# First Price Package Auction with Many Traders *

**Yasuhiro Shirata** †

**June 8, 2011**

**Abstract**

We study a first price package auction with many buyers and *many sellers* in a decentralized networked market. We show that any payoff vector in equilibria with profit-target strategies is efficient and in bidder optimal core relative to an exogenously given network. Each buyer earns a less payoff in the bidder optimal core than the VCG payoff. Furthermore, any efficient network is stable if each buyer unilaterally links with sellers. However, it is not pairwise stable if each link is formed by bilateral agreement in general.

## 1 Introduction

A package auction is a selling problem of the seller where each buyer bids on bundles of multiple items (package). The theory of the package auction nowadays play an important role in the real economy. For example, U.S. and U.K. governments sells their bundles of spectrum under the guidance of the auction theorists.

In their seminal paper, Bernheim and Whinston [3] first analyzes the package auction in which only one seller exists. They show that there exist equilibria where each bidder is truth-telling, and the corresponding equilibrium payoffs are in the bidder-optimal frontier of the core. Ausubel and Milgrom [2] show that in their dynamic ascending proxy package auction with a single seller, the same results holds true. This paper extends the static first price package auction model with a single seller to that with multiple sellers.

One natural class of mechanisms to allocate goods between multiple buyers and multiple sellers is a class of *centralized mechanisms*. Each centralized mechanism (e.g. the VCG mechanism and the Double auction) assumes the existence of the unique auctioneer (or the unique market maker) who can collect all messages from all buyers and all sellers, compute an array of trades and prices, and impose them. If the market maker exists, using the VCG mechanism yields the efficient allocation.

However, we assume that there exists no market maker. In many real exchange markets (e.g. a wholesaler-retailer market and a manufacturer-supplier market), there is no market maker. Furthermore, in such markets, an emergence of market makers is prohibited by the government from the view of antitrust law.

*I am very grateful to Professor Akira Okada for his guidance and encouragement. I also thank seminar participants at Hitosubashi University and GTW 2010. Preliminary Draft: Please do not cite or circulate without author's permission.

†Graduate School of Economics, Hitotsubashi University, 2-1 Naka, Kunitachi, Tokyo, Japan.
E-mail: ed081004@g.hit-u.ac.jp

Thus, we study a class of *decentralized mechanisms*. The decentralized mechanism assumes that each buyer's message is a collection of separate messages sent to different sellers–one for each seller, and all actions of a particular seller and her final allocation are independent of messages that the buyers send to other sellers (Peters and Severinov [8]).

Peters and Severinov [8] first analyzes a decentralized auction with many sellers where buyers have single-unit demands and sellers have single-unit supplies. They show that the existence of a symmetric perfect Bayesian equilibrium resulting in the Vickrey outcome. Anwar et al. [1] support their prediction by testing the data from competing auctions in eBay.

Furthermore, we embed a network structure into a market. We observe that a large two-sided market with many traders is often networked. Each trader cannot encounter any other trader because of opportunity cost. Kranton and Minehart [5] and Corominas-Bosch [4] characterize a relation to the competitive equilibrium in networked markets using centralized mechanisms.

We show that there exist equilibria where each buyer bids truthfully given trading histories, and any corresponding equilibrium payoff vector is efficient and in bidder optimal core relative to an exogenously given network. This result is an extension of Milgrom [6]. He also shows that the equilibrium outcome is unique and the VCG outcome if there is a single seller and goods are substitutes for all buyers. However, we show that in any equilibrium, each buyer earns strictly less payoff than the VCG payoff if there are multiple sellers.

The rest of paper is organized as follows. Section 2 introduces a networked market with many buyers and many sellers. Section 3 develops a first price package auction. Section 4 provides our main results. We show that the set of equilibrium payoffs is the same as the bidder-optimal core relative to an exogenously given network. Section 5 investigates a relation to the VCG mechanisms. Section 6 examines stability of networks. Section 7 discusses our results.

## 2 The Model

### 2.1 A Pure Exchange Networked Market

A (pure exchange) networked market consists of buyers $I$ indexed by $i = 1, ..., b$ and sellers $J$ indexed by $j = 1, ..., s$, commodities indexed by $n = 1, ..., N$, money, and network $g \in 2^{I \times J}$. We denote each package or bundle of commodities by $x \in \mathbb{R}^N_+$ and the set of all bundles by $X$. Each seller $j$ who has endowments $\omega_j \in \mathbb{R}^N_+$ is characterized valuation function $v_j$ over $X$. Each buyer who has no endowments is characterized by valuation function $v^i$ over $X$. Valuation functions $v^i$ and $v_j$ satisfies *free disposal*, i.e. $v^i(x) \geq v^i(x')$ and $v_j(x) \geq v_j(x')$ for all $x, x' \in X$ with $x \geq x'$. We assume that all agents know $v^i$, $v_j$, and $\omega_j$ for all $i$ and $j$ (*Complete information*).

All trading between buyers and sellers are restricted by bipartite network $g \subset \{ij\}_{i \in I, j \in J}$. Each buyer $i$ and seller $j$ can engage to trade if a link between $i$ and $j$, $ij$, is formed (i.e. $ij \in g$). Let $L^i(g)$ be a set of sellers who are connected to $i$, and $L_j(g)$ be a set of buyers who are connected to $j$. We denote an allotment from seller $j$ to buyer $i$ by $x^i_j = (x^i_{j,1}, ..., x^i_{j,N})$.

**Definition 1.** An allocation $x = (x^1_1, x^1_2, ..., x^b_s) \in \mathbb{R}^b \times \mathbb{R}^s$ is $g$-feasible if

$$\begin{aligned}\sum_j \omega_j &\geq \sum_{i,j} x^i_j,\\ x^i_j &\geq 0 \text{ for all } ij \in g, \text{ and}\\ x^i_j &= 0 \text{ for all } ij \notin g.\end{aligned}$$

Let $F(g)$ be the set of feasible allocations if $g$ is formed.

We assume the transferable quasi-linear payoff function for all buyers and sellers. A Pareto-efficient allocation relative to $g$ is defined as follows: An allocation $\bar{x}(g) = (\bar{x}^1(g), ..., \bar{x}^b(g), \bar{x}_1(g), ..., \bar{x}_s(g))$ in the networked market $g$ is $g$-efficient if

$$\bar{x}(g) \in \arg\max_{x \in F(g)} \sum_i v^i(x^i) + \sum_j v_j(x_j).$$

## 3 First price package auctions in a networked market

We consider a decentralized mechanism in which each seller $j$ sells $j$'s endowments using a first price package auction. First price package auctions are organized as follows:

Step 1. All sellers are ordered at random. Set $j = 1$.

Step 2. Each buyer $i$ bids a payment schedule $t^i_j(x^i_j)$ to each seller $j$ if $j \in L^i(g)$ simultaneously, where $x^i_j = (x^i_{j1}, ..., x^i_{jN})$ is an allocation from $j$ to $i$.

Step 3. Seller $j$ allocates goods to buyers $x_j = (x^0_j, x^1_j, ..., x^b_j)$ ($x^i_j \equiv (0, ..., 0)$ for all $i \notin L_j(g)$), where $x^0_j$ is an allocation to $j$. By the resource constraint, $\sum_{i=0}^b x^i_j = \omega_j$

Step 4. If $j + 1 \leq s$, then the auction goes back to Step 2 with $j = j + 1$. Otherwise, it ends.

In each round, each buyer $i$ bids a payment schedule $t^i_j(x^i_j)$ to seller $j \in L^i(g)$ simultaneously where $x^i_j = (x^i_{j1}, ..., x^i_{jN})$ is an allocation from $j$ to $i$. Then, seller $j$ allocate its goods to buyers $x_j = (x^0_j, x^1_j, ..., x^b_j)$ ($x^i_j \equiv (0, ..., 0)$ for all $i \notin L_j(g)$), where $x^0_j$ is an allocation to himself. Finally, each buyer $i$ pays sellers for an allocation to $i$, $x^i = (x^i_1, ..., x^i_s)$, according to schedules $t^i = (t^i_1, ..., t^i_s)$. Note that the payment from $i$ to $j$ depends only on the allocation from $j$ to $i$. The payoff functions for buyer $i$ and seller $j$ over a profile $(t, x)$ are described by

$$\Pi^i(t, x) = v^i(\sum_{j \in L^i(g)} x^i_j) - \sum_{j \in L^i(g)} t^i_j(x^i_j) \tag{1}$$

$$\Pi_j(t, x) = v_j(x^0_j) + \sum_{i \in L_j(g)} t^i_j(x^i_j), \tag{2}$$

where $t = (t^1, ..., t^b)$ is a strategy profile of buyers, and $x = (x_1, ..., x_s)$ is a strategy profile of sellers.

### 3.1 Profit-Target Strategies

Let $t_j = (t_j^1, ..., t_j^b)$ be a bidding profile to $j$. Given $t_j$, independently of other sellers, each seller $j$ maximizes its payoff function $\Pi_j$. Let $x_j^*(t_j) \in \arg\max_{x_j} \Pi_j(t, x)$ and $X_j^*(t_j)$ be the set of $x_j^*(t)$. Without loss of generality, we assume that every seller $j$ automatically chooses $x_j^*(t_j)$ for any $t_j$. Then, a payoff function of buyer $i$ is rewritten as

$$\Pi^i(t) = v^i(\sum_j x_j^{i*}(t_j)) - \sum_{j \in L^i(g)} t_j^i(x_j^{i*}(t_j)). \tag{3}$$

Let $h_j = (x_1^*(t_1), ..., x_{j-1}^*(t_{j-1}))$ be a trading history in round $j$.

**Definition 2.** Bidding strategy $t_j^i$ is a $\pi_j^i$-profit-target strategy to $j$ for history $h_j$ if for all $x_j^i$,

$$t_j^i(x_j^i) = \max[0, f_j^i(x_j^i) - \pi_j^i],$$

where $f_j^i(x_j^i) = v^i(x_j^i + \sum_{l<j} x_l^{i*}(t_l)) - v^i(\sum_{l<j} x_l^{i*}(t_l))$. The bidding strategy $t^i$ is $\pi^i$-profit-target strategy for history $h_s$ if $t_j^i$ is $\pi_j^i$-profit-target strategy for all $j \in L^i(g)$, where $\pi^i = (\pi_1^i, ..., \pi_s^i)$.

The profit target strategy for buyer $i$ reveals the $i$'s valuation function truthfully. Given $i$'s allotment in all rounds $l < j$, buyer $i$ pays money that the increase of true valuation minuses constant target profit $\pi_j^i$ for any package $x_j^i$ in each round $j$.

**Proposition 1.** *For any bids by other buyers $t^{-i}$, let $t^i \in \arg\max_{t^i} \Pi^i(t^i, t^{-i})$ and*

$$\bar{\pi}_j^i = v^i(x_j^{i*}(t) + \sum_{l<j} x_l^{i*}(t)) - v^i(\sum_{l<j} x_l^{i*}(t)) - t_j^i(x_j^{i*}(t))$$
$$f_j^i(x_j^i) = v^i(x_j^i + \sum_{l<j} x_l^{i*}(t)) - v^i(\sum_{l<j} x_l^{i*}(t)).$$

*Then, the $\bar{\pi}^i$-profit-target strategy is a best response of buyer $i$.*

*Proof.* Let $\bar{t}^i$ be the $\bar{\pi}^i$-profit-target strategy. Then $t_j^i(x_j^{i*}) = \bar{t}_j^i(x_j^{i*})$ for all $j$. For all $j$ and $x_j$, since $\bar{t}_j^i(x_j^i) = \max[0, f_j^i(x_j^i) - \bar{\pi}_j^i]$,

$$\sum_{i \in L_j(g)} t_j^i(x_j^i) + v_j(x_j^0) \le \sum_{i \in L_j(g)} t_j^i(x_j^{i*}) + v_j(x_j^{0*})$$
$$\sum_{k \ne i} t_j^k(x_j^k) + \bar{t}_j^i(x_j^i) + v_j(x_j^0) \le \sum_{k \ne i} t_j^k(x_j^{k*}) + \bar{t}_j^i(x_j^{i*}) + v_j(x_j^{0*}).$$

Thus, given $(\bar{t}_j^i, t_j^{-i})$, seller $j$ holds to choose $x_j^*(t^i, t^{-i})$ for all $j$. Hence, $\Pi^i(\bar{t}^i, t^{-i}) = v^i(\sum_{j \in L^i(g)} x_j^{i*}) - \sum_{j \in L^i(g)} t_j^i(x_j^{i*}) = \Pi^i(t^i, t^{-i})$. □

Proposition 1 holds for any $v$, $b$, and $s$. This implies that the continuation of the $\bar{\pi}^i$-profit-target strategy is a best response for any subgame.

## 4 Equilibrium and Bidder-Optimal Core

Let $M = I \cup J$, and $S \subset M$. Fix $g$. Let $g_S$ be a subnetwork for coalition $S$ such that $ij \in g_S$ if $i, j \in S$ and $ij \in g$, and $ij \notin g$ otherwise. Let $F(g_S)$ be the set of $g_S$-feasible allocations. We define a cooperative game $(M, g, w)$. The characteristic function $w$ is defined as for all $S \subset M$,

$$w(S|g) = \max_{x \in F(g_S)} \sum_{i \in S} v^i(\sum_{j \in S} x_j^i) + \sum_{j \in S} v_j(x_j^0). \tag{4}$$

The payoff for $i$ and $j$ is denoted by $\phi^i$ and $\phi_j$, respectively. Obviously, $w(S|g)$ is super-additive. Thus, the core is non-empty and defined as follows:

**Definition 3.** The core of $(M, g, w)$ is given by

$$\begin{aligned} Core(M, g, w) = \{\phi | \sum_{i \in I} \phi^i + \sum_{j \in J} \phi_j \leq w(M|g)\} \\ \cap \{\phi | \sum_{i \in I \cap S} \phi^i + \sum_{j \in J \cap S} \phi_j \geq w(S|g) \forall S \subset M\}. \end{aligned}$$

By definition, each payoff vector in core is $g$-efficient. When $s = 1$, the set of payoffs supported by a SPE with profit-target strategies coincides with the bidder optimal core (Milgrom [6, Theorem 8.7]). The following proposition shows the above result can be extended to cases $s \geq 2$.

**Definition 4** (Milgrom [6])**.** A payoff vector $\phi$ is *bidder-optimal* relative to $g$ if $\phi \in Core(M, g, w)$ and there exists no $\phi' \in Core(M, g, w)$ with $\phi'^i \geq \phi^i$ for all $i \in I$ and $\phi'^i > \phi^i$ for some $i \in I$. The set of bidder-optimal payoff vectors relative to $g$ is the *bidder-optimal core* relative to $g$.

**Proposition 2.** *Suppose $\phi$ is bidder optimal relative to $g$. Then, the profile $t^*$ of $\pi^i$-profit-target strategies and the corresponding $x^*(t^*)$ yielding the payoff vector $\phi$ constitutes a subgame perfect equilibrium (SPE). Conversely, suppose the profile $t^*$ of $\pi^i$-profit-target strategies and the corresponding $x^*(t)$ yielding the payoff vector $\phi$ constitutes a SPE. Then, $\phi$ is bidder optimal relative to $g$.*

*Proof.* First, we show that if $\phi$ is bidder optimal then the profile $t^*$ of $\pi^i$-profit-target strategies and the corresponding $x^*$ constitutes SPE. Since $\phi$ is in $Core(M, g, w)$, we obtain $\sum_{k \in I \cap S} \phi^k + \sum_{l \in J \cap S} \phi_l \geq w(S|g)$ for any $S \subset M$. By Proposition 1, it suffice to show that there is no deviation to another $\pi'^i$-profit-target strategy.

Suppose that buyer $i \in L_j(g)$ deviates to $t'^i$ with $\pi'^i = (\pi_j^i + \delta, \pi_{-j}^i)$ for some $j \in L^i(g)$. If $x_j^{i*}(t_j^*) = 0$ then $x_j^{i*}(t_j'^i, t_j^{-i*}) = 0$. Thus, the deviation is not profitable.

Suppose $x_j^{i*}(t_j^*) > 0$. Since $\phi$ is bidder optimal, the payoff vector $(\phi^i + \delta, \phi_j - \delta, \phi^{-i}, \phi_{-j})$ is not in $Core(M, g, w)$. Thus, there exists coalition $S$ such that $i \notin S$, $j \in S$, and

$$\sum_{k \in S} \phi^k + \sum_{l \in S} \phi_l \geq w(S|g) > \sum_{k \in S} \phi^k + \sum_{l \in S} \phi_l - \delta.$$

Let $t_j^k$ be $\pi_j^k$-profit-target strategy to $j$ for $k$. Then, we obtain

$$\begin{aligned}
\max_{\{x_j | x_j^k = 0 \forall k \notin S\}} \sum_{k \in S} t_j^k(x_j^k) + v_j(x_j^0) &\geq \max_{\{x_j | x_j^k = 0 \forall k \notin S\}} \sum_{k \in S} f_j^k(x_j^k) - \pi_j^k + v_j(x_j^0) \\
&\geq \sum_{l \in S} \max_{\{x_l | x_l^k = 0 \forall k \notin S\}} \sum_{k \in S} [f_l^k(x_l^k) - \pi_l^k + v_l(x_l^0)] - \sum_{l \in S \setminus \{j\}} \phi_l \\
&\geq \max_{\{x | x_l^k = 0 \forall k \notin S\}} \sum_{k, l \in S} [v^k(\sum_{l \in S} x_l^k) + v_l(x_l^0)] - \sum_{k \in S} \phi^k - \sum_{l \in S \setminus \{j\}} \phi_l \\
&= w(S|g) - \sum_{k \in S} \phi^k - \sum_{l \in S \setminus \{j\}} \phi_l \\
&> \phi_j - \delta \\
&= \max_{x_j} \sum_{k \in I} t_j^k(x_j^k) + v_j(x_j^0) - \delta \\
&\geq \max_{\{x_j | x_j^i > 0\}} \sum_{k \in I} t_j^i(x_j^i) + v_j(x_j^0) - \delta.
\end{aligned}$$

The second inequality holds since $\phi_l = \max_{x_l} \sum_{k \in I} [f_l^k(x_l^k) - \pi_l^k + v_l(x_l^0)]$ for all $l \in S$. This inequality shows that seller $j$ excludes buyer $i$ since $i \notin S$. Then, the payoff for $i$ decreases since $\phi$ is bidder-optimal. Hence, there is no profitable deviation for any buyer $i$.

To show the converse, suppose that the profile $t^*$ of $\pi^i$-profit-target strategies and the corresponding $x^*(t^*)$ constitutes a SPE with SPE payoff vector $\phi$.

First, we show that $\phi \in Core(M, g, w)$. Suppose $\phi \notin Core(M, g, w)$. Then, there exists coalition $S$ such that $\sum_{i \in I \cap S} \phi^i + \sum_{j \in J \cap S} \phi_j < w(S|g)$. Since $t_j^{i*}(x_j^i) = \max[0, f_j^i(x_j^i) - \pi_j^i]$, we obtain

$$\begin{aligned}
\sum_{j \in J \cap S} \phi_j &= \sum_{j \in J \cap S} \max_{x_j} \sum_{i \in I} t_j^{i*}(x_j^i) + v_j(x_j^0) \\
&\geq \sum_{j \in J \cap S} \max_{\{x_j | x_j^i = 0 \forall i \notin S\}} \sum_{i \in I} t_j^{i*}(x_j^i) + v_j(x_j^0) \\
&\geq \sum_{j \in J \cap S} \max_{\{x_j | x_j^i = 0 \forall i \notin S\}} \sum_{i \in I \cap S} f_j^i(x_j^i) - \pi_j^i + v_j(x_j^0) \\
&= [\sum_{j \in J \cap S} \max_{\{x_j | x_j^i = 0 \forall i \notin S\}} \sum_{i \in I \cap S} f_j^i(x_j^i) + v_j(x_j^0)] - \sum_{i \in I \cap S} \phi^i \\
&= w(S|g) - \sum_{i \in I \cap S} \phi^i > \sum_{j \in J \cap S} \phi_j.
\end{aligned}$$

This is a contradiction. Hence, $\phi \in Core(M, g, w)$.

Next, we will show that $\phi$ is bidder optimal. Let $\tilde{x}$ be a corresponding allocation to $\phi$. Suppose that $\phi$ is not bidder optimal. Then, there exists $(i, j)$ and $\delta > 0$ such that $\tilde{\phi} = (\phi_j - 2\delta, \phi^i + 2\delta, \phi^{-i}, \phi_{-j}) \in Core(M, g, w)$. Suppose that $i$ deviates to $\hat{\pi}^i$-profit-target strategy such that $\hat{\pi}^i = (\pi^i_{-j}, \pi^i_j + \delta)$ with $\sum_l \pi^i_l = \phi^i$, denoted by $\hat{t}^i = (\hat{t}^i_j, t^{i*}_{-j})$. Then, we obtain

$$\begin{aligned}
&\max_{x_j} \sum_{k \neq i} t^{k*}_j(x^k_j) + \hat{t}^i_j(x^i_j) + v_j(x^0_j) \\
&\quad \geq \sum_{k \neq i} t^{k*}_j(\tilde{x}^k_j) + \hat{t}^i_j(\tilde{x}^i_j) + v_j(\tilde{x}^0_j) \\
&\quad = \phi_j - \delta > \phi_j - 2\delta.
\end{aligned}$$

Let $S^j$ be any coalition such that $l \in S^j$ for all seller $l \leq j$ and $l \notin S^j$ for all seller $l > j$. Since $(\phi_j - 2\delta, \phi^i + 2\delta, \phi^{-i}, \phi_{-j})$ is in $Core(M, g, w)$,

$$\begin{aligned}
\phi_j - 2\delta &\geq \max_{\{S^j | i \notin S^j\}} w(S^j|g) - \sum_{k \in I \cap S^j} \phi^k - \sum_{l \in J \cap S^j \setminus \{j\}} \phi_l \\
&= \max_{\{S^j | i \notin S^j\}} \left[ \max_{x} \sum_{k \in I \cap S^j} v^k( \sum_{l \in J \cap S^j} x^k_l) + \sum_{l \in J \cap S^j} v_l(x^0_l) \right] - \sum_{k \in I \cap S^j} \phi^k - \sum_{l \in J \cap S^j \setminus \{j\}} \phi_l \\
&\geq \max_{\{S^j | i \notin S^j\}} \left[ \max_{x_j} \sum_{k \in I \cap S^j} v^k(x^k_j + \sum_{l<j} \tilde{x}^k_l) + \sum_{l<j} v_l(\tilde{x}^0_l) + v_j(x^0_j) \right] - \sum_{k \in I \cap S^j} \phi^k - \sum_{l<j} \phi_l \\
&= \max_{x_j} \left[ \max_{\{S^j | i \notin S^j\}} \sum_{k \in I \cap S^j} (v^k(x^k_j + \sum_{l<j} \tilde{x}^k_l) + v_j(x^0_j) - \pi^k_j - v^k(\sum_{l<j} \tilde{x}^k_l) \right] \\
&= \max_{x_j} \left[ \sum_{k \neq i} \max[0, v^k(x^k_j + \sum_{l<j} \tilde{x}^k_l) - v^k(\sum_{l<j} \tilde{x}^k_l) - \pi^k_j] + v_j(x^0_j) \right] \\
&= \max_{x_j} \left[ \sum_{k \neq i} t^{k*}_j(x^k_j) + v_j(x^0_j) \right].
\end{aligned}$$

The fourth equality holds since $\phi^k = \sum_{l \in S^j} \pi^k_l$ and $\phi_l = \sum_k v^k(\sum_{m \leq l} \tilde{x}^k_m) - v^k(\sum_{m<l} \tilde{x}^k_m) + v_l(\tilde{x}^0_l) - \pi^k_l$. This inequality implies that seller $j$ decreases his payoff by rejecting $i$'s offer. Then, since the $i$'s deviation is accepted by $j$, it is profitable for $i$. This contradicts with the assumption. □

**Remark 1.** Consider the following two stage simultaneous-bidding game. In Stage 1, each buyer bids schedules to all linked sellers. In Stage 2, each seller decides an allocation. Then, any profile of on-the-path bids and allocations in an SPE of the first package auction constitutes a SPE in the above simultaneous-bidding game. Thus, any bidder-optiaml core payoff vector is supported by an SPE in the simultaneous-bidding game.

By Proposition 2, $x^*(t^*)$ is $g$-efficient for any $g$, where $t^*$ is an SPE bidding profile of $\pi^i$-profit-target strategies. Thus, any $g$-efficient outcome is implemented (in an SPE with profit-target strategies) by the first price package auction.

When $s = 1$, the set of above SPEs with profit-target strategies yielding the bidder-optimal payoff vector is equal to the set of coalition-proof Nash equilibria (Bernheim and Whinston [3]). Thus, the bidder optimal core is equal to the coalition-proof Nash equilibrium payoff vectors. This result also holds when $s \geq 2$.

**Definition 5.** Fix a networked market $(v^i, v_j, \omega_j, g)_{i.j \in M}$.

(i) In a first price package auction with a single buyer and a single seller, $(t^{1*}, x_1^*(t^{1*}))$ is a perfectly coalition-proof Nash equilibrium (PCPNE) if it is an SPE.

(ii) (a) For a first price package auction with the set $S$ of buyers and sellers, $(t^{i*}, x_j^*(t^*))_{i,j \in S}$ is perfectly self-enforcing if for all coalition $T \subset S$, $(t^{i*}, x_j^*(t^*))_{i,j \in T}$ is a PCPNE in the auction given profile $(t^{i*}, x_j^*(t^*))_{i,j \in S \setminus T}$, and if the restriction of $(t^{i*}, x_j^*(t^*))_{i,j \in S}$ to any proper subgame constitutes a PCPNE.

(b) For any auction with the set $S$ of buyers and sellers, $(t^{i*}, x_j^*(t^*))_{i,j \in S}$ is a PCPNE if it is perfectly self-enforcing, and it does not Pareto-dominated by another perfectly self-enforcing profile.

**Proposition 3.** *The bidder-optimal core is equal to the set of PCPNE payoff vectors.*

*Proof.* We first show that any PCPNE payoff vector is in the bidder-optimal core. Suppose that there exists a PCPNE payoff vector $\psi$ that is not in the bidder optimal core. Let $t$ be the corresponding bidding profile. Then, there is $\psi'$ that dominates $\psi$ for buyers in some coalition $S$. Let $t'$ be a bidding profile such that buyer $i$ and seller $j$ in $S$ obtains $\psi'^i$ and $\psi'_j$ respectively. Then, each $i \in S$ deviates $t'^i$. Since buyers in $S$ have deviation, it is not a PCPNE. This is a contradiction. Therefore, any PCPNE payoff vector is bidder-optimal.

We next show the converse by induction. In any auction with single pair of a buyer and a seller, it is obvious that the unique SPE is a PCPNE. For any bidder-optimal payoff vector $\phi$, there is an SPE with profit-target strategies yielding $\phi$ in any auction with $S$ of buyers and sellers. Suppose that any SPE yielding $\phi$ is perfectly coalition-proof for an auction with $n$ buyers and $m$ sellers. Consider an auction with $m+1$ sellers, and take an SPE profile $t$ yielding bidder-optimal $\phi$. Any proper subgame is an auction with $l \leq m$ sellers. Given trading $x_j^*(t)$ in any single period $j$, $t$ is a PCPNE since it is equivalent to an auction with $m$ sellers and $n$ buyers in which $v^i(\sum_l x_l^i + x_j^{i*})$ $(l \neq j)$. Thus, $t$ is perfectly self-enforcing. Since $\phi$ is bidder-optimal, there is no Pareto-dominating perfectly self-enforcing profile. By induction, any SPE yielding $\phi$ is a PCPNE for any $s \in \mathbb{N}$.

In any auction with one seller and any number of buyers, any SPE yielding $\phi$ is a PCPNE by Bernheim and Whinston [3]. Thus, applying the above argument for any auction with any number of buyers shows that any bidder optimal payoff vector is a PCPNE payoff vector. $\square$

## 5 Substitutes and the VCG outcome

This section studies a relation to between bidder optimal core, which is implemented by the decentralized first price package auction mechanism, and the VCG outcome, which is implemented by the centralized VCG mechanism. For simplification, we assume that each seller has no valuation over any package ($v_j \equiv 0$). Milgrom [6] shows that, when $s = 1$, the bidder-optimal payoff vector is unique and coincide with the VCG payoff vector if valuations satisfy concavity and a substitute condition.[1]

Formally, we denote a price vector over commodities $N$ by $p = (p^n)_{n \in N}$ ($p^n \in \mathbb{R}_+$). A demand correspondence for agent $i$ is given by $D^i(p) = \arg\max v^i(\sum_j y_{ij}) - \sum_j p \cdot y_{ij}$. Valuation $v^i$ is linear-substitute if whenever $p^n \leq \tilde{p}^n$, $p^{-n} = \tilde{p}^{-n}$, and $x \in D^i(p)$, there exists $\tilde{x} \in D^i(\tilde{p})$ such that $x^{-n} \leq \tilde{x}^{-n}$.

In the VCG mechanism, there is a unique planner. Each buyer $i$ reports valuation $\hat{v}^i$ to the planner (valuations of sellers are known). Then, the planner imposes a $g$-feasible allocation of commodities $x_V(\hat{v})$ and transfer $t_V(\hat{v})$. It is well-known that each buyer $i$ earns $i$'s marginal contribution $w(M|g) - w(M \setminus \{\, i \,\}\,|g)$ in the VCG outcome. We denote buyer $i$'s VCG payoff by $\phi_V^i$ and $i$'s bidder optimal payoff by $\phi_B^i$ for $i \in I$.

**Proposition 4.** *For any $i \in I$, $\phi_V^i \geq \max \phi_B^i$.*

*Proof.* Any payoff vector such that some buyer $i$'s payoff is strictly greater than $i$'s marginal contribution is not in core. Thus, $i$'s VCG payoff is greater than or equal to $i$'s maximum payoff in the bidder optimal core. □

If Proposition 4 holds with equalities, then the payoff equivalence might be true as $s = 1$. In the following example, equalities hold.

**Example 1.** Let $I = \{1,2,3\}$, $J = \{1,2\}$, $N = 1$, and $g = g^c$ (complete bipartite graph). Suppose that $\omega_j = 1$ for all $j$. Each buyer has a valuation function given in Table 1. Each $v_i$ is concave and linear-substitute valuation for $i = 1,2,3$. The marginal contribution of buyer 1, 2, and 3 are given by $2,1,0$, respectively. Hence, $(\phi_V^1, \phi_V^2, \phi_V^3) = (2,1,0)$. The bidder optimal payoff vector for buyers is uniquely given by $(\phi_B^1, \phi_B^2, \phi_B^3) = (2,1,0)$. This example demonstrates that each buyer $i = 1,2$ earns $i$'s marginal contribution, which is the VCG outcome.

The next example, however, shows a strict inequity even if the substitute condition holds for buyers.

**Example 2.** Let $I = \{1,2\}$, $J = \{1,2\}$, $N = 1$, and $g = g^c$. Suppose that $\omega_j = 1$ for all $j$. The valuation function for buyers are also given in Table 1. The marginal contributions of buyer 1 and 2 are 3 and 1 respectively. Thus, $(\phi_V^1, \phi_V^2) = (3,1)$. However, the set of bidder optimal payoff vector

[1]We additionally require concavity since goods are divisible. If we consider multiple indivisible goods, the strong-substitute property is sufficient. The statement holds true when valuation functions are concave nonlinear-substitute valuations since the linear-substitute valuation and the nonlinear-substitute valuation are equivalent for concave valuations. See Milgrom and Strulovici [7].

| $x$ | $v^1$ | $v^2$ | $v^3$ |
|---|---|---|---|
| $\leq 1$ | $10x$ | $9x$ | $8x$ |
| $\leq 2$ | $10+8(x-1)$ | $9+7(x-1)$ | $8+6(x-1)$ |
| $>2$ | 18 | 16 | 14 |

Table 1: Valuation functions for buyers

for buyers $(\phi_B^1, \phi_B^2)$ is $\{(2,1)\}$. This example demonstrates that each buyer $i = 1,2$ earns less payoff than $i$'s marginal contribution while goods are substitutes for buyer $i = 1,2$.

Thus, the payoff equivalence does not hold when $s \geq 2$ even if commodities are substitutes for all buyers. Note that in the above two examples, the price for a commodity is 8 in the bidder optimal core, which is equal to a competitive price. However, the following example shows that this relation does not hold.

**Example 3.** Let $I = \{1,2,3\}$, $J = \{1,2\}$, $N = 1$, and $g = g^c$. Suppose that $\omega_j = 2$ for all $j$. The valuation function for buyers are also given in Table 1. The minimum competitive price is 7. The corresponding competitive payoff vector $(\phi_C^1, \phi_C^2, \phi_C^2) = (4,2,1)$. However, the set of bidder optimal payoff vector for buyers is $\{(5,3,1)\}$. This example demonstrates that each buyer $i = 1,2,3$ earns greater payoff than $i$'s maximum competitive payoff.

## 6 Stable Network

We have investigated the allocation problem given networks. This section discusses an efficiency and a stability of networks. Let $W(g) = w(M|g) - L(g)$, where $L$ is a link cost function to form network $g$. We assume that $L(g) = \sum_i l^i \eta^i(g) + \sum_j l_j \eta_j(g)$, where $\eta^i$ and $\eta_j$ are a number of links of $i$ and $j$ on $g$ for all $i$ and $j$, respectively. The network $g$ is efficient if $g \in \arg\max W(g)$. This implies that $w(M|g+ij) - w(M|g) \leq L(g+ij) - L(g)$ for all $ij \notin g$ and $w(M|g) - w(M|g-ij) \geq L(g) - L(g-ij)$ for all $ij \in g$ if $g$ is efficient.

First, we develop a model of unilateral formation of networks. Each buyer $i$ unilaterally form link with $j$ with whom $i$ want to link at cost $l^i > 0$ and $l_j = 0$ for all $i$ and $j$. A network $g$ is stable if

(i) for $ij \in g$, $\phi^i(g) - \phi^i(g-ij) \geq L^i(g) - L^i(g-ij)$, and

(ii) for $ij \notin g$, $\phi^i(g+ij) - \phi^i(g) \leq L^i(g+ij) - L^i(g)$.

**Proposition 5.** *Any efficient network is stable if buyers unilaterally form links.*

*Proof.* Since $\phi^i$ is the marginal contribution of $i$, $\phi^i(g+ij) - \phi^i(g) = w(M|g+ij) - w(M|g)$. Thus, every efficient network is stable. □

Next, we model a bilateral formation of networks. A link $ij$ is formed if and only if both $i$ and $j$ agree with forming link $ij$. We allow side-payments between $i$ and $j$ to form link $ij$. A network $g$ satisfies pairwise stability with side-payments if

(i) for $ij \in g$, $[\phi^i(g) - \phi^i(g - ij)] + [\phi_j(g) - \phi_j(g - ij)] \geq L(g) - L(g - ij)$, and

(ii) for $ij \notin g$, $[\phi^i(g + ij) - \phi^i(g)] + [\phi_j(g + ij) - \phi_j(g)] \leq L(g + ij) - L(g)$.

If an efficient network is pairwise stable, then the payoff increase of seller $j$ by forming new link $ij$ is smaller than decrease of social welfare; $\phi_j(g + ij) - \phi_j(g) \leq [L(g + ij) - L(g)] - [w(M, g + ij) - w(M, g)] = W(g) - W(g + ij)$. Thus, for any $ij \notin g$, if $\phi_j(g + ij) - \phi_j(g) > W(g) - W(g + ij)$ then an efficient network $g$ is not pairwise stable.

**Example 4** (Efficient but not pairwise stable network)**.** Suppose that all buyers are symmetric ($v^i = v$) for all $i$, all sellers are symmetric and obtain no gain from commodities ($v_j \equiv 0$ and $\omega_j = \omega$) for all $j$, and $b = s$. Let the link cost $l$ be relatively low such that $v(2\omega) - v(\omega) > 2l$. Then, the unique architecture of an efficient network is $g = \{11, 22, ..., ss\}$. Given $g$, each seller obtains no surplus ($\phi_s(g) = 0$). If the last seller $s$ forms link $is$ with $is \notin g$, then $s$ earns $\phi_s(g + is) = v(2\omega) - v(\omega)$. Since $\phi_s(g + ij) - \phi_s(g) = v(2\omega) - v(\omega) > 2l$ for any $ij \notin g$, it is profitable for $s$. Hence, the efficient network is not pairwise stable.

A sufficient condition that efficient network is pairwise stable is given as follows.

**Proposition 6.** *Suppose the complete network $g^c$ is an efficient network. Then, it is pairwise stable.*

*Proof.* Since $\phi^i(g + ij) - \phi^i(g) = w(M|g + ij) - w(M|g)$ and $\phi_j(g + ij) \geq \phi_j(g)$ for all $g$, any efficient network satisfies the condition (i). Since there is no $ij \notin g^c$, the condition (ii) is satisfied. □

## 7 Concluding remarks

We have studied decentralized trading in networked two-sided network. It is shown that the results Bernheim and Whinston [3] holds true. There exist equilibria where each buyer bids truthfully given trading histories, and any corresponding equilibrium payoff vector is in bidder optimal core relative to an exogenously given network. Our analysis has the following two limitations.

The first is *information structure*. Throughout the paper, we have assumed complete information among traders. Buyers and sellers, however, usually have private information for their valuations or endowments in auctions. The second is *optimality*. We have assumed that all sellers sell endowments by the first price package auction. However, it would not be an optimal selling mechanism for sellers. Solving an optimal auction to sell multiple goods is an open question even when an auction with one seller. Analyzing the private information and the optimal mechanism in decentralized two-sided markets is left for future research.

## References


[1] S. Anwar, R. McMillan, and M. Zheng. Bidding behavior in competing auctions: Evidence from eBay. *European Economic Review*, 50(2):307–322, Feb. 2006. 2

[2] L. M. Ausubel and P. R. Milgrom. Ascending auctions with package bidding. *The B.E. Journal of Theoretical Economics*, 1(1), Aug. 2002. 1

[3] B. D. Bernheim and M. D. Whinston. Menu auctions, resource allocation, and economic influence. *The Quarterly Journal of Economics*, 101(1):1–32, Feb. 1986. 1, 8, 11

[4] M. Corominas-Bosch. Bargaining in a network of buyers and sellers. *Journal of Economic Theory*, 115(1):35–77, 2004. 2

[5] R. E. Kranton and D. F. Minehart. A theory of Buyer-Seller networks. *The American Economic Review*, 91(3):485–508, June 2001. 2

[6] P. Milgrom. *Putting Auction Theory to Work*. Cambridge University Press, 1 edition, 2004. ISBN 0521536723. 2, 5, 9

[7] P. Milgrom and B. Strulovici. Substitute goods, auctions, and equilibrium. *Journal of Economic Theory*, 144(1):212–247, 2009. 9

[8] M. Peters and S. Severinov. Internet auctions with many traders. *Journal of Economic Theory*, 130 (1):220–245, 2006. 2