal{U}_1$. Now we define a new solution for smaller population $\mathcal{N}\backslash U_1$. For each $Y\subset\mathcal{X}$ and $\tilde{\succsim}\in\mathcal{R}^{\mathcal{N}\backslash U_1}$, let $\psi^1_{\mathcal{N}\backslash U_1}(Y,\tilde{\succsim})\equiv\phi(Y,\hat{\succsim})$, where, for each $i\in\mathcal{N}\backslash U_1$, $\hat{\succsim}_i=\tilde{\succsim}_i$, and, for each $j\in U_1$, his preference is indifferent for all the alternatives, i.e., for each $x,y\in\mathcal{X}$, $x\hat{\sim}_j y$. We show that $\psi^1_{\mathcal{N}\backslash U_1}$ satisfies St.Unanimity, IA-Independence, and LA-Stability.

First, we show St.Unanimity. Let $x\in X$ and $\tilde{\succsim}\in\mathcal{R}^{\mathcal{N}\backslash U_1}$. Suppose that, for each $i\in\mathcal{N}\backslash U_1$ and $y\in X$, $x\tilde{\succsim}_i y$, and there exists $j\in\mathcal{N}\backslash U_1$ such that $x\tilde{\succ}_i y$. Then, by the construction of $\hat{\succsim}$, we also have that, for each $i\in\mathcal{N}$ and $y\in X$, $x\hat{\succsim}_i y$, and $x\hat{\succ}_j y$. Therefore, by St.Unanimity of ϕ, we have that $\psi^1_{\mathcal{N}\backslash U_1}(X,\tilde{\succsim})=\phi(X,\hat{\succsim})=\{x\}$. Second, we show that IA-Independence holds. Let $Y\subset\mathcal{X}$, and $\tilde{\succsim},\tilde{\succsim}'\in\mathcal{R}^{\mathcal{N}\backslash U_1}$ be such that $\tilde{\succsim}=_Y\tilde{\succsim}'$. Then, by construction, we also have that $\hat{\succsim}=_Y\hat{\succsim}'$. By IA-Independence of ϕ, we have that $\phi(Y,\hat{\succsim})=\phi(Y,\hat{\succsim}')$. It implies that $\psi^1_{\mathcal{N}\backslash U_1}(Y,\tilde{\succsim})=\psi^1_{\mathcal{N}\backslash U_1}(Y,\tilde{\succsim}')$. Finally, we show that LA-Stability holds. Let $Y\subset X$ and $\tilde{\succsim}\in\mathcal{R}^{\mathcal{N}\backslash U_1}$. By LA-Stability of ϕ, if the intersection is non-empty, we have that $\phi(Y,\hat{\succsim})=\phi(X,\hat{\succsim})\cap Y$. This implies that, if the intersection is non-empty, we have that $\psi^1_{\mathcal{N}\backslash U_1}(Y,\tilde{\succsim})=\psi^1_{\mathcal{N}\backslash U_1}(X,\tilde{\succsim})\cap Y$.

Now $\psi^1_{\mathcal{N}\backslash U_1}$ satisfies St. Unanimity, IA-Independence, and LA-Stability. Let $\tilde{\succsim}\in\mathcal{R}^{\mathcal{N}\backslash U_1}$ be such that, for each $i\in\mathcal{N}\backslash U_1$, $\tilde{\succsim}_i=\succsim_i$ Then, by Lemma 4.2, there exists $U^*\subset\mathcal{N}\backslash U_1$ such that $\psi^1_{\mathcal{N}\backslash U_1}(Top_{(U_1)}(X,\succsim),\tilde{\succsim})\subset Top_{(U^*)}(Top_{(U_1)}(X,\succsim),\tilde{\succsim})$. As shown in the proof of 4.2, we have that $U^*\in\mathcal{U}^{\mathcal{N}\backslash U_1}$ and it is unique. We name this set U_2 and this ultrafilter $\mathcal{U}_2$.

Now we know that $\phi(X,\succsim)\subset Top_{(U_1)}(X,\succsim)$. By LA-Stability, we have that

$$\begin{aligned}
\phi(Top_{(U_1)}(X,\succsim),\succsim)&=\phi(X,\succsim)\cap Top_{(U_1)}(X,\succsim)\\
&=\phi(X,\succsim).
\end{aligned}$$

On the other hand, we also have that $\psi^1_{\mathcal{N}\backslash U_1}(Top_{(U_1)}(X,\succsim),\tilde{\succsim})=\phi(Top_{(U_1)}(X,\succsim),\hat{\succsim})$. Note that, for each $i\in U_1$ and $x,y\in Top_{(U_1)}(X,\succsim)$, $x\sim_i y$. Thus, by construction of $\hat{\succsim}$, $\succsim=_{Top_{(U_1)}(X,\succsim)}\hat{\succsim}$. By IA-

Independence of ϕ, $\phi(Top_{(U_1)}(X,\succsim),\hat{\succsim}) = \phi(Top_{(U_1)}(X,\succsim),\succsim)$. Therefore

$$\begin{aligned}\phi(X,\succsim) &= \phi(Top_{(U_1)}(X,\succsim),\succsim)\\ &= \phi(Top_{(U_1)}(X,\succsim),\hat{\succsim})\\ &= \psi^1_{\mathcal{N}\setminus U_1}(Top_{(U_1)}(X,\succsim),\tilde{\succsim})\\ &\subset Top_{(U_2)}(Top_{(U_1)}(X,\succsim),\tilde{\succsim})\\ &= Top_{(U_2)}(Top_{(U_1)}(X,\succsim),\succsim).\end{aligned}$$

We can repeat this process until we number all the elements in $\mathscr{P}$. $\square$

We say that the partition $\{U_k\}_{k\in\{1,\cdots,K\}}$ above is the *ordered preference partition* on $\mathcal{N}$ w.r.t. $\succsim\in\mathcal{R}^{\mathcal{N}}$, and $\{\mathcal{U}_k\}_{k\in\{1,\cdots,K\}}$ is the *ordered preference hierarchy of ultrafilters* w.r.t. $\succsim\in\mathcal{R}^{\mathcal{N}}$.

Next we show the existence of a tie-breaking rule;

Lemma 4.4. *There exists $\rho\in\mathcal{R}$, such that for each $X\subset\mathcal{X}$ with $X\neq\varnothing$ and $\succsim\in\mathcal{R}^{\mathcal{N}}$, $\phi(X,\succsim) = Top(T^*_K(X,\succsim),\rho)$.*

Proof. Let ρ be a relationship on $\mathcal{X}$ such that, for each $x,y\in\mathcal{X}$, $x\ \rho\ y$ iff there exist $\succsim\in\mathcal{R}^{\mathcal{N}}$ such that $\{x,y\}\subset T^*_K(\mathcal{X},\succsim)$ and $x\in\phi(\mathcal{X},\succsim)$. First we show that ρ is a weak preference.

Let $x,y\in\mathcal{X}$. Then we can find $\tilde{\succsim}\in\mathcal{R}^{\mathcal{N}}$ be such that, for each $i\in\mathcal{N}$ and $z\in\mathcal{X}\setminus\{x,y\}$, $x\tilde{\sim}_i y$ and $x\tilde{\succ}_i z$. Then $T^*_K(\mathcal{X},\tilde{\succsim}) = \{x,y\}$. Since $\phi(\mathcal{X},\tilde{\succsim})\subset T^*_K(\mathcal{X},\tilde{\succsim})$ and $\phi(\mathcal{X},\tilde{\succsim})\neq\varnothing$, we have $x\in\phi(\mathcal{X},\tilde{\succsim})$ or $y\in\phi(\mathcal{X},\tilde{\succsim})$. That is, $x\ \rho\ y$ or $y\ \rho\ x$. This means that ρ is complete.

Next we show transitivity. Let $x,y,z\in\mathcal{X}$ be such that $x\ \rho\ y$ and $y\ \rho\ z$. Then there exists $\succsim\in\mathcal{R}^{\mathcal{N}}$ such that $\{x,y\}\subset T^*_K(\mathcal{X},\succsim)$ and $x\in\phi(\mathcal{X},\succsim)$. By the same way, there also exists $\succsim'\in\mathcal{R}^{\mathcal{N}}$ such that $\{y,z\}\subset T^*_K(\mathcal{X},\succsim')$ and $y\in\phi(\mathcal{X},\succsim')$. Then, by LA-Stability, $x\in\phi(\{x,y\},\succsim)$ and $y\in\phi(\{y,z\},\succsim')$. Now let $\succsim^*\in\mathcal{R}^{\mathcal{N}}$ be such that, for each $i\in\mathcal{N}$ and $w\in\mathcal{X}\setminus\{x,y,z\}$, $x\sim^* y\sim^*_i z$ and $z\succ^*_i w$. Then $T^*_K(\mathcal{X},\succsim^*) = \{x,y,z\}$. Therefore all we have to show is that $x\in\phi(\mathcal{X},\succsim^*)$. Suppose not. Note that $\succsim^* =_{\{y,z\}} \succsim'$. Thus $y\in\phi(\{y,z\},\succsim') = \phi(\{y,z\},\succsim^*)$. By LA-Stability, $y\in\phi(\{x,y,z\},\succsim^*)$. By-LA-Stability again, we have that $\phi(\{x,y\},\succsim^*) = \phi(\{x,y,z\},\succsim^*)\cap\{x,y\} = \{y\}$. However, since $\succsim^* =_{\{x,y\}}\succsim$, we have that $\phi(\{x,y\},\succsim) = \phi(\{x,y\},\succsim^*)$. Therefore $x\in\phi(\{x,y\},\succsim) = \phi(\{x,y\},\succsim^*)$. It is a contradiction. Thus ρ is a weak preference over $\mathcal{X}$.

Finally we show that, for each $X\subset\mathcal{X}$ with $X\neq\varnothing$ and $\succsim\in\mathcal{R}^{\mathcal{N}}$, $\phi(X,\succsim) = Top(T^*_K(X,\succsim),\rho)$. First suppose that there exist $X\subset\mathcal{X}$ with $X\neq\varnothing$ and $\succsim\in\mathcal{R}^{\mathcal{N}}$ such that $\phi(X,\succsim)\nsubseteq Top(T^*_K(X,\succsim),\rho)$. Then there exists $x\in\phi(X,\succsim)$ such that $x\notin Top(T^*_K(X,\succsim),\rho)$. Let $y\in Top(T^*_K(X,\succsim),\rho)$. Then $\{x,y\}\subset T^*_K(X,\succsim)$ and $x\in\phi(\{x,y\},\succsim)$. Now let $\succsim'\in\mathcal{R}^{\mathcal{N}}$ be such that, for each $i\in\mathcal{N}$ and $z\in\mathcal{X}\setminus\{x,y\}$, $x\sim'_i y$ and $x\succ'_i y$. Then $\succsim =_{\{x,y\}}\succsim'$. It implies that $\phi(\{x,y\},\succsim) = \phi(\{x,y\},\succsim')$. Therefore $x\in\phi(\{x,y\},\succsim')$. Now, since $T_K(\mathcal{X},\succsim') = \{x,y\}$, we have $x\in\phi(\{x,y\},\succsim') = \phi(\mathcal{X},\succsim')$. This implies that $x\ \rho\ y$. Since $y\in Top(T^*_K(X,\succsim),\rho)$, we have $x\in Top(T^*_K(X,\succsim),\rho)$. It contradicts the hypothesis.

Next we suppose that there exists $X\subset\mathcal{X}$ with $X\neq\varnothing$ and $\succsim\in\mathcal{R}^{\mathcal{N}}$ such that $\phi(X,\succsim)\nsupseteq Top(T^*_K(X,\succsim),\rho)$. Then there exists $x\in Top(T^*_K(X,\succsim),\rho)$ such that $x\notin\phi(X,\succsim)$. Let $y\in\phi(X,\succsim)$. Then, LA-Stability,

$\phi(\{x,y\},\succsim) = \{y\}$. Note that $x \; \rho \; y$. Therefore there exists $\succsim' \in \mathcal{R}^{\mathcal{N}}$ such that $\{x,y\} \subset T^*_K(\mathcal{X},\succsim')$ and $x \in \phi(\mathcal{X},\succsim')$. Since $\{x,y\} \subset T^*_K(X,\succsim)$, we have that $\succsim =_{\{x,y\}} \succsim'$. Therefore $\phi(\{x,y\},\succsim') = \phi(\{x,y\},\succsim) = \{y\}$. However, we also have that $x \in \phi(\mathcal{X},\succsim') = \phi(\{x,y\},\succsim')$. It is a contradiction. $\square$

The series of "dictatorial coalitions" shown in the above lemmas varies across the agents' preferences. However we show that there is some order coherent across dictatorial coalitions.

Definition 4.5. *A family $\mathscr{U}$ is a hierarchical family of ultrafilters over $\mathcal{N}$ if $\mathscr{U}$ is minimal in terms of set inclusion among the families of subsets of $2^{\mathcal{N}}$ satisfying that*

$$(1) \textit{ There exists } \mathcal{U} \in \mathscr{U} \textit{ s.t. } \mathcal{U} \textit{ is an ultrafilter over } \mathcal{N},$$
$$(2) \textit{ For each } \mathcal{U} \in \mathscr{U} \textit{ s.t. } \mathcal{U} \textit{ is an ultrafilter over } N \subset \mathcal{N}, \textit{ and for each } U \in \mathcal{U},$$
$$\textit{if } N \backslash U \neq \varnothing, \textit{ there exists } \tilde{\mathcal{U}} \in \mathscr{U} \textit{ s.t. } \tilde{\mathcal{U}} \textit{ is an ultrafilter over } N \backslash U.$$

Definition 4.6. *A finite subfamily $\{\mathcal{U}_k\}_{k=1}^K \subset \mathscr{U}$ is a K-th order hierarchical series of ultrafilters in $\mathscr{U}$ if (i) for each $k \in \{1,\cdots,K\}$, there exists $N_k \subset \mathcal{N}$ s.t. $\mathcal{U}_k$ is an ultrafilter over N_k, (ii) $N_1 = \mathcal{N}$ and, for each $k \geq 2$, there exists $U_{k-1} \in \mathcal{U}_{k-1}$ s.t. $N_k = N_{k-1} \backslash U_{k-1}$, and (iii) $|\mathcal{U}_K| = 1$.*

We say that such a series of sets $\{U_l\}_{k=1}^{K-1}$ above is a *series of coalitions*. Note that each hierarchical family of ultrafilters $\mathscr{U}$ over $\mathcal{N}$ consists of the hierarchical series of ultrafilters originated from a unique ultraliter $\mathcal{U}_1$ over $\mathcal{N}$ as in Figure 4 below.

Hereafter we consider each hierarchical family of ultrafilters as the set of hierarchical series of ultrafilters with the same initial ultrafilter. We say that $\mathscr{U}$ is a K-th order hierarchical family of ultrafilters if the largest order of the hierarchical series of ultrafilters in $\mathscr{U}$ is K. And, for each $\{\mathcal{U}_k\}_{k\in\{1,\cdots,K\}} \in \mathscr{U}$, let $\{U_k\}_{k\in\{1,\cdots,K\}}$ be the series of coalitions associated with $\{\mathcal{U}_k\}_{k\in\{1,\cdots,K\}}$.

Next we define an "order" of a hierarchical family of ultrafilters;

Definition 4.7. *A K-th order hierarchical family of ultrafilters $\mathscr{U}$ over $\mathcal{N}$ satisfies order preservation if, for each $\{\mathcal{U}_k\}_{k\in\{1,\cdots,K\}}$ and $\{\mathcal{U}'_l\}_{l\in\{1,\cdots,L\}} \in \mathscr{U}$, the associated series of coalitions $\{U_k\}_{k=1}^K$ and $\{U'_l\}_{l=1}^L$ satisfy that*

$$\textit{For each } n,m \in \mathbb{N} \textit{ with } 1 \leq n < m \leq K,$$
$$(1) \textit{ if there exists } l_n \in \mathbb{N} \textit{ s.t. } U_n \cap U'_{l_n} \in \mathcal{U}_n \textit{ and } U_n \cap U'_{l_n} \notin \mathcal{U}'_{l_n},$$
$$\textit{then there exists } n' < n \textit{ s.t. } U_{n'} \cap U'_{l_n} \in \mathcal{U}'_{l_n},$$
$$\textit{and } (2) \textit{ if there exist } l_n, l_m \in \mathbb{N} \textit{ with } l_m < l_n \textit{ s.t. } U_m \cap U'_{l_m} \in \mathcal{U}_m \textit{ and } U_n \cap U'_{l_n} \in \mathcal{U}_n,$$
$$\textit{then } U_m \cap U'_{l_m} \notin \mathcal{U}'_{l_m}.$$

The intuition is depicted in Figure 2;

Figure 2: Order preservation

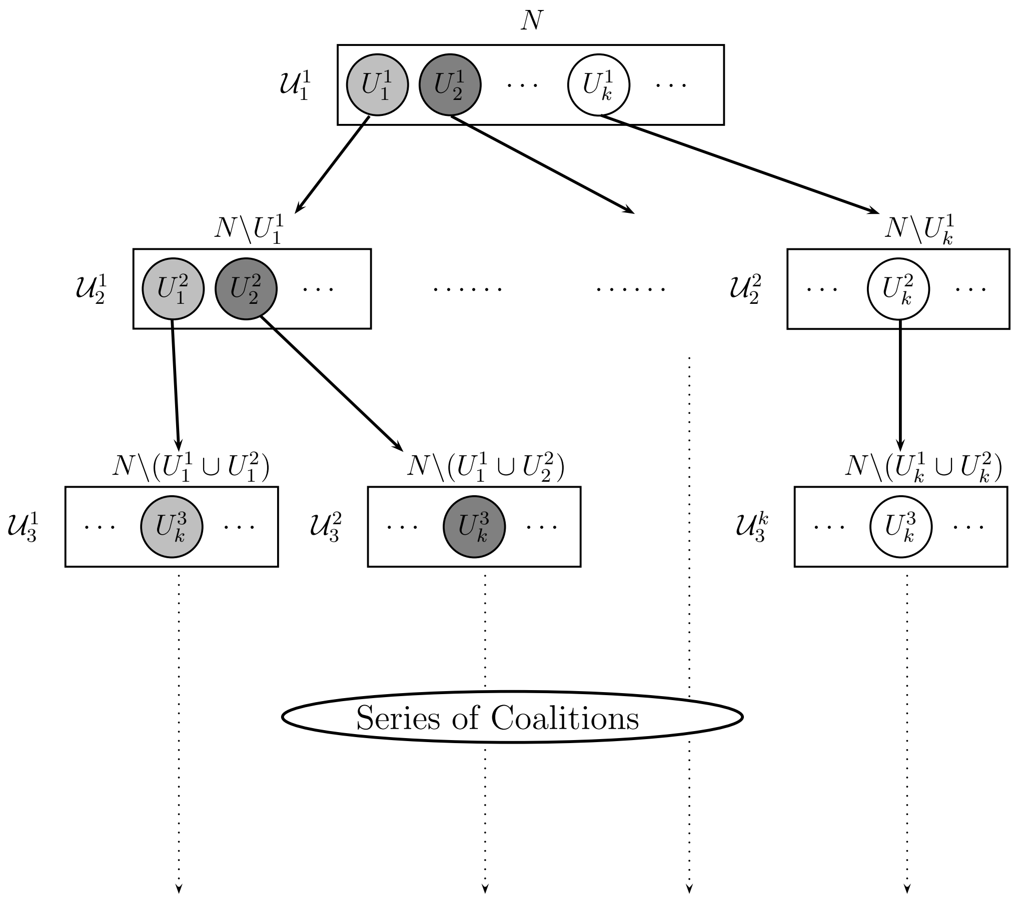

Figure 1: Illustration for A Hierarchical Family of Ultrafilters

Definition 4.8. *(Generalized serial dictatorship): A solution ϕ satisfies generalized serial dictatorship if there exists a tie-breaking rule $\rho \in \mathcal{R}$ and a hierarchical family of ultrafilters $\mathscr{U}$, such that (1) $\mathscr{U}$ satisfies order preservation, for each $X \subset \mathcal{X}$ with $X \neq \varnothing$ and $\succsim \in \mathcal{R}^{\mathcal{N}}$, (2) the ordered preference partition, $\{U_K\}_{l\in\{1,\cdots,K\}}$, is a series of coalition in $\mathscr{U}$, and (3) $\phi(X,\succsim) = Top(T^*_K(X,\succsim),\ \rho)$.*

From Lemma 4.3 and 4.4, we show that the next theorem.

Theorem 4.9. *If ϕ satisfies St.Unanimity, IA-Independence, and LA-Stability, then ϕ satisfies generalized serial dictatorship.*

Proof. Suppose that ϕ satisfy St.Unanimity, IA-Independence, and LA-Stability. Then, as shown in Lemma 4.3 and 4.4, there exists a tie breaking rule $\rho \in \mathcal{R}$, and, for each $\succsim \in \mathcal{R}^{\mathcal{N}}$, the ordered preference partition $\{U_k\}_{k\in\{1,\cdots,K\}}$ and its associated hierarchy of $\{\mathcal{U}_k\}_{k\in\{1,\cdots,K\}}$ satisfy that, for each $X \subset \mathcal{X}$ with $X \neq \varnothing$, $\phi(X,\succsim) = Top(T^*_K(X,\succsim),\ \rho)$.

Let $\mathscr{U}^\phi$ be as follows;

$$\mathscr{U}^\phi \equiv \{\{\mathcal{U}_k\}_{k\in\{1,\cdots,K\}} \mid \exists \succsim \in \mathcal{R}^{\mathcal{N}},\ \{\mathcal{U}_k\}_{k\in\{1,\cdots,K\}} \text{ is an ordered preference partition w.r.t } \succsim\}.$$

Then $\mathscr{U}^\phi$ is a hierarchical family of ultrafilters. We have to show that $\mathscr{U}^\phi$ satisfies order preservation.

Let $\succsim, \succsim' \in \mathcal{R}$, and $\{U_k\}_{k=1}^K$ and $\{U'_l\}_{l=1}^L$ be their ordered preference partitions respectively. Let $\{\mathcal{U}_k\}_{k=1}^K$ and $\{\mathcal{U}'_l\}_{l=1}^L$ be their associated hierarchical series of ultrafilters. Suppose that $\mathscr{U}^\phi$ does not satisfy order preservation. Without loss of generality, we suppose that the condition is violated for $\{U_k\}_{k=1}^K$ at $k = m$. Let $l_m \in \{1,\cdots,L\}$ be such that $U_m \cap U'_{l_m} \in \mathcal{U}_m$. Note that such l_m is unique. By the hypothesis, we have that (1) $U_m \cap U'_{l_m} \notin \mathcal{U}'_{l_m}$ and there exists $n > m$ such that $U_n \cap U'_{l_m} \in \mathcal{U}'_{l_m}$, or (2)$U_m \cap U'_{l_m} \in \mathcal{U}'_{l_m}$ and there exists $\tilde{n} < m$ and $l_{\tilde{n}} > l_m$ such that $U_{\tilde{n}} \cap U'_{l_{\tilde{n}}} \in \mathcal{U}_{\tilde{n}}$.

First, suppose that (1) holds. Let $V_m \equiv U_m \cap U'_{l_m}$ and $V'_{l_m} \equiv U_n \cap U'_{l_m}$. By construction, V_m is a dictatorial coalition for population $\mathcal{N}\setminus\bigcup_{k=1}^{m-1} U_k$. On the other hand, $V_m \cap \bigcup_{l=1}^{l_m-1} U'_l = \varnothing$. By the same reason as V_m, V'_{l_m} is a dictatorial coalition for population $\mathcal{N}\setminus\bigcup_{l=1}^{l_m-1} U_l$. We also have that $V'_{l_m} \cap \bigcup_{k=1}^{n-1} U_k = \varnothing$. Now let $N^* \equiv \mathcal{N}\setminus\{(\bigcup_{l=1}^{l_m-1} U'_l) \cup (\bigcup_{k=1}^{n-1} U_l)\}$, and $\mathcal{U}^*$ be the dictatorial ultrafilter. Then, it must be the case that both $V_m \in \mathcal{U}^*$ and $V'_{l_m} \in \mathcal{U}^*$, and so $V_m \cap V'_{l_m} \neq \varnothing$. However, since $V'_{l_m} \subset U_n$ and $V_m \subset U_m$, we have that $V'_{l_m} \cap V_m = \varnothing$. It is a contradiction.

Next, suppose that (2) holds. Let $V_{\tilde{n}} \equiv U_{\tilde{n}} \cap U'_{l_{\tilde{n}}}$. Then, by the same argument as above, $V_{\tilde{n}}$ is a dictatorial coalition for population $\mathcal{N}\setminus\bigcup_{k=1}^{\tilde{n}-1} U_k$ and $V_{\tilde{n}} \cap \bigcup_{l=1}^{l_{\tilde{n}}-1} U'_l = \varnothing$. On the other hand, V_m is a dictatorial coalition for population $\mathcal{N}\setminus\bigcup_{l=1}^{l_m-1} U'_l$ and $V_m \cap \bigcup_{l=1}^{\tilde{n}} U_k = \varnothing$. Let $\tilde{\mathcal{U}}$ be the dictatorial ultrafilter for a population $\tilde{N} \equiv \mathcal{N}\setminus\{(\bigcup_{k=1}^{\tilde{n}-1} U_k) \cup (\bigcup_{l=1}^{l_{\tilde{n}}-1} U'_l)\}$. Then, it must be the case that both $V_m \in \tilde{\mathcal{U}}$ and $V_{\tilde{n}} \in \tilde{\mathcal{U}}$, and so $V_m \cap V_{\tilde{n}} \neq \varnothing$. However, since $V_m \subset U_m$ and $V_{\tilde{n}} \subset U_{\tilde{n}},\ V_m \cap V_{\tilde{n}} = \varnothing$. It is a contradiction. Thus $\mathscr{U}^\phi$ satisfies order preservation. □

Finally we show that St.Unanimity, IA-Independence, and LA-Stability are the necessary condition for the generalized serial dictatorship;

Theorem 4.10. *If a solution ϕ satisfies the generalized serial dictatorship, then ϕ satisfies St.Unanimity, IA-Independence, and LA-Stability.*

The proof is done through the following three lemmas:

Lemma 4.11. *If a solution ϕ satisfies the generalized serial dictatorship, ϕ satisfies St.Unanimity.*

Proof. Let $X \subset \mathcal{X}$, $x \in X$ and $\succsim \in \mathcal{R}^{\mathcal{N}}$. Suppose that, for each $i \in \mathcal{N}$ and $y \in X$, $x \succsim_i y$, and there exists $j \in \mathcal{N}$ such that $x \succ_j y$. Let $\{U_k\}_{k=1}^{K}$ be the ordered preference partition for $\succsim$. Then, for each $k \in \{1, \cdots, K\}$, $x \in T_k^*(X, \succsim)$. On the other hand, for each $y \in X$ with $y \neq x$, there exists $k_y \in \{1, \cdots, K\}$ such that $x \succ_{U_{k_y}} y$, therefore $y \notin T_{k_y}^*(X, \succsim)$. It implies that $T_K^*(X, \succsim) = \{x\}$ and $\phi(\mathcal{N}, X, \succsim) = \{x\}$. $\square$

Lemma 4.12. *If a solution ϕ satisfies the generalized serial dictatorship, ϕ satisfies IA-Independence.*

Proof. Let $\mathscr{U}$ be the hierarchical family of ultrafilters governing ϕ. Let $X \subset \mathcal{X}$ and $\succsim, \succsim' \in \mathcal{R}^{\mathcal{N}}$ with $\succsim =_X \succsim'$. Let $\{U_k\}_{k \in K}$ and $\{U_l'\}_{l \in L}$ be their ordered preference partitions respectively, and $\{\mathcal{U}_k\}_{k=1}^{K}$ and $\{\mathcal{U}_l'\}_{l=1}^{L}$ be the associated hierarchical series of ultrafilters. Note that $\mathcal{U}_1 = \mathcal{U}_1'$ by construction. Now we can construct a subsequence of $\{U_k\}_{k \in K}$, say, $\{U_{\pi(n)}\}_{n=1}^{\tilde{K}}$, such that (1) $\pi(n) = 1$, and, (2) for each $n \in \{2, \cdots, \tilde{K}\}$, $\pi(n)$ is the first number after $\pi(n-1)$ such that $\succsim_{U_{\pi(n)}} \neq_X \succsim_{U_{\pi(n-1)}}$. A family of sets $\{T_{\pi(n)}^*(X, \succsim)\}_{n=1}^{\tilde{K}}$ is defined as before;

$$
\begin{aligned}
& T_{\pi(0)}^*(X, \succsim) \equiv X, \\
& T_{\pi(1)}^*(X, \succsim) \equiv Top_{(U_{\pi(1)})}(X, \succsim), \\
\text{for } \forall\, n \in \{2, \cdots, \tilde{K}\}, \quad & T_{\pi(n)}^*(X, \succsim) \equiv Top_{(U_{\pi(n-1)})}(T_{\pi(n-1)}^*(X, \succsim), \succsim).
\end{aligned}
$$

By construction, $\bigcap_{k \in \{1, \cdots, K\}} T_k^*(X, \succsim) = \bigcap_{n \in \{1, \cdots, \tilde{K}\}} T_{\pi(n)}^*(X, \succsim)$. Similarly we can also define a subsequence $\{U'_{\pi'(m)}\}_{m=1}^{\tilde{L}}$, so that $\bigcap_{l \in \{1, \cdots, L\}} \tilde{T}_l^*(X, \succsim') = \bigcap_{m \in \{1, \cdots, \tilde{L}\}} \tilde{T}_{\pi'(m)}^*(X, \succsim')$. Here $\{\tilde{T}_{\pi'(m)}^*(X, \succsim')\}_{m=1}^{\tilde{L}}$ is defined as $\{T_{\pi(n)}^*(X, \succsim)\}_{n=1}^{\tilde{K}}$.

Next we show that, for each $1 \leq n \leq \min\{\tilde{K}, \tilde{L}\}$, $\succsim_{U_{\pi(n)}} =_X \succsim'_{U'_{\pi'(n)}}$ using mathematical induction. First, we have that $U_{\pi(1)} = U_1$ and $U'_{\pi'(1)} = U'_1$. Since both U_1, $U'_1 \in \mathcal{U}_1$, we have $U_{\pi(1)} \cap U'_{\pi'(1)} \neq \varnothing$. Therefore $\succsim_{U_{\pi(1)}} =_X \succsim'_{U'_{\pi'(1)}}$. Next, let $n > 1$. Suppose that, for each $\tilde{n} \leq n$, $\succsim_{U_{\pi(\tilde{n})}} =_X \succsim'_{U'_{\pi'(\tilde{n})}}$. Then there exists an unique $l^* \in \{1, \cdots, L\}$ such that $U_{\pi(n+1)} \cap U'_{l^*} \in \mathcal{U}_{\pi(n+1)}$. Then $\succsim_{U_{\pi(n+1)}} =_X \succsim'_{U'_{l^*}}$. Now, we have that, for each $k < \pi(n+1)$, $\succsim_{U_k} \neq_X \succsim_{U_{\pi(n+1)}}$. Therefore, for each $k < \pi(n+1)$, $\succsim_{U_k} \neq_X \succsim_{U'_{l^*}}$. It implies that, for each $k < \pi(n+1)$, $U_k \cap U'_{l^*} = \varnothing$. By order preservation, $U_{\pi(n+1)} \cap U'_{l^*} \in \mathcal{U}'_{l^*}$.

Next we want to show that $\succsim'_{l^*} =_X \succsim'_{\pi'(n+1)}$. Since, for each $\tilde{n} \leq n$, $\succsim_{U_{\pi(\tilde{n})}} =_X \succsim'_{U'_{\pi'(\tilde{n})}}$, we also have that, for each $\tilde{n} \leq n$, $\succsim'_{U'_{\pi'(\tilde{n})}} \neq_X \succsim'_{U'_{l^*}}$. Suppose that $\succsim'_{l^*} \neq_X \succsim'_{\pi'(n+1)}$. Then there exists $l' \in \{\pi'(n)+1, \cdots, l^* - 1\}$ such that $\succsim'_{U'_{\pi'(n)}} \neq_X \succsim'_{U'_{l'}} \neq_X \succsim'_{U'_{l^*}}$. Therefore, for each $k \leq \pi(n+1)$, $U_k \cap U'_{l'} = \varnothing$. It implies that there exists $k' > \pi(n+1)$, $U_{k'} \cap U'_{l'} \in \mathcal{U}'_{l'}$. It contradicts order preservation. Thus, $\succsim'_{l^*} =_X \succsim'_{\pi'(n+1)}$. This implies that $\succsim_{\pi(n+1)} =_X \succsim'_{\pi'(n+1)}$.

From the above result, we have that $\tilde{K} = \tilde{L}$, and $T^*_{\pi(\tilde{K})} = T^*_{\pi'(\tilde{L})}$. Thus $\phi(X, \succsim) = \phi(X, \succsim')$. $\square$

Lemma 4.13. *If a solution ϕ satisfies the generalized serial dictatorship, ϕ satisfies LA-Stability.*

Proof. Suppose solution ϕ satisfies the generalized serial dictatorship with a hierarchical family of ultrafilters $\mathscr{U}$ and a tie-braking rule ρ. Let $Y \subset X \subset \mathcal{X}$ and $\succsim \in \mathcal{R}^{\mathcal{N}}$. Let $\{U_k\}_{k=1}^K$ be the ordered preference partition for $\succsim$. Suppose that $\phi(\mathcal{N}, X, \succsim) \cap Y \neq \varnothing$. Then, for each $k \in \{1, \cdots, K\}$, $T^*_k(X, \succsim) \cap Y \neq \varnothing$. Therefore, by construction, $T^*_k(Y, \succsim) = T^*_k(X, \succsim) \cap Y$. So we have that

$$\begin{aligned}
\phi(\mathcal{N}, X, \succsim) \cap Y &= Top(T^*_K(X, \succsim), \rho) \cap Y \\
&= Top(T^*_K(X, \succsim) \cap Y, \rho) \\
&= Top(T^*_K(Y, \succsim), \rho) \\
&= \phi(\mathcal{N}, Y, \succsim).
\end{aligned}$$

Thus LA-Stability is satisfied. $\square$

5 Free and Fixed Ultrafilters

Theorem 5.1. *For each $N \subset \mathbb{N}$ and $X \subset \mathcal{X}$ with $3 \leq |X| < \infty$, the decisive ultrafilter $\mathcal{U}^N_X$ resulting from $\phi(N, X, \cdot)$ is free if and only if there exists an finitely additive measure μ on 2^N such that (1) for each $E \subset N$, $\mu(E) = 0$ or 1, and, for each singleton $\{i\} \subset N$, $\mu(\{i\}) = 0$, and (2) the social order $P(\cdot)$ corresponding to $\phi(N, X, \cdot)$ satisfies that, for each $x, y \in X$ and each $\succsim \in \mathcal{R}$, $xP(\succsim)y$ if $\mu(\{i \in N \mid x \succ_i y\}) = 1$.*

Proof. ($\Leftarrow$) Let $\tilde{\mathcal{U}} \equiv \{E \subset N \mid \mu(E) = 1\}$. We show that $\tilde{\mathcal{U}} = \mathcal{U}^N_X$. Let $E \in \tilde{\mathcal{U}}$. By definition, for each $x, y \in X$ and each $\succsim \in \mathcal{R}$, if $x \succ_E y$, then $xP(\succsim)y$. It implies that $E \in \mathcal{U}^N_X$, so $\tilde{\mathcal{U}} \subset \mathcal{U}^N_X$. Next, let $F \in \mathcal{U}^N_X$. Then, for each $x, y \in X$ and each $\succsim \in \mathcal{R}$, if $x \succ_F y$ and $y \succsim_{N \setminus F} x$, then $xP(\succsim)y$. This means that, for $\succsim^* \in \mathcal{R}$ with $\{i \in N \mid x \succ^*_i y\} = F$, $xP(\succsim^*)y$. Therefore $\mu(F) = 1$, i.e., $F \in \tilde{\mathcal{U}}$. We have $\mathcal{U}^N_X \subset \tilde{\mathcal{U}}$. By definition, there exists no $i \in N$ such that $\{i\} \in \tilde{\mathcal{U}}$. Thus $\tilde{\mathcal{U}} = \mathcal{U}^N_X$ is free.

($\Rightarrow$) Let a set function $\mu : 2^N \to \{0, 1\}$ be such that $\mu(E) = 1$ iff $E \in \mathcal{U}^N_X$. Then, it is well known that μ is a finitely additive measure on 2^N.[2] Since $\mathcal{U}^N_X$ is free, for each $i \in N$, $\mu(\{i\}) = 0$. By definition, it is obvious that, for each $x, y \in X$ and each $\succsim \in \mathcal{R}^N$, if $\mu(\{i \in N \mid x \succ_i y\}) = 1$, i.e., $\{i \in N \mid x \succ_i y\} \in \mathcal{U}^N_X$, then $xP(\succsim)y$. $\square$

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[2] See Jech, etc

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