A UNIFYING IMPOSSIBILITY THEOREM

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Abstract. This paper considers social choice correspondences assigning a choice set to each non-empty subset of social alternatives. A social choice correspondence is Independent of Infeasible Alternatives when choices from subsets depend only on preferences over the subsets. It is Independent of Losing Alternatives if choices out of all possible alternatives remain chosen whenever they are available. With more than three social alternatives and the universal preference domain, any social choice correspondence that is weakly Pareto optimal, independent of infeasible and losing alternatives is serially dictatorial. A number of known impossibility theorems — including the Arrow's Impossibility Theorem, Gibbard-Satterthwaite Theorem and the Dutta-Jackson-Le Breton impossibility theorem under strategic candidacy — follow as corollaries.

1. Introduction

This paper considers social choices defined on all subsets of social alternatives. We impose three axioms on these social choice correspondences:

Weak Pareto Optimality: Only the unique weakly Pareto dominant alternative within the subset is chosen whenever there is one.

Independence of Infeasible Alternatives: Choices from subsets depend only on the preferences over the subsets.

Independence of Losing Alternatives: Choices out of the set of all social alternatives remain chosen at the same preference profile whenever they are available.

With more than three social alternatives and the universal preference domain, any social choice correspondence that satisfies our three axioms is serially dictatorial. Remarkably,

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many known impossibility theorems — including the Arrow's Impossibility Theorem (Arrow, 1963), the Gibbard-Satterthwaite Theorem (Gibbard, 1973; Satterthwaite, 1975), the Muller-Satterthwaite Theorem (Muller and Satterthwaite, 1977), the Jackson-Srivastava characterization of game-theoretic solutions that implement only dictatorial social choice functions (Jackson and Srivastava, 1996), the Grether-Plott Theorem (Grether and Plott, 1982) and the Dutta-Jackson-Le Breton impossibility theorem under strategic candidacy (Dutta, Jackson, and Le Breton, 2001)[1]— are corollaries of our main theorem. In other words, Pareto optimality and our independence axioms logically underpin all these impossibility theorems.

We show by Example 2.3 that our three axioms do not imply Monotonicity (a chosen alternative remains chosen whenever its relative ranking has improved), which is the counterpart of Arrow's Independence of Irrelevant Alternatives for social choice functions (see Reny (2001) for the parallel). In this sense our axioms are weaker than the requirements in the classical impossibility theorems. Closest to our theorem are two impossibility theorems on social choice functions defined on subsets of social alternatives: Theorem 7 of Campbell (1979) and the Grether-Plott Theorem (Grether and Plott, 1982). Both of them assume the Arrow's Choice Axiom, which is stronger than our Independence of Losing Alternatives (see Section 2). Moreover, both of their proofs appeal to the Arrow's Impossibility Theorem, while we provide a direct proof.

Among the many proofs of the classical impossibility theorems, Reny (2001) provides a unified proof of both Arrow's Impossibility Theorem and the Gibbard-Satterthwaite Theorem using techniques from Barberá (1980, 1983) and Geanakoplos (2005). In terms of the use of "top sets", our proof resembles that in Sen (2001). All these proofs invoke monotonicity, which is not directly implied by our axioms. Recently, Vohra (2011) proves the Arrow's impossibility theorem, Gibbard-Satterthwaite Theorem and the impossibility theorem under strategic candidacy using integer programming techniques. Unlike us, he uses Arrow's Theorem to prove the other two theorems.

2. Model

Let X be the set of all possible social alternatives, $|X| \geq 3$ and finite. Denote the set of all complete, transitive binary relations on X as R. Let $P \subset R$ be the set of all complete, transitive and anti-symmetric binary relations on X. We abuse notations to use N as both the set of all individuals and its cardinality, which is finite.

A (weak) preference of individual $i \in N$ is denoted by $\succsim_i \in R$. The symbols $\succ_i$ and $\sim_i$ will have their usual derived meanings. Let $\succsim = (\succsim_1, \ldots, \succsim_N) \in R^N$ be a preference profile.

[1]See also Ehlers and Weymark (2003); Eraslan and McLennan (2004); Rodíguez-Álvarez (2006).

Similarly, a strict preference of an individual $i \in N$ and a strict preference profile are denoted as $\succ_i \in P$ and $\succ \in P^N$ respectively.

Two preference profiles $\succsim, \succsim' \in R^N$ *agree on* Y if they induce the same preference ordering on the subset of alternatives $Y \subseteq X$. Given a subset of alternatives $Y \subseteq X$ and a preference profile $\succsim \in R^N$, we say $\succsim^Y \in R^N$ *takes* Y *to the top from* $\succsim$ if $\succsim$ and $\succsim^Y$ agree on Y and $y \succ_i^Y z$ for all i whenever $y \in Y$ and $z \notin Y$.

A *social choice correspondence* is a mapping $f : (2^X \setminus \emptyset) \times R^N \to 2^X \setminus \emptyset$ such that $f(Y, \succsim) \subseteq Y$ for all $Y \subseteq X$ and all $\succsim \in R^N$. We impose the following axioms on a social choice correspondence:

Definitions 2.1 (Main Axioms)**.** A social choice correspondence f is:

Weakly Pareto Optimal (WP) if $f(Y, \succsim) = \{y\}$ whenever y is uniquely weakly Pareto dominant in $Y \subseteq X$ (i.e., for all $y' \in Y \setminus \{y\}$, $y \succsim_i y'$ for all $i \in N$ with at least one individual having a strict preference);

Independent of Infeasible Alternatives (IIA) if $f(Y, \succsim) = f\left(Y, \succsim'\right)$ whenever $\succsim$ and $\succsim'$ agree on $Y \subseteq X$;

Independent of Losing Alternatives (ILA) if $f(Y, \succsim) = f(X, \succsim) \cap Y$ whenever the intersection is non-empty.

Notice that ILA puts restrictions only when the subset Y has common elements with the choices out of X, the set of *all* social alternatives. Thus it is weaker than Arrow's Choice Axiom (Arrow, 1959), which replaces the X in the definition of ILA with any $Z \supseteq Y$.[2]

One may wonder how our axioms relate to Monotonicity (also known as Strong Positive Association), defined for correspondences[3] as follows:

Definition 2.2 (Monotonicity)**.** A social choice correspondence f is *monotonic* if $Y \subseteq f(X, \succsim') \subseteq f(X, \succsim)$ whenever $Y \subseteq f(X, \succsim)$ and $y \succsim_i x$ implies $y \succsim'_i x$ (with $y \succ_i x$ implies $y \succ'_i x$) for all $y \in Y$, all $x \in X$ and all $i \in N$.

As a trivial corollary of our main theorem, any social choice correspondence that is WP, IIA and ILA is monotonic on the universal preference domain. Yet this implication is not true on restricted preference domains, as the following example illustrates.

Example 2.3. There are 3 alternatives, $X = \{x, y, z\}$ and 3 individuals, $N = \{1, 2, 3\}$. The preference domain admits only two strict preference profiles, $\succ$ and $\succ'$, as depicted in Figure 1(a). Note that $\succ'$ differs from $\succ$ only by moving x above y for individual 3. The

[2]Arrow's Choice Axiom has also been known as Independence of Nonoptimal Alternatives (Karni and Schmeidler, 1976), Strong Stability (Campbell, 1979) and the Weak Axiom of Revealed Preference (Grether and Plott, 1982).

[3]This definition coincides with that in Muller and Satterthwaite (1977) when f is a function.

$\succ$			$\succ'$		
1	2	3	1	2	3
x	y	z	x	y	z
y	z	y	y	z	x
z	x	x	z	x	y

(a) Preference Profiles

Y	$f(Y, \succ)$	$f(Y, \succ')$
X	x	y
$\{x, y\}$	x	y
$\{x, z\}$	x	x
$\{y, z\}$	y	y

(b) Social Choices

Figure 1. Preferences and Social Choices for Example 2.3

social choice function given by Figure 1(b) satisfies WP, IIA and ILA but not monotonicity. To see this, first note that WP has no bite in this example. The two profiles $\succ$ and $\succ'$ agree only on $\{x, z\}$ and $\{y, z\}$. IIA is satisfied since choices from these subsets are equal across preference profiles. ILA is satisfied since for all subsets Y containing $x = f(X, \succ)$, $f(Y, \succ) = x$; and similarly for all subsets Y containing $y = f(X, \succ')$, $f(Y, \succ') = y$. Yet monotonicity is violated as $f(X, \succ) = x$ and the ranking of x improves from $\succ$ to $\succ'$ but $f(X, \succ') = y$.

For any subset of alternatives $Y \subseteq X$ and any individual preference $\succsim_i \in R$, let

$$T(Y, \succsim_i) = \{y \in Y : y \succsim_i y' \text{ for all } y' \in Y\}$$

be the "top set" — the set of favorite alternatives — within Y according to $\succsim_i$. Let $\pi = (\pi_1, \ldots, \pi_N)$ be a permutation of the set of all individuals. Write also $\pi^k = (\pi_1, \ldots, \pi_k)$ for any $k \leq N$. Given a preference profile $\succsim$ and π^k, write $\succsim^{\pi^k} = \left(\succsim_{\pi_1}, \ldots, \succsim_{\pi_k}\right)$ as the preferences of individuals in π^k. For all subsets Y and all preference profiles $\succsim$, define the kth iteration of the top set operator as

$$T^0(Y) = Y$$

$$T^k(Y, \succsim^{\pi^k}, \pi^k) = T(T^{k-1}(Y, \succsim^{\pi^{k-1}}, \pi^{k-1}), \succsim_{\pi_k}).$$

As a shorthand, write $Y_k = T^k(Y, \succsim^{\pi^k}, \pi^k)$ when the preference profile and π^k are clear from the context.

Definition 2.4 (Serial Dictatorship). A social choice correspondence f is *serially dictatorial* if there exists a permutation of individuals π and a tie-breaking rule $\rho \in R$ such that

$$f(Y, \succsim) = T\left(T^N(Y, \succsim, \pi), \rho\right) \quad \text{for all } Y \subseteq X, \text{ all } \succsim \in R^N.$$

If the preference domain admits only strict preferences (i.e., the preference domain is P^N), serial dictatorship implies the existence of an individual $i \in N$ such that $f(Y, \succ) = T(Y, \succ_i)$

Type x	Type y	Type xy
x	y	$x\,y$
$\cdot$	$\cdot$	$\cdot$
y	$\cdot$	$\cdot$
$\cdot$	x	$\cdot$
$\cdot$	$\cdot$	$\cdot$

(a) Profile $\succsim$

Type x	Type y	Type xy
x	y	$x\,y$
$\cdot$	x	$\cdot$
y	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$

(b) Profile $\succsim'$

Type x	Type y	Type xy
x	y	$x\,y$
y	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$
$\cdot$	x	$\cdot$
$\cdot$	$\cdot$	$\cdot$

(c) Profile $\succsim''$

FIGURE 2. Preference Profiles for Proof of Lemma 3.3

for all $Y \subseteq X$ and all $\succ \in P^N$. In this case we say the social choice correspondence (in fact, function) f is *dictatorial*.

3. Main Theorem

Theorem 3.1. *Any social choice correspondence that is weakly Pareto optimal, independent of infeasible alternatives and independent of losing alternatives is serially dictatorial.*

The rest of this section contains the proof. Section 3.1 gives two preliminary lemmata. They will be used in Section 3.2 and 3.3, which construct the permutation π and the tie-breaking rule ρ, respectively. As a corollary of the argument in Section 3.2, if the preference domain is P^N, any social choice correspondence that is WP, IIA and ILA is dictatorial.

3.1. **Preliminary Results.**

We first give two preliminary results for the proof of the main theorem.

Lemma 3.2. *If the social choice correspondence f satisfies WP and ILA, then $x \notin f(X, \succsim)$ whenever x is weakly Pareto dominated at $\succsim$ (i.e., there exists a $y \in X$ such that $y \succsim_i x$ for all $i \in N$ with at least one strict preference).*

Proof. Suppose y weakly Pareto dominates x at $\succsim$. WP implies $f(\{x, y\}, \succsim) = \{y\}$. Thus $x \notin f(X, \succsim)$ or else ILA will be violated. □

Lemma 3.3. *Let f be a WP, IIA and ILA social choice correspondence and $\succsim$ be a preference profile. Let $S \subseteq N$ be the set of individuals who are not indifferent between all alternatives at $\succsim$. Then*

$$f(X, \succsim) \subseteq \bigcup_{i \in S} T(X, \succsim_i)$$

whenever $\left|\bigcup_{i \in S} T(X, \succsim_i)\right| \leq 2$.

Proof. The case of $\left|\bigcup_{i \in S} T(X, \succsim_i)\right| = 1$ follows from WP. So let $\bigcup_{i \in S} T(X, \succsim_i) = \{x, y\}$ and suppose by contradiction that $z \in f(X, \succsim)$ for some $z \neq x, y$.

Since $\bigcup_{i\in S} T(X, \succsim_i) = \{x, y\}$, there are three types of individuals in S: those whose favorite is x (Type x), y (Type y) and those whose favorites are x and y (Type xy) (see Figure 2(a)). Construct a new preference profile $\succsim'$ by moving, for all Type y individuals, the ranking of x up to just below y, keeping all else unchanged (Figure 2(b)). Similarly, construct $\succsim''$ by moving y to just below x for all Type x individuals (Figure 2(c)). Observe that

(1) $\succsim$, $\succsim'$ and $\succsim''$ agree on $\{x, y\}$;
(2) $\succsim$ and $\succsim'$ agree on $\{y, z\}$; and
(3) $\succsim$ and $\succsim''$ agree on $\{x, z\}$.

Since $z \in f(X, \succsim)$, ILA requires $z \in f(\{y, z\}, \succsim)$. By observation (2) and IIA $z \in f\left(\{y, z\}, \succsim'\right)$. Meanwhile, all alternatives other than x and y are weakly Pareto dominated by x at $\succsim'$. Lemma 3.2 says none of them can be chosen out of X at $\succsim'$. This means $y \notin f(X, \succsim')$. Otherwise $z \in f\left(\{y, z\}, \succsim'\right) \neq \{y\} = f\left(X, \succsim'\right) \cap \{y, z\}$, which violates ILA. Therefore $f\left(X, \succsim'\right) = \{x\}$. By a similar argument using the subset $\{x, z\}$, $f\left(X, \succsim''\right) = \{y\}$.

Now ILA requires $f\left(\{x, y\}, \succsim'\right) = \{x\}$ and $f\left(\{x, y\}, \succsim''\right) = \{y\}$. This contradicts IIA in light of observation (1). □

3.2. **Serial Dictators.** Given a WP, IIA and ILA social choice correspondence f, we construct in this subsection the permutation of individuals π such that for all $k \geq 0$,

$$f(Y, \succsim) \subseteq T^k(Y, \succsim^{\pi^k}, \pi^k) \quad \text{for all } Y \subseteq X, \text{ all } \succsim \in R^N. \tag{1}$$

The case for $k = 0$ follows by definition. Now suppose π^{k-1} is defined and Equation (1) holds for $k-1$. We construct π_k that satisfies Equation (1) for k in 3 steps: Step 1 identifies a group of individuals containing the π_k we look for. Step 2 shows that whenever this group of individuals have the same preferences over $Y \subseteq X$, the social choice out of Y is always a subset of their favorites in $Y_{k-1} = T^{k-1}\big(Y, \succsim^{\pi^{k-1}}, \pi^{k-1}\big)$. Step 3 shrinks this group to a singleton, giving us the desired π_k. Step 4 is given for cases when the above 3-step proof is infeasible (this happens if $N < 3$ or $k > N-2$).

Step 1. If $N < 3$ or $k > N-2$ proceed directly to Step 4. Otherwise, construct a preference profile $\succsim^*$ where (1) $\pi_1, \ldots \pi_{k-1}$ are indifferent between all alternatives; (2) remaining individuals have one of the Condorcet preferences in Figure 3; and (3) all individual preferences in the Condorcet cycle are assigned to at least one individual.

Since all alternatives other than x, y and z are weakly Pareto dominated by each of them, by Lemma 3.2, $f(X, \succsim^*) \subseteq \{x, y, z\}$. Without loss assume $x \in f(X, \succsim^*)$. Let $S \subseteq N$ be the set of individuals who have the Type x preference in the Condorcet cycle.

Step 2. We show whenever individuals in S have the same preference over $Y \subseteq X$, the social choice from Y is a subset of their favorites in $Y_{k-1} = T^{k-1}(Y, \succsim^{\pi^{k-1}}, \pi^{k-1})$. We break

Type x	Type y	Type z
x	y	z
y	z	x
z	x	y
$\vdots$	$\vdots$	$\vdots$

FIGURE 3. Condorcet Cycle

Group S	Type y	Type z
x	z	z
y	y	x
z	x	y
$\vdots$	$\vdots$	$\vdots$

(a) Profile $\succsim^1$

Group S	Not S
x	z
y	$\cdot$
z	$\cdot$
$\vdots$	$\vdots$

(b) Profile $\succsim^2$

Group S	Not S
y	z
x	$\cdot$
z	$\cdot$
$\vdots$	$\vdots$

(c) Profile $\succsim^3$

FIGURE 4. Preference Profiles for Step 2.1

this step into three smaller steps: Step 2.1 proves this for $Y = \{x, y\}$ (the top 2 alternatives in the Type x Condorcet preference); Step 2.2 for any two-element subset Y; and Step 2.3 for any subset $Y \subseteq X$.

Step 2.1. We show whenever individuals in S have the same preferences over $\{x, y\}$, the social choice from $\{x, y\}$ is a subset of their favorites in $\{x, y\}_{k-1}$. This is trivial if $\{x, y\}_{k-1}$ is a singleton or if $x \sim_i y$ for all $i \in S$.

Construct a preference profile $\succsim^1$ from $\succsim^*$ by switching the rankings of y and z of all Type y individuals in the Condorcet cycle, keeping everything else unchanged (Figure 4(a)). Notice that $\succsim^*$ and $\succsim^1$ agree on $\{x, z\}$. Since x is chosen out of X at the Condorcet profile, IIA and ILA implies $x \in f\left(\{x, z\}, \succsim^1\right) = f\left(\{x, z\}, \succsim^*\right)$.

Now take a preference profile $\succsim^2$ where (1) $\pi_1, \ldots, \pi_{k-1}$ are indifferent between all alternatives; (2) all individuals in S have the Type x Condorcet preference; and (3) all other individuals' favorite is z (Figure 4(b)). Again $\succsim^1$ and $\succsim^2$ agree on $\{x, z\}$ so by IIA $x \in f\left(\{x, z\}, \succsim^2\right)$. Meanwhile, Lemma 3.3 requires $f\left(X, \succsim^2\right) \subseteq \{x, z\}$. By ILA $x \in f\left(X, \succsim^2\right)$. Applying ILA once more gives $f\left(\{x, y\}, \succsim^2\right) = \{x\}$. Notice that $\succsim^2$ puts no restriction on the relative ranking between x and y for individuals not in $\{\pi_1, \ldots, \pi_{k-1}\} \cup S$. IIA therefore implies $f\left(\{x, y\}, \succsim\right) = \{x\}$ whenever $\pi_1, \ldots, \pi_{k-1}$ are indifferent between x and y and all individuals in S strictly prefer x to y.

Next construct a preference profile $\succsim^3$ from $\succsim^2$ by switching the positions of x and y in $\succsim^2_S$ and keeping everything else unchanged (Figure 4(c)). Since $\succsim^2$ and $\succsim^3$ agree on $\{x, z\}$, IIA requires $x \in f\left(\{x, z\}, \succsim^3\right)$. However, Lemma 3.3 dictates $f\left(X, \succsim^3\right) \subseteq \{y, z\}$. This means

$\succsim^4_S$	$\succsim^4_{-S}$
w	y
z	$\cdot$
x	$\cdot$
y	$\cdot$
$\vdots$	$\vdots$

FIGURE 5. Preference Profile for Step 2.2

$z \notin f\left(X, \succsim^3\right)$ (otherwise ILA would be violated). Hence $f\left(X, \succsim^3\right) = \{y\}$. Applying ILA once more gives $f\left(\{x, y\}, \succsim^3\right) = \{y\}$. Since $\succsim^3$ puts no restriction on the relative ranking between x and y for individuals not in $\{\pi_1, \ldots, \pi_{k-1}\} \cup S$, IIA implies $f(\{x, y\}, \succsim) = \{y\}$ whenever $\pi_1, \ldots, \pi_{k-1}$ are indifferent between x and y and all individuals in S strictly prefers y to x.

Step 2.2. We show whenever individuals in S have the same preferences over a two-element subset $\{w, z\} \subseteq X$, the social choice from $\{w, z\}$ is a subset of their favorites in $\{w, z\}_{k-1}$. The statement is trivial if $\{w, z\}_{k-1}$ is a singleton or if $w \sim_i z$ for all $i \in S$. There is also nothing to prove if $\{w, z\} = \{x, y\}$. So without loss assume $y \notin \{w, z\}$.

Create a preference profile $\succsim^4$ where (1) $\pi_1, \ldots, \pi_{k-1}$ are indifferent between all alternatives; (2) all individuals in S have w as their unique favorite and strictly prefer x to y; and (3) all other individuals' unique favorite is y (Figure 5). By Lemma 3.3, $f\left(X, \succsim^4\right) \subseteq \{w, y\}$. However, all individuals in S strictly prefer x to y. By Step 2.1 $f(\{x, y\}, \succsim) = \{x\}$. ILA then requires $f\left(X, \succsim^4\right) = \{w\}$, which implies $f\left(\{w, z\}, \succsim^4\right) = \{w\}$. Since $\succsim^4$ puts no restriction on the relative ranking between w and z for individuals not in $\{\pi_1, \ldots, \pi_{k-1}\} \cup S$, IIA implies $f(\{w, z\}, \succsim) = \{w\}$ whenever $\pi_1, \ldots, \pi_{k-1}$ are indifferent between w and z and all individuals in S strictly prefer w to z.

Switching the names of w and z gives $f(\{w, z\}, \succsim) = \{z\}$ whenever $\pi_1, \ldots, \pi_{k-1}$ are indifferent between w and z and all individuals in S strictly prefer z to w.

Step 2.3. We show whenever individuals in S have the same preferences over $Y \subseteq X$, the social choice from Y is a subset of their favorites in Y_{k-1}.

Given a preference profile $\succsim$ at which individuals in S have the same preferences over Y, obtain $\succsim^Y$ by taking Y to the top from $\succsim$ (see Section 2). If $k > 1$, the induction hypothesis (Equation (1)) ensures $f\big(X, \succsim^Y\big) \subseteq Y_{k-1}$. Otherwise, $f\big(X, \succsim^Y\big) \subseteq Y = Y_0$ since all alternatives not in Y are Pareto dominated (Lemma 3.2). Moreover, $\pi_1, \ldots, \pi_{k-1}$ are indifferent between all alternatives in Y_{k-1}.

Now if y is group S's favorite in Y_{k-1} and $y' \in Y_{k-1}$ is not, Step 2.2 requires $f\big(\{y, y'\}, \succsim^Y\big) = \{y\}$. ILA means $y' \notin f\big(X, \succsim^Y\big)$. Thus $f\big(X, \succsim^Y\big)$ is a subset of group S's favorites in Y_{k-1}

at $\succsim^Y$. Applying ILA once more means the social choice set out of Y at $\succsim^Y$ is also a subset of group S's favorites in Y_{k-1}. Complete the proof by noting $f(Y, \succsim) = f\big(Y, \succsim^Y\big)$ (IIA) and group S's favorites in Y_{k-1} are the same at $\succsim$ and $\succsim^Y$.

Remark on Step 2. Step 2 implies the choice out of X at the Condorcet profile is a singleton. For if not there will be at least two disjoint subsets of individuals that can get their favorites out of Y_{k-1} whenever preferences over Y within each group are the same. Contradictions arise when the preferences of these groups conflict with each other.

Step 3. If S is a singleton, letting $\pi_k = S$ completes our induction step. Otherwise, construct a preference profile $\succsim^{**}$ where (1) $\pi_1, \ldots \pi_{k-1}$ are indifferent between all alternatives; (2) all individuals in S have either the Type x or Type y Condorcet preference (with both types assigned to at least one individual); and (3) all remaining individuals get the Type z Condorcet preference.

Notice that only Type z individuals rank z above y in the Condorcet cycle (see Figure 3). Since $y \succ_i^{**} z$ for all $i \in S$, Step 2 implies $z \notin f(\{y, z\}, \succsim^{**})$. By ILA and Lemma 3.2, $f(X, \succsim^{**}) \subseteq \{x, y\}$.

Define $S_2 \subset S$ as the set of individuals whose favorite is chosen out of X at $\succsim^{**}$. Repeat Step 2. Proceeding this way gives us a strictly decreasing sequence of subsets $S_n \subset \cdots \subset S_2 \subset S$ such that each S_n group gets their favorites out of Y_{k-1} whenever they have the same preference over Y. Since N is finite, S_n must be a singleton at some finite n. Setting $\pi_k = S_n$ completes our induction proof.

Step 4. Step 1 is infeasible when there are two or fewer individuals not assigned to the permutation π (this happens when $N < 3$ or $k > N - 2$). This step takes care of such cases.

When there are only two individuals left, construct the Condorcet profile $\succsim^*$ as in Step 1 without using the Type z preference. Since only preference Type z ranks z above y in the Condorcet cycle, z is weakly Pareto dominated by y at such $\succsim^*$. Lemma 3.2 requires $f\left(X, \succsim^*\right) \subseteq \{x, y\}$. Proceed with the same argument as above.

When there is only one individual left, construct π by appending the last individual to π^{N-1}. Given $Y \subseteq X$ and $\succsim \in R^N$, construct $\succsim^Y$ by taking Y to the top from $\succsim$. The induction hypothesis ensures $f\big(X, \succsim^Y\big) \subseteq Y_{N-1}$. Now if y is π_N's favorite in Y_{N-1} and $y' \in Y_{N-1}$ is not, y weakly Pareto dominates y'. By WP and ILA, the social choice out of X at $\succsim^Y$ is a subset of π_N's favorites in Y_{N-1} at $\succsim^Y$. Applying ILA once more the social choice set out of Y at $\succsim^Y$ is also a subset of π_N's favorites in Y_{N-1}. Complete the proof by noting $f\big(Y, \succsim\big) = f\big(Y, \succsim^Y\big)$ (IIA) and π_N's favorites in Y_{N-1} at $\succsim^Y$ are the same as $T^N\big(Y, \succsim, \pi\big)$.

Remark. If the preference domain is P^N, the $k = 1$ step does not require any indifference. Since the existence of π_1 implies dictatorship when only strict preferences are admitted, we have the following corollary:

Corollary 3.4. *Any social choice correspondence $f : (2^X \setminus \emptyset) \times P^N \to (2^X \setminus \emptyset)$ that is weakly Pareto optimal, independent of infeasible alternatives and independent of losing alternatives is dictatorial.*

3.3. **Tie-breaking Rule.** It remains to find the tie-breaking rule $\rho \in R$ such that

$$f(Y, \succsim) = T(T^N(Y, \succsim, \pi), \rho) \quad \text{for all } Y \subseteq X, \text{ all } \succsim \in R^N.$$

So let $\sim$ denote the preference profile in which all individuals are indifferent between all alternatives. Given a social choice correspondence f, define a binary relation ρ on X such that for all $x, y \in X$,

$$x \, \rho \, y \quad \text{if and only if} \quad x \in f(\{x, y\}, \sim).$$

Claim 3.5. The binary relation ρ is complete and transitive. That is, $\rho \in R$.

Proof. Completeness: For any $x, y \in X$, $f(\{x, y\}, \sim) \neq \emptyset$.[4] Thus it must either be $x \, \rho \, y$ or $y \, \rho \, x$.

Transitivity: Take any $x, y, z \in X$ and suppose $x \, \rho \, y$ and $y \, \rho \, z$. Construct $\succsim^\circ$ by taking $\{x, y, z\}$ to the top from $\sim$. IIA implies x and y are chosen out of $\{x, y\}$ and $\{y, z\}$ respectively at $\succsim^\circ$. We claim $x \in f(X, \succsim^\circ)$. Suppose not, then since $x \in f(\{x, y\}, \succsim^\circ)$ ILA requires $y \notin f(X, \succsim^\circ)$. Applying ILA once more means $z \notin f(X, \succsim^\circ)$. But this contradicts Lemma 3.2 since all alternatives other than x, y and z are strictly Pareto dominated at $\succsim^\circ$. Now by ILA and IIA we have $x \in f(\{x, z\}, \succsim^\circ) = f(\{x, z\}, \sim)$. Therefore $x \, \rho \, z$. □

Fix $Y \subseteq X$ and $\succsim \in R^N$. Construct $\succsim^Y$ by taking Y to the top from $\succsim$. Our argument in Section 3.2 ensures $f\big(X, \succsim^Y\big) \subseteq Y_N$.

We first show $f\big(Y, \succsim^Y\big) \subseteq T\big(Y_N, \rho\big)$. If y is a favorite in Y_N according to ρ and $y' \in Y_N$ is not, $f(\{y, y'\}, \sim) = \{y\}$. Since both y and y' are in Y_N, $\sim$ and $\succsim^Y$ agree on $\{y, y'\}$. IIA requires $f\big(\{y, y'\}, \succsim^Y\big) = \{y\}$. ILA then means y' cannot be chosen out of X at $\succsim^Y$. Applying ILA once more gives the desired set inclusion.

Next we show $f\big(Y, \succsim^Y\big) \supseteq T\big(Y_N, \rho\big)$. If y is a favorite in Y_N according to ρ and y' is a social choice out of X at $\succsim^Y$, then $y \in f(\{y, y'\}, \sim)$. Since both y and y' are in Y_N, $\sim$ and $\succsim^Y$ agree on $\{y, y'\}$. IIA requires $y \in f\big(\{y, y'\}, \succsim^Y\big)$. ILA means y is also chosen out of X at $\succsim^Y$. Applying ILA once more gives the desired set inclusion.

Finally, by IIA we get $f\big(Y, \succsim\big) = f\big(Y, \succsim^Y\big) = T\big(Y_N, \rho\big)$. This completes the proof of Theorem 3.1.

[4] $f(\{x, x\}, \sim) = f(\{x\}, \sim) = \{x\}$ for all $x \in X$.

4. Arrow's Impossibility Theorem

In Arrow (1963) framework, a *social welfare function* is a mapping $F : R^N \to R$. Given a permutation of individuals $\pi = (\pi_1, \ldots, \pi_N)$ and a weak preference $\rho \in R$, write $\succsim_{\pi_{N+1}} = \rho$ and define the lexicographic ordering $L(\succsim, \pi, \rho) \in R$ such that $x \; L(\succsim, \pi, \rho) \; y$ if and only if whenever $y \succ_{\pi_k} x$, there exists an $l < k$ such that $x \succ_{\pi_l} y$.

Definitions 4.1. A social welfare function F is:

Weakly Pareto if x is strictly preferred to y according to $F(\succsim)$ whenever x weakly Pareto dominates y under $\succsim$;

Independent of Irrelevant Alternatives if $F(\succsim)$ and $F(\succsim')$ agree on $\{x, y\}$ whenever $\succsim$ and $\succsim'$ agree on the same set;

Serially Dictatorial if there exists a permutation of individuals π and a tie-breaking rule $\rho \in R$ such that $F(\succsim) = L(\succsim, \pi, \rho)$ for all $\succsim \in R^N$.

If the preference domain is P^N, serial dictatorship implies the existence of an individual $i \in N$ such that $F(\succ) = \succ_i$ for all $\succ \in P^N$. In this case we say F is *dictatorial*.

Given a social welfare function F, define the induced social choice correspondence as follows: for all subsets of alternatives $Y \subseteq X$, all preference profiles $\succsim \in R^N$,

$$f(Y, \succsim) = T(Y, F(\succsim)). \tag{2}$$

The following proposition relates properties of the social welfare function and those of the induced social choice correspondence.

Proposition 4.2. *If the social welfare function F is weakly Pareto and independent of irrelevant alternatives, then the social choice correspondence defined in Equation* (2) *satisfies WP, IIA and ILA.*

Proof. WP: Let y be uniquely weakly Pareto dominant in $Y \subseteq X$ at $\succsim \in R^N$. Since F is weakly Pareto, y is strictly preferred to all other $y' \in Y$ according to $F(\succsim)$. Equation (2) implies $f(Y, \succsim) = \{y\}$.

IIA: Suppose $\succsim$ and $\succsim'$ agree on $Y \subseteq X$. Independence of irrelevant alternatives means $F(\succsim)$ and $F(\succsim')$ agree on all pairs $y, y' \in Y$. By Equation (2), $f(Y, \succsim) = f\left(Y, \succsim'\right)$.

ILA: Suppose $f(X, \succsim) \cap Y \neq \emptyset$. If $x \in f(X, \succsim) \cap Y$ and $y \in Y$ is not, then x is strictly preferred to y according to $F(\succsim)$. By Equation (2), $x \in f(Y, \succsim)$ and y is not. Thus $f(Y, \succsim) = f(X, \succsim) \cap Y$. □

By Theorem 3.1, f is serially dictatorial. Thus there exists a permutation of individuals π and a tie-breaking rule ρ such that $f(\{x, y\}, \succsim) = T(T^N(\{x, y\}, \succsim, \pi), \rho)$ for all $x, y \in X$. Equation (2) implies $x \in f(\{x, y\}, \succsim)$ is equivalent to $x \; F(\succsim) \; y$. Meanwhile, $x \in$

$T(T^N(\{x,y\},\succsim,\pi),\rho)$ is equivalent to $x\,L(\succsim,\pi,\rho)\,y$. Arrow's Impossibility Theorem follows immediately:

Corollary 4.3 (Arrow's Impossibility Theorem)**.** *Any social welfare function $F : R^N \to R$ (respectively, $F : P^N \to R$) that is weakly Pareto and independent of irrelevant alternatives is serially dictatorial (dictatorial).*

5. Muller-Satterthwaite Theorem and Implementation

To avoid confusion with our social choice correspondence, we call a social choice correspondence defined only on the set of all social alternatives an *overall social choice correspondence*, which is a mapping $f^* : R^N \to 2^X \setminus \emptyset$.

Definitions 5.1. An overall social choice function f^* is
Onto if for every $x \in X$ there is a $\succsim \in R^N$ such that $f^*(\succsim) = x$;
Strategy-Proof if $f^*(\succsim) \succsim_i f^*(\succsim'_i, \succsim_{-i})$ for all $\succsim$, all i and all $\succsim'_i$.

Definitions 5.2. An overall social choice correspondence f^* is
Weakly Pareto if $f^*(\succsim) = \{x\}$ whenever x is uniquely weakly Pareto dominant at $\succsim$;
Monotonic if $Y \subseteq f^*(\succsim') \subseteq f^*(\succsim)$ whenever $Y \subseteq f^*(\succsim)$ and $y \succsim_i x$ implies $y \succsim'_i x$ (with $y \succ_i x$ implies $y \succ'_i x$) for all $y \in Y$, all $x \in X$ and all $i \in N$;
Serially Dictatorial if there exists a permutation of individuals π and a tie-breaking rule $\rho \in R$ such that $f^*(\succsim) = T(T^N(X,\succsim,\pi),\rho)$ for all $\succsim \in R^N$.

If the preference domain is P^N, Serial Dictatorship implies the existence of an individual $i \in N$ such that $f^*(\succ) = T_i(X,\succ_i)$ for all $\succ \in P^N$. In this case we say f^* is *dictatorial.*

Lemma 5.3 (Muller-Satterthwaite)**.** *Let the preference domain be P^N. Any onto and strategy-proof overall social choice function is weakly Pareto and monotonic.*

Proof. See Muller and Satterthwaite (1977). □

Whenever implementation is concerned, we restrict f^* to be a function. A *mechanism* $M = (A, g)$ consists of an action profile space $A = \prod_{i \in N} A_i$ and an outcome function $g : A \to X$. Let $\mathcal{M}$ be the set of all mechanisms. An *equilibrium concept* is a mapping $E : \mathcal{M} \times R^N \to 2^A$. The *equilibrium outcome* correspondence associated with E is given by

$$O_E(M,\succsim) = \{x \in X : \exists\, a \in E(M,\succsim) \text{ s.t. } g(a) = x\}.$$

An overall social choice function f^* is *implemented* via equilibrium concept E and mechanism M if $O_E(M,\succsim) = f^*(\succsim)$ for all $\succsim \in R^N$.

Definitions 5.4 (Jackson-Srivastava)**.** Let $M = (A, g)$ be a mechanism. Take $a \in A$, $\succsim \in R^N$ and two groups of individuals $S, S' \subseteq N$. The action profile $a' = (a'_S, a_{-S})$ is an (S, S')-*improvement from* a *at* $\succsim$ if $g(a') \succsim_i g(a)$ for all $i \in S'$ with at least one strict preference. A pair of groups (S, S') is *responsive* with respect to mechanism M under equilibrium concept E if $a \notin E(M, \succsim)$ whenever there exists an (S, S')-improvement from a at $\succsim$.

An equilibrium concept E satisfies *direct breaking* with respect to M if whenever $O_E(M, (\succsim'_i, \succsim_{-i})) \neq O_E(M, \succsim)$, for each $a \in E(M, \succsim)$ there exists a responsive pair of groups (S, S') with respect to M under E and an (S, S')-improvement from a at $(\succsim'_i, \succsim_{-i})$.

Jackson and Srivastava (1996) show that iterative elimination of strictly dominated strategies, Nash equilibrium and Strong equilibrium satisfy direct breaking with respect to all mechanisms, while undominated strategies satisfies direct breaking with respect to all bounded mechanisms[5].

The next lemma adapts Jackson and Srivastava's result to the weak preference domain.

Lemma 5.5. *Suppose mechanism M implements an overall social choice function f^* via equilibrium concept E. If E satisfies direct breaking with respect to M, then f^* is monotonic.*

Proof. Let $f^*(\succsim) = x$ and consider $\succsim'$ such that $x \succsim_i y$ implies $x \succsim'_i y$ (with $x \succ_i y$ implies $x \succ'_i y$) for all $y \in X$ and all $i \in N$. We claim $O_E(M, (\succsim'_i, \succsim_{-i})) = x$. Suppose not and let $a \in E(M, \succsim)$. Since E satisfies direct breaking, there exists an (S, S')-improvement $a' = (a'_S, a_{-S})$ from a at $(\succsim'_i, \succsim_{-i})$. If $j \in S' \setminus \{i\}$, this means $g(a') \succsim_j x$ with at least one j having a strict preference if $i \notin S'$. In addition, $g(a') \succsim'_i x$ implies $g(a') \succsim_i x$ (with strict preference implies strict preference). Therefore a' is also an (S, S')-improvement from a at $\succsim$. This contradicts (S, S') being responsive. Repeat the same argument with another individual starting at $(\succsim'_i, \succsim_{-i})$. Proceeding this way we reach $\succsim'$ and the social choice remains x. $\square$

One implication of weak Pareto optimality and monotonicity is that a Pareto dominated alternative can never be chosen.

Lemma 5.6. *If an overall social choice correspondence f^* is weakly Pareto and monotonic, then $x \notin f^*(\succsim)$ whenever x is weakly Pareto dominated at $\succsim$.*

Proof. Suppose by contradiction that y weakly Pareto dominates x at $\succsim$ but $x \in f^*(\succsim)$. Construct $\succsim'$ by taking $\{x, y\}$ to the top from $\succsim$. Monotonicity requires $x \in f^*(\succsim')$ but weak Pareto optimality requires $f^*(\succsim') = \{y\}$. $\square$

[5] Jackson and Srivastava (1996) admit only strict preferences but their proof that various solution concepts satisfy direct breaking extends immediately. When weak preference is allowed, we require $g(a_i, a_{-i}) \succ_i g(a'_i, a_{-i})$ whenever $g(a_i, a_{-i}) \neq g(a'_i, a_{-i})$ in addition to the standard definition for a_i to dominate a'_i.

Given an overall social choice correspondence f^*, extend it to our unified domain by defining for all $Y \subseteq X$ and all $\succsim \in R^N$

$$f(Y, \succsim) = f^*(\succsim^Y), \tag{3}$$

where $\succsim^Y$ is a preference profile that takes Y to the top from $\succsim$. Lemma 5.6 guarantees $f(Y, \succsim) \subseteq Y$ so f is a valid social choice correspondence[6]. In addition, if f^* is a function, so is f.

Proposition 5.7. *If an overall social choice correspondence f^* is weakly Pareto and monotonic, then the social choice correspondence f defined in Equation* (3) *satisfies WP, IIA and ILA.*

Proof. WP: Follows from Lemma 5.6.

IIA: Suppose $\succsim, \succsim' \in R^N$ agree on $Y \subseteq X$. Then $\succsim^Y$ and $\succsim'^Y$ differ only by the ranking among alternatives not in Y, which are all ranked below $f^*(\succsim^Y) \subseteq Y$ in both $\succsim^Y$ and $\succsim'^Y$. Monotonicity and Equation (3) require $f(Y, \succsim) = f^*(\succsim^Y) = f^*(\succsim'^Y) = f(Y, \succsim')$.

ILA: Suppose $f^*(\succsim) = f(X, \succsim)$ has a non-empty intersection with $Y \subseteq X$. Then for all $y \in f^*(\succsim) \cap Y$ and all $x \in X$, all individuals (strictly) prefer y to x at $\succsim^Y$ if they (strictly) prefer y to x at $\succsim$. Monotonicity requires $f^*(\succsim) \cap Y \subseteq f^*(\succsim^Y) \subseteq f^*(\succsim)$. Meanwhile, Lemma 5.6 means $f^*(\succsim^Y) \subseteq Y$ so $f^*(\succsim^Y) \subseteq f^*(\succsim) \cap Y$. By Equation (3), $f(Y, \succsim) = f^*(\succsim^Y) = f(X, \succsim) \cap Y$. $\square$

A generalized version of the Muller-Satterthwaite Theorem follows immediately:

Corollary 5.8 (Muller-Satterthwaite Theorem)**.** *Any overall social choice correspondence $f^* : R^N \to 2^X \setminus \emptyset$ (respectively, $f^* : P^N \to 2^X \setminus \emptyset$) that is weakly Pareto and monotonic is serially dictatorial (dictatorial).*

Together with Lemma 5.3, this gives the Gibbard-Satterthwaite Theorem when all preferences are strict and f^* is a function:

Corollary 5.9 (Gibbard-Satterthwaite Theorem)**.** *Any onto and strategy-proof overall social choice function $f^* : P^N \to X$ is dictatorial.*

One can then extend it to weak preference domains using the techniques in Barberá and Peleg (1990).

The Jackson-Srivastava characterization on equilibrium concepts that lead to impossibility theorems[7] is also straight-forward:

[6]If f^* is implemented by a mechanism $M = (A, g)$, this extension does not restrict the range of g. In particular, it does not means M implements $f(Y, \cdot)$ using only alternatives in $Y \subseteq X$.
[7]We thank Matthew Jackson for pointing us to this theorem.

Corollary 5.10 (Jackson and Srivastava (1996)). *Suppose a mechanism M implements a weakly Pareto*[8] *overall social choice function $f^* : R^N \to X$ via equilibrium concept E. Then f^* is serially dictatorial if and only if E satisfies direct breaking with respect to M.*

Proof. If: Follows from Lemma 5.5 and Corollary 5.8.

Only if: Let $\pi = (1, \ldots, N)$ be the sequence of serial dictators and ρ be the tie-breaking rule. Add a dummy individual $N+1$ whose only preference is ρ. Suppose $O_E(M, \succsim) = x$ and $O_E(M, (\succsim_i', \succsim_{-i})) = y \neq x$. Then let $k \geq i$ be the individual who vetoes x under the new preference profile. Notice that k strictly prefers y to x. Thus any $a' \in E(M, (\succsim_i', \succsim_{-i}))$ is a (N, π^k)-improvement from any $a \in E(M, \succsim)$ at the new preference profiles. Serial dictatorship implies (N, π^k) is responsive with respect to M under E.[9] Therefore E satisfies direct breaking with respect to M. □

Due to the remarks after Definitions 5.4, Corollary 5.10 implies any weakly Pareto overall social choice function that is implementable via iterative elimination of strictly dominated strategies, Nash equilibrium, Strong equilibrium or undominated strategies (in a bounded mechanism) is serially dictatorial.

6. Strategic Candidacy

Strategic candidacy concerns the effect of a unilateral withdrawal of candidacy on the election outcome. Hence the social choice is defined on subsets of social alternatives with at least $|X| - 1$ elements. More generally, one can consider social choices defined on some $\mathcal{X} \subseteq 2^X$. Say $\mathcal{X}$ satisfies *k-set feasibility* if for all subsets $Y \subseteq X$, $Y \in \mathcal{X}$ if and only if $|Y| \geq k$. A *voting procedure* is a correspondence $\hat{f} : \mathcal{X} \times R^N \to 2^X \setminus \emptyset$ such that $\hat{f}(Y, \succsim) \subseteq Y$ for all $Y \in \mathcal{X}$ and all $\succsim \in R^N$.[10]

Definitions 6.1. A voting procedure $\hat{f} : \mathcal{X} \times R^N \to 2^X \setminus \emptyset$ satisfies

k-set Feasibility if $\mathcal{X}$ is k-set feasible;

Unanimity if $\hat{f}(Y, \succsim) = \{y\}$ whenever y is uniquely weakly Pareto dominant in $Y \in \mathcal{X}$;

Independence of Irrelevant Alternatives if $\hat{f}(Y, \succsim) = \hat{f}(Y, \succsim')$ whenever $\succsim$ and $\succsim'$ agree on $Y \in \mathcal{X}$;

Arrow's Choice Axiom if $\hat{f}(Y, \succsim) = \hat{f}(Z, \succsim) \cap Y$ whenever $Y, Z \in \mathcal{X}$, $Y \subseteq Z$ and the intersection is non-empty;

[8] Jackson and Srivastava (1996) requires f^* to be onto instead of weakly Pareto. The two assumptions are substitutable when all preferences are strict. With weak preference, however, that f^* is onto and E satisfies direct breaking guarantee only strict but not necessarily weak Pareto optimality of f.

[9] Strictly speaking, this is not true since we add a dummy individual. Nevertheless, this complication arises only when all individuals are indifferent between x and y. One can modify the definition of responsive groups to be completely rigorous.

[10] We do not allow for candidate voters (social alternatives which are also individuals).

Strong Candidate Stability if $\hat{f}(Y, \succsim) = \hat{f}(X, \succsim) \cap Y$ whenever $|Y| \geq |X| - 1$ and the intersection is non-empty;

Serial Dictatorship if there exists a permutation of individuals π and a tie-breaking rule $\rho \in R$ such that $\hat{f}(Y, \succsim) = T(T^N(Y, \succsim, \pi), \rho)$ for all $Y \in \mathcal{X}$, all $\succsim \in R^N$.

If only strict preferences are admitted, serial dictatorship implies the existence of an individual $i \in N$ such that $f(Y, \succ) = T(Y, \succ_i)$ for all $Y \in \mathcal{X}$, all $\succ \in P^N$. In this case we say $\hat{f}$ is *dictatorial*.

Obviously, if $\hat{f}$ is k-set feasible for some $k < |X|$ and satisfies Arrow's choice axiom, $\hat{f}$ is strongly candidate stable.

Lemma 6.2. *If the voting procedure $\hat{f}$ is unanimous, independent of irrelevant alternatives and strongly candidate stable, then $f(X, \succsim) \subseteq Y$ whenever each alternative $y \in Y$ weakly Pareto dominates each $x \notin Y$ at $\succsim$.*

Proof. See Eraslan and McLennan (2004, Lemma 1, p. 41-42). □

Lemma 6.3. *Let $\hat{f}$ be an unanimous, independent of irrelevant alternatives (IIA in this lemma) and strongly candidate stable (SCS) voting procedure. Then $\hat{f}(X, \succsim^Y) = \hat{f}(X, \succsim) \cap Y$ whenever $\succsim^Y$ takes Y to the top from $\succsim$ and the intersection is non-empty.*

Proof. Fix Y, $\succsim$ and $\succsim^Y$. Let $Z = X \setminus Y$. Enumerate the elements of Z as $z_1, \ldots, z_K$. Define $Z_0 = \emptyset$ and $Z_k = \{z_1, \ldots, z_k\}$. Construct $\succsim^0 = \succsim$ and for all $k > 0$ a preference profile $\succsim^k$ such that: (1) $\succsim^k$ and $\succsim^{k-1}$ agree on $X \setminus \{z_k\}$; (2) $\succsim^k$ and $\succsim^Y$ agree on Z_k; and (3) all $z \in Z_k$ are strictly Pareto dominated by all $x \notin Z_k$. By construction, $\succsim^K = \succsim^Y$.

We claim that for all $k \geq 0$,

$$\hat{f}(X, \succsim^k) = \hat{f}(X, \succsim) \setminus Z_k. \tag{4}$$

The case of $k = 0$ is trivial. Suppose Equation (4) holds for $k - 1$. Since $\hat{f}(X, \succsim) \cap Y \neq \emptyset$, the induction hypothesis means $\hat{f}(X, \succsim^{k-1}) \neq \{z_k\}$. Thus

$$\begin{aligned}
\hat{f}(X, \succsim^k) &= \hat{f}(X, \succsim) \cap (X \setminus \{z_k\}) && \text{(Lemma 6.2, using } Y = X \setminus Z_k\text{)} \\
&= \hat{f}(X \setminus \{z_k\}, \succsim^k) && \text{(SCS)} \\
&= \hat{f}(X \setminus \{z_k\}, \succsim^{k-1}) && \text{(IIA)} \\
&= \hat{f}(X, \succsim^{k-1}) \cap (X \setminus \{z_k\}) && \text{(SCS)} \\
&= \left(\hat{f}(X, \succsim) \setminus Z_{k-1}\right) \setminus \{z_k\} && \text{(Induction hypothesis)} \\
&= \hat{f}(X, \succsim) \setminus Z_k.
\end{aligned}$$

Therefore $\hat{f}(X, \succsim^Y) = \hat{f}(X, \succsim) \setminus Z = \hat{f}(X, \succsim) \cap Y$. □

Given a voting procedure $\hat{f}$, extend it to our unified domain by defining for all $Y \subseteq X$ and all $\succsim \in R^N$,

$$f(Y, \succsim) = \hat{f}(X, \succsim^Y), \tag{5}$$

where $\succsim^Y$ is a preference profile that takes Y to the top from $\succsim$. Lemma 6.2 guarantees $f(Y, \succsim) \subseteq Y$ so f is a valid social choice correspondence. Lemma 6.3 and Arrow's choice axiom ensure $\hat{f}(X, \succsim^Y) = \hat{f}(X, \succsim) \cap Y = \hat{f}(Y, \succsim)$ for all $Y \in \mathcal{X}$, so f is indeed an extension of $\hat{f}$.

Proposition 6.4. *If a voting procedure $\hat{f}$ is unanimous, independent of irrelevant alternatives and strongly candidate stable, then the social choice correspondence f defined in Equation* (5) *satisfies WP, IIA and ILA.*

Proof. WP: If y is uniquely weakly Pareto dominant in $Y \subseteq X$ at $\succsim$, then y is uniquely weakly Pareto dominant in X at any $\succsim^Y$ taking Y to the top from $\succsim$. Unanimity and Equation (5) ensure $f(Y, \succsim) = \hat{f}(X, \succsim^Y) = \{y\}$.

IIA: If $\succsim$ and $\succsim'$ agree on $Y \subseteq X$, then $\succsim^Y$ takes Y to the top from $\succsim'^Y$ as well. Lemma 6.2 guarantees both $\hat{f}(X, \succsim^Y)$ and $\hat{f}(X, \succsim'^Y)$ are subsets of Y. Applying Lemma 6.3 we get $f(Y, \succsim) = \hat{f}(X, \succsim^Y) = \hat{f}(X, \succsim'^Y) \cap Y = \hat{f}(X, \succsim'^Y) = f(Y, \succsim')$.

ILA: Follows from Lemma 6.3. $\square$

The Grether-Plott Theorem[11] is immediate from our main theorem:

Corollary 6.5 (Grether and Plott (1982))**.** *If a voting procedure $\hat{f} : \mathcal{X} \times R^N \to 2^X \setminus \emptyset$ (respectively, $\hat{f} : \mathcal{X} \times P^N \to 2^X \setminus \emptyset$) satisfies k-set feasibility for some $k < |X|$, unanimity, independence of irrelevant alternatives and Arrow's choice axiom, it is serially dictatorial (dictatorial).*

The impossibility theorem under strategic candidacy follows as a special case:

Corollary 6.6 (Dutta, Jackson, and Le Breton (2001))**.** *If a voting procedure $\hat{f} : \mathcal{X} \times R^N \to 2^X \setminus \emptyset$ (respectively, $\hat{f} : \mathcal{X} \times P^N \to 2^X \setminus \emptyset$) is* $(|X| - 1)$*-set feasible, unanimous, independent of irrelevant alternatives and strongly candidate stable, it is serially dictatorial (dictatorial).*

7. Conclusion

This paper proposes a unifying impossibility theorem. The two independence conditions we propose underlie the axioms of a number of classical impossibility theorems. In other words, even if one finds the axioms of these impossibility theorems disputable, our theorem indicates that any alternative set of axioms that implies ours leads to dictatorship.

[11] We thank John Weymark for pointing us to this theorem.

Several extensions are possible. First, one may restrict the preference domain. For instance, one can modify our definitions and proof to prove the Gibbard-Satterthwaite Theorem under the set of all continuous preferences over a compact metric space of social alternatives (c.f.: Barberá and Peleg, 1990). Another possibility is to allow randomized social choices (c.f.: Benôit, 2002). Infinite sets of social alternatives can be handled by a careful choice of subsets over which the social choices are defined. Finally, one may consider using the ultrafilter method of Kirman and Sondermann (1972) to analyze the case with infinitely many individuals when only strict preferences are admitted.

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