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STV Elections by Feedback Counting: An Approach from Cooperative Game Theory

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We describe the mechanism of feedback methods of counting single transferable votes, such as *Meek's method*, in the framework of *n*-person strategic games. We show that the games are in the class introduced by Masuzawa (*International Journal of Game Theory* 32:2003 and 37:2008), and that for any given finite domain of keep value, the algorithm by Masuzawa (2008) correctly maximizes the set of winners and minimizes the corresponding keep values. Starting at zero, our algorithm increases the keep value of any candidate until the surplus becomes positive, while the prevailing method decreases it and does not necessarily attain the maximum set of winners.

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STV Elections by Feedback Counting:
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We describe the mechanism of feedback methods of counting single transferable votes, such as *Meek's method*, in the framework of n-person strategic games. We show that the games are in the class introduced by Masuzawa (*International Journal of Game Theory* 32:2003 and 37:2008), and that for any given finite domain of keep value, the algorithm by Masuzawa (2008) correctly maximizes the set of winners and minimizes the corresponding keep values. Starting at zero, our algorithm increases the keep value of any candidate until the surplus becomes positive, while the prevailing method decreases it and does not necessarily attain the maximum set of winners.

Keywords Single transferable vote, Meek's method, New Zealand Method, Efficient algorithm, N-person prisoner's dilemma, Cooperative game theory.

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1 Introduction

The single transferable vote (STV) is a voting system designed to achieve proportional representation through preferential voting. In this system, the procedure consists of the repetition of two stages, called the surplus transfer stage and the eliminating stage in this paper. In the surplus transfer stage, any candidate whose number of votes exceeds the quota is elected as a winner, and the surplus is distributed to other candidates according to the preferences. If all seats are filled, then the procedure is completed. Otherwise, the procedure enters the eliminating stage. Then, some candidates are chosen as losers and all of their votes are transferred by some rule. After this, the procedure goes back to the surplus transfer stage, where new winners are elected.

Many kinds of methods of surplus transfer have been discussed and implemented. In the most traditional method, the votes to be transferred are randomly selected. To remove the randomness, in the Gregory method and its variants, any single vote is partitioned into fractions and they are chosen proportionally to be transferred. In such methods, since any vote is not transferred to any elected candidate, the order of surplus transfer affects the results. In implementing these methods, a detailed rule must be fixed, sometimes without persuasive reason.

Meek (1969, 1970), Woodall (1983), and Warren (1994) proposed *feedback methods* in which any candidate has his/her *keep value* according to which he/she retains his/her votes and the remainder is transferred even to elected candidates. The winners are determined by the keep value vector such that (i) the sum of votes credited to any candidate does not exceed the quota only if his/her keep value is 1 and (ii) the surplus of any winner is decreased as much as possible. The feedback methods are formalized by simple principles without any reference to complicated details and carried out quite easily by machine. Meek's method is adopted in some local assemblies of New Zealand. In Schedule 1A of Local Electoral Regulations 2001, "New Zealand method of counting single transferable votes" is prescribed

(See Hill, 2006).

In this paper, we will describe the feedback mechanism of transferring votes by a strategic form game in which any candidate, as a player, chooses his/her keep value as a strategy and gets the fractions of votes credited to him/her as a payoff. Then, it will be shown that the game satisfies *the punishment-dominance condition* introduced by Masuzawa (2003), which implies remarkable properties in game theory and has a wide range of applications. Further, we show that for any given finite domain of keep values, the algorithm introduced in Masuzawa (2008) maximizes the set of candidates with positive surplus and minimizes the corresponding keep values. As will be shown, note that the prevailing method, such as New Zealand method, does not maximize the set of winners.

The remainder of this paper is organized as follows. In Section 2, we describe the situation by a game and briefly review the theory of games with punishment-dominance condition. In Section 3, from the theory, we derive an algorithm for determining the maximum set of winners and the minimum keep value vector. Surprisingly, in our derived method, the keep values are increasingly updated: any candidate increases his/her keep value; in contrast, in the prevailing methods, any candidate decreases it. In Section 4, we show that the difference between our method and the prevailing ones affects the determination of the winners.

2 Preliminaries

2.1 Description by Strategic Games

We consider a class of games to analyze the procedure of counting single transferable votes. The set of players, N, are all candidates. Any candidate, i, chooses his/her *keep value*, $a^i \in [0,1]$, of votes, the remainder of which is transferred to the others or wasted. The domain of a^i is confined to a subset, X^i, of $[0,1]$ such that $0,1 \in X^i$. By a^S we denote the list of the keep values over a set of candidates $S \subset N$, whose i-th coordinate is a^i for all $i \in S$. The payoff of a candidate is defined by the *surplus*: $s^i(a^N) = v^i(a^N) - q(a^N)$, where $v^i(a^N)$ is the summation of the *fractions of votes* credited to him, and $q(a^N)$ is the threshold of votes to be elected as a winner, called the *quota*.

In Meek's rule, for any whole vote, α, which ranks $i_1, i_2, \ldots, i_k, \ldots$ in this order, the fraction of α transferred to candidate i_k is $m^\alpha(i_k) := (1-a^{i_1})(1-a^{i_2})\ldots(1-a^{i_{k-1}})a^{i_k}$. On the other hand, in Warren's rule, the fraction transferred to candidate i_k is $w^\alpha(i_k) := \min\{a^{i_k}, \max\{0, 1-\sum_{j<k} a^{i_j}\}$. Then $v^{i_k}(a^N)$ is defined as the sum of the fractions candidate i_k received: the summation of $m^\alpha(i_k)$ over all votes α in Meek's, and that of $w^\alpha(i_k)$ in Warren's.

We assume that the number of votes retained by any winner must exceed the quota. We use, for example, $(\sum_{i\in N} v^i(a^N))/(\text{seats} + 1)$ rather than (the minimum unit of the fraction of vote $+ \sum_{i\in N} v^i(a^N))/(\text{seats} + 1)$ as Droop's quota[1].

2.2 Games with Punishment-Dominance

In sum, a feedback counting mechanism of STV election is specified by a strategic form game $(N, (X^i)_{i\in N}, (v^i)_{i\in N}, q)$ where the payoff is defined by $s^i(a^N) = v^i(a^N) - q(a^N)$. We assume that the game has the following properties:

Assumption 1
For all $a^N \in \prod_{i\in N} X^i$, all $i \in N$, and all $x^i, y^i \in X^i$,

(i) any decrease of the keep value does not decrease any other's votes:
if $x^i > y^i$ then $v^j(a^{N\setminus\{i\}}, x^i) \leq v^j(a^{N\setminus\{i\}}, y^i)$ for all $j \in N \setminus \{i\}$, and

(ii) any decrease of the keep value does not increase the quota:
if $x^i > y^i$ then $q(a^{N\setminus\{i\}}, x^i) \geq q(a^{N\setminus\{i\}}, y^i)$.

From Assumption 1, it can be easily shown that the game satisfies the following.

[1]See Lundell and Hill (2007) and Janson (2011).

Condition 1 (Punishment-Dominance Condition)
For all $i \in N$, all $x^i, y^i \in X^i$, all $j \in N \setminus \{i\}$, and all $a^{N\setminus\{i\}} \in \prod_{j' \in N\setminus\{i\}} X^{j'}$,

$$\text{if } x^i > y^i \text{ then, } s^j(a^{N\setminus\{i\}}, x^i) \leq s^j(a^{N\setminus\{i\}}, y^i).$$

This condition implies important properties of cooperative solutions and the situations with monotone externality, such as n-person prisoners' dilemma game, public good provision game, and Cournot oligopoly game, all satisfy this condition[2].

Masuzawa (2008) has given an efficient algorithm to compute the cooperative solutions in cases where X^i is finite. Note that if we assume the minimum unit of the scale, $h > 0$, the domain, X^i, is finite. In such a finite case, for all $i \in N$ and all $a^i \in X^i \setminus \{1\}$, we can define $\boldsymbol{succ}(a^i) := \min\{c \in X^i : c > a^i\}$. Similarly, $\boldsymbol{pre}(a^i) := \max\{c \in X^i : c < a^i\}$ is well-defined for all $a^i \in X^i \setminus \{0\}$. We assume, hereafter, the following.

Assumption 2
For all $i \in N$, X^i is finite and $0, 1 \in X^i$.

Note that for computation by any given real machine, this assumption is indispensable.

The basic sub-algorithm in Masuzawa (2008) finds the solutions of a class of problems called MGSP (minimum guaranteeing problem) for all vectors of real numbers, $(r^i)_{i\in N}$:

Problem 1
Find the minimum of vectors $a^N \in \prod_{i\in N} X^i$ such that

$$\text{for all } i \in N, \;\; s^i(a^N) > r^i \quad \text{or} \quad a^i = 1. \tag{1}$$

If $a^N \in \prod_{i\in N} X^i$ satisfies the constraint, a^N is called *feasible*. Note that at least one feasible vector, $(1, 1, \ldots, 1)$, exists, and that a feasible vector, a^{*i}, is the minimum if $a^i \geq a^{*i}$ for all $i \in N$ and all feasible vectors, a^N. It can be easily shown that the minimum necessarily exists, and is attained by the following algorithm.

[2] See Masuzawa (2003).

Algorithm 1

1. $\boldsymbol{a}^i := 0$ for all $i \in N$

2. **while** $\boldsymbol{a}^N$ is not feasible **do begin**

   (a) $\boldsymbol{S} := \{i \in N : s^i(\boldsymbol{a}^N) > r^i \text{ or } \boldsymbol{a}^i = 1\}$;

   (b) $\boldsymbol{a}^i := \boldsymbol{succ}(\boldsymbol{a}^i)$ for all $i \in N \setminus \boldsymbol{S}$ **end**

Justification To justify the algorithm, let a^{*N} be feasible. It suffices to show that $\boldsymbol{a}^N \leq a^{*N}$ at any time during the computation. To see this, note that if $\boldsymbol{a}^i = a^{*i}$ and $\boldsymbol{a}^N \leq a^{*N}$, then, from Condition 1, $s^i(\boldsymbol{a}^N) \geq s^i(a^{*N})$, and thus $i \in \boldsymbol{S}$.

Order-independence Note that we can replace step (2b) by
(2b') $\boldsymbol{a}^i := \boldsymbol{succ}(\boldsymbol{a}^i)$ for **some** $i \in N \setminus \boldsymbol{S}$
since this change preserves the property that $\boldsymbol{a}^N \leq a^{*N}$ at any time during the computation. Further, since $s^i(\boldsymbol{a}^{N\setminus\{i\}}, \boldsymbol{a}^i)$ is decreasing unless $\boldsymbol{a}^i$ is updated, the update of $\boldsymbol{a}^i$ into

$$\min\{a^i \geq \boldsymbol{a}^i : s^i(\boldsymbol{a}^{N\setminus\{i\}}, a^i) > r^i \text{ or } a^i = 1\}$$

does not violate the relation, $\boldsymbol{a}^N \leq a^{*N}$, for all feasible vectors a^{*N}.

Weak inequality If we substitute for (1) of Problem 1,

$$\text{for all } i \in N, \;\; s^i(a^N) \geq r^i \quad \text{or} \quad a^i = 1 \tag{2}$$

then, to recover the exactness of the algorithm, we only have to replace (2a) of Algorithm 1 by
(2a") $\boldsymbol{S} := \{i \in N : s^i(\boldsymbol{a}^N) \geq r^i \text{ or } \boldsymbol{a}^i = 1\}$.

3 Determination of the winners

Now, let us derive an algorithm to determine the winners from the theory of games with punishment-dominance condition. In the surplus transfer stage, any candidate with a non-positive surplus is not allowed to transfer any of his/her votes: if $s^i(a^N) \leq 0$, then $a^i = 1$. Given $a^N \in \prod_{i \in N} X^i$, the winners are determined by $W := \{i \in N : s^i(a^N) > 0\}$. To obtain the maximum set of winners, consider the following problem:

Problem 2 (Minimum keep value problem)
Find the minimum of vectors a^N such that

$$\text{for all } i \in N, \;\; s^i(a^N) > 0 \quad \text{or} \quad a^i = 1.$$

Let $a_1^N, a_2^N, \ldots, a_k^N$ be feasible vectors, and define $S_k = \{i : s^i(a_k^N) > 0\}$. Then from Condition 1, for all $i \in \cup_j S_j$, $s(a_*^N) > 0$, where a_*^N is defined by $a_*^i = \min\{a_j^i : j = 1, 2, \ldots, k\}$. Thus, there exists a set of players $S \subset N$ and a feasible vector a_*^N such that $S = \{i \in N : s^i(a^{*N}) > 0\} = \cup\{i \in N : s^i(a^N) > 0\}$, where the union is taken over all feasible vectors, a^N. In particular, by Assumption 2, we can take a^{*N} as the minimum. In other words, Problem 2 determines the maximum set of winners where there is no transfer from the non-winners.

Since Problem 2 is an example of Problem 1, the solution of Problem 2 is obtained by the following algorithm.

Algorithm 2

1. $\boldsymbol{a}^i := 0$ for all $i \in N$
2. **while** $\boldsymbol{a}^N$ is not feasible **do begin**
   (a) $\boldsymbol{S} := \{i : s^i(\boldsymbol{a}^N) > 0 \text{ or } \boldsymbol{a}^i = 1\}$;
   (b) $\boldsymbol{a}^i := \boldsymbol{succ}(\boldsymbol{a}^i)$ for all $i \in N \setminus \boldsymbol{S}$ **end**

Acceleration

Both Meek's and Warren's rules satisfy the following assumption.

Assumption 3
For all $a^N \in \prod_{i \in N} X^i$, all $i \in N$, and all $x^i, y^i \in X^i$,

$$\text{if } x^i > y^i, \text{ then } s^i(a^{N \setminus \{i\}}, x^i) \geq s^i(a^{N \setminus \{i\}}, y^i).$$

By Assumption 3 and the order-independence previously discussed, instead of $\boldsymbol{succ}(a^i)$, we can update $\boldsymbol{a}^i$ into $\min\{a^i : s^i(\boldsymbol{a}^{N\setminus\{i\}}, a^i) > 0 \text{ or } a^i = 1\}$, which can be obtained by the binary method. Sometimes, this modification can speed up computation. Further, in specific situations, some criteria are available for acceleration. For example, in Meek's rule, one can update the keep value by rule:

$$\boldsymbol{a}^i := \min\left\{a^i \in X^i : a^i > \frac{\boldsymbol{a}^i \cdot q(\boldsymbol{a}^N)}{v^i(\boldsymbol{a}^N)} \text{ or } a^i = 1\right\}.$$

After Eliminating Losers

If all seats are not filled in the surplus transfer stage, then, in the subsequent eliminating stage, some of candidates become losers, denoted by L, and all of their votes are transferred: for all $i \in L$, $a^i := 0$. Typically, in every eliminating stage, the candidate with the fewest votes is added to the set of losers. After this, the surplus transfer stage is repeated again and other candidates are newly elected by the following.

Problem 3
Find the minimum of vectors b^N such that

$$\begin{aligned}
&\text{for all } i \in N \setminus L, \;\; s^i(b^N) > 0 \quad \text{or} \quad b^i = 1;\\
&\text{for all } i \in L, \qquad\;\; s^i(b^N) > 0 \quad \text{or} \quad b^i = 1 \quad \text{or} \quad b^i = 0.
\end{aligned}$$

Let b^{*N} be the minimum. First, note that for all $i \in L$, $b^{*i} = 0$ from Condition 1. Second, if a^{*N} is the solution of Problem 2, then it also satisfies the constraint of Problem 3. It follows that $a^{*i} \geq b^{*i}$ for all $i \in N$. In other words, the minimum keep value vector is decreasing in the iteration of the surplus transfer stages. Third, one can easily see that the following algorithm finds additional winners.

Algorithm 3

1. $\boldsymbol{b}^i := 0$ for all $i \in N$
2. **while** $\boldsymbol{b}^N$ is not feasible **do begin**

(a) $\boldsymbol{S} := \{i : s^i(\boldsymbol{b}^N) > 0 \text{ or } \boldsymbol{b}^i = 1\} \cup D$;

(b) $\boldsymbol{b}^i := \boldsymbol{succ}(\boldsymbol{b}^i)$ for all $i \in N \setminus \boldsymbol{S}$ **end**

4 Decreasing Keep Value

4.1 Decreasing Method

While the keep value of any candidate is increasing in Algorithm 2, it is decreasing in the prevailing methods: at first, any candidate retains all of his/her votes, $\boldsymbol{a}^i := 1$, and any candidate with positive surplus iteratively decreases it as much as it remains positive. In other words, the prevailing method is formalized as follows.

Algorithm 4

1. $\boldsymbol{c}^i := 1$ for all $i \in N$;
2. $\boldsymbol{S} := \emptyset$;
3. **while** $\boldsymbol{S} \neq N$ **do begin**
   (a) $\boldsymbol{S} := \{i : s^i(\boldsymbol{c}^{N\setminus\{i\}}, \boldsymbol{pre}(\boldsymbol{c}^i)) \leq 0 \text{ or } \boldsymbol{c}^i = 0\}$;
   (b) $\boldsymbol{c}^i := \boldsymbol{pre}(\boldsymbol{c}^i)$ for all $i \in N \setminus \boldsymbol{S}$ **end**

Consider $s^{*i}(d^N)$ defined by

$$s^{*i}(d^N) := \begin{cases} q(c^{N\setminus\{i\}}, 0) & \text{if } d^i \neq 1 \\ -s^i(c^{N\setminus\{i\}}, \boldsymbol{pre}(c^i)) & \text{otherwise,} \end{cases}$$

where $c^i := 1 - d^i$ for all $i \in N$. Then, the same condition as Condition 1 is satisfied for s^{*i}. It follows that Algorithm 4 is the solution of the following problem:

Problem 4
Find the maximum of vectors c^N such that

$$\text{for all } i \in N,\ \ s^i(c^{N\setminus\{i\}}, \boldsymbol{pre}(c^i)) \leq 0 \quad \text{or} \quad c^i = 0.$$

Note that the same discussion on the order-independence is also applicable to this case. Thus, adopting the updating rule

$$\boldsymbol{c}^i := \min\left\{c^i \in X^i : c^i > \frac{\boldsymbol{c}^i q(\boldsymbol{c}^N)}{v^i(\boldsymbol{c}^i)}\right\}$$

for acceleration, we obtain, as variations of Algorithm 4, the algorithm for Meek's method by Hill, Wichmann and Woodall (1987) and that of New Zealand Local Electoral Regulations 2001.

4.2 Difference Between the Two Methods

Let a^{*N} be the solution of Problem 2 and c^{*N} that of Problem 4. It can be easily shown that c^{*N} satisfies the constraint of Problem 2. Then, by the definition of the minimum,

$$c^{*i} \geq a^{*i}, \text{ for all } i \in N.$$

The difference between a^{*N} and c^{*N} may affect the determination of the winners as well as that of the losers. To see this, consider the case where the set of candidates is $\{A, B, C, D, E\}$, the number of seats is four, and the list of votes is:

$51 : A, B, C;$	$51 : B, A, C$	$40 : C$
$47 : D, E;$	$43 : E.$	

In Meek's rule with $X^i = \{0, 0.1, 0.2, \ldots, 1\}$, by Algorithm 4, we obtain $c^{*N} = (1, 1, 1, 1, 1, 1)$ and the winners are A, B and D, while by Algorithm 2, we obtain $a^{*N} = (0.7, 0.7, 1, 1, 1, 1)$ and C is also an winner. Following Algorithm 4, if, in every eliminating stage, the candidate with the lowest votes is eliminated, then C is ultimately determined to be a loser.

To analyze the difference, consider the matrix below, which denotes

$s^A(a^N)$ for $a^N = (a^A, a^B, 1, 1, 1, 1)$:

a^A \ a^B	1	0.9	0.8	0.7	0.6
1	4.6	9.7	14.8	19.9	25
0.9	−0.9	4.09	8.68	13.27	17.86
0.8	−5.6	−1.52	2.56	6.64	10.72
0.7	−10.7	−7.13	−3.56	**0.01**	3.58
0.6	−15.8	−12.74	−9.68	−6.62	−3.56

For example, if $(a^A, a^B) = (0.7, 0.9)$ then $s^A(a^N) = -7.13$. If candidate A decreases his/her keep value from $(a^A, a^B) = (1, 1)$ individually, the surplus becomes negative. Thus, the prevailing method never attains $(a^A, a^B) = (0.7, 0.7)$. On the other hand, the coalition of A and B can decrease their keep values and keep their surpluses positive. Roughly speaking, the difference arises from the fact that in our method, coalitions of candidates as well as individual candidates jointly transfer their positive surpluses and keep them positive.

5 Concluding Remarks

In this paper, we formulate the feedback mechanism of STVs by games with punishment-dominance. By the framework, we propose a new method of finding the winners and the corresponding keep value vector and compare this method with the prevailing one. We show that our method yields the correct solution for any given minimum unit of the scale.

As a secondary result, we obtain the non-procedural definition of the outcome of STV for the existing method as well as the new method (Problems 2 and 4). While procedural definitions of outcome (the algorithms themselves) are indispensable, they are too complicated to persuade many people to adopt STV elections. Thus, the simplicity of non-procedural definitions will, I think, contribute to spreading and legislating STV.

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