Continuous-Time Optimal Contracts with Costly Information Disclosure: Analysis of Strategic Default Probability

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Abstract

This paper studies the behavior of strategic corporate default when corporate cash-flows are unobservable directly to outside investors but are verifiable at a cost of disclosure. In particular, this paper looks at two types of default: liquidation and restructuring. Using the mathematical method of impulse control, this paper shows that an optimal contract takes the form of a debt contract that permits a debtor's ex-post strategic default. In equilibrium, restructuring occurs only when the corporate outcome is currently poor but is expected to recover enough in the future. After one or more restructurings occur, the defaulting firm is liquidated eventually. Quantitatively, expected probability of strategic restructuring takes the form of an exponential distribution. Keywords: Costly information disclosure, Liquidation, Restructuring, Default Probability, Impulse control. JEL Classification: C73, D82, G33.

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1 Introduction

Information disclosure is important in actual corporate practices when corporate information is unobservable directly to outside investors. However, the quantitative effect of disclosure on default behavior has not been made clear enough. In previous literature, Duffie and Lando (2001) and Yu (2005) study the effect of corporate accounting transparency (or, disclosure quality) on credit spreads by incorporating incomplete information into Merton (1974)'s contingent-claim models.[1] However, under exogenous, symmetric security structures in the contingent-claim models, their models lack investigation into the role of strategic information disclosure in optimally designed defaultable contracts.[2]

The purpose of this paper is to examine strategic information disclosure and strategic default in optimal contracts under informational asymmetry. In particular, this paper looks at equilibrium expected default probability in a continuous-time environment with Markov income shocks and costly verifiable information. As a consequence, it shows that an ex-ante optimal contract takes the form of a debt contract that permits a debtor's ex-post strategic default. The default is a discontinuous, downward jump of equilibrium payment path as forgiveness,[3] and enables the contracting players to keep their contractual relationship beyond the debtor's temporary poor performance and to hold the possibility that they will obtain higher cash-flows after the debtor's future recovery. In equilibrium, creditors expect the strategic debtor to default based on an explicit (i.e., closed-form) exponential distribution, in which the default probability is decreasing (increasing) in the disclosure cost when the cost is low (high, respectively).

This model structure is an infinite-horizon, continuous-time version of costly state verification (CSV) models, which are explored seminally by Townsend (1979). The infinite-horizon game consists of a series of very-fine-grid component games, each of which is similar to the standard 2- or 3-period CSV game.[4] Specifically, there exists two risk-averse players: one firm and one lender. The lender invests in the firm's project. The firm's cash-flow process from the project is uncertain, and its realization is privately observable only to the firm but is verifiable to the lender via a costly disclosure technology.[5] The cash-flows are allocated at the end of each component game according to contract terms. In contrast to the standard 2- or 3-period CSV model, the contract may be continued beyond default in this dynamic CSV model. In equilibrium, the firm decides whether to continue, to liquidate, or to restructure the contract strategically from a dynamic perspective.

This model extends Wang (2005)'s infinite-horizon discrete-time CSV model mainly in two points. First, the income process is Markov, whereas Wang assumes individually and independently distributed (i.i.d.) income shocks.[6] The Markovian income shocks are obviously more realistic than i.i.d. shocks. Second, this model has a continuous-time structure by formalizing the dynamic CSV game as a continuous limit of discrete-time games with fine grids. The best feature of the continuous-time model is tractability based on the well-established mathematical theory of stochastic processes.[7] This method makes complex, dynamic Bayesian games tractable so as to achieve complete characterizations of the equilibrium.[8]

More specifically, I solve for the optimal contract via the impulse control method of Bensoussan and Lions (1982) and Øksendal and Sulem (2005). Restructuring in default is characterized by impulse control. Contrary to continuous control problems, the timing, number, size, and intensity of jumps (i.e., "impulses") are decision variables. Precisely, under some regularities, the firm's default behavior is characterized by one equation:

$$\max\left(\sup_{\mu_S}\left\{f(y)+Lu(y)\right\},\mathcal{M}u(y)-u(y)\right)=0\quad\text{for all } y\in\mathcal{S}\tag{1.1}$$

where u and f represent the firm's continuation utility and instantaneous utility, respectively, each of which is a funciton of the state vector y in a solvency space $\mathcal{S}$; Lu denotes the generator of the state process $Y\in\mathcal{S}$; μ_S denotes the drift coefficient of the payment process, which is controlled continuously;[9] $\sup_{\mu_S}\{f(y)+Lu(y)\}=0$ stands for the Hamilton-Jacobi-Bellman (HJB) equation when the firm keeps the payment promise according to contract terms; $\mathcal{M}u(y)$ stands for the firm's optimally restructured utility after the firm's bad shape y is disclosed at a cost of disclosure. Eq.(1.1) implies that, when the current utility under the contract is higher than the restructured utility (i.e., $\mathcal{M}u(y)<u(y)$), the firm commits to the contract, and his utility evolves based on the HJB equation; when y is low such that $\mathcal{M}u(y)=u(y)$, the firm discloses his bad condition and requests restructuring. When the lender expects the firm's future recovery, she accepts the request; otherwise, the contract is terminated. Using the result of Eq.(1.1), this paper obtains equilibrium default probability in a closed form.

This paper is in line with a large literature on dynamic optimal contracting using recursive methods under asymmetric information environments, which started with a seminal paper of Green (1987). Recently, DeMarzo and Fishman (2007) study a dynamic optimal capital structure in finitely horizontal discrete time when a borrower privately observes independent cash flows from his investment project and is able to enjoy costly diversion from them. Tchistyi (2005) extends DeMarzo and Fishman's model into a two-state Markov chain model. Furthermore, DeMarzo and Sannikov (2006) extend Tchistyi's model into an infinitely horizontal continuous-time framework. The paper of DeMarzo and Sannikov is close to mine in the sense of studying an optimal long-term contract under Markovian technological environments in continuous time. The difference is that they study costly diversion, whereas this paper explores costly disclosure and strategic default.

This paper is organized as follows. The next section defines an environment. Section 3 solves for an optimal contract and characterizes it qualitatively. Section 4 shows quantitative results. The final section concludes.

2 Environment

2.1 Set-up

Consider a stochastic economy with a single non-storable consumption good in infinite-horizon continuous time with time parameter $t \in [0, \infty)$. There is also single storable capital. Assume that there are no capital depreciation and no capital accumulation, in order to confine our attention simply to level-stationary equilibrium. Uncertainty is governed by a complete filtered space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, P)$, which satisfies the usual conditions. In particular, the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is generated by a one-dimensional standard Brownian motion B.

There are two infinitely-lived risk-averse agents: an entrepreneurial firm and a representative lender, indexed by $i = 1, 2$, respectively. Individual preferences over deterministic consumption sequences $\{\gamma_i(t), t \geq 0\}$ (for $i = 1, 2$) are representable by $\int_{t=0}^{\infty} \exp(-\delta t) \left\{-\frac{\exp(-\alpha\gamma_1(t))}{\alpha}\right\} \mathrm{d}t$, and by $\int_{t=0}^{\infty} \exp(-\delta t) \log(\gamma_2(t))\, \mathrm{d}t$, where $\delta \in (0, 1)$ is their common discount factor.[10] The firm's and the lender's autarky utility are constants $\bar{U}$ and $\bar{V}$, respectively, and work as reservation utility.[11] Agent i ($\in \{1, 2\}$)'s information set, denoted by $\{\mathcal{F}_t^i\}_{t\geq 0}$, is generated by the processes distinguishable to agent i up to time t – call it agent i's filtration. As I will specify below, $\{\mathcal{F}_t^2\}_{t\geq 0}$ is no finer than either $\{\mathcal{F}_t^1\}_{t\geq 0}$ since the firm has informational advantages in this model. Let $E\left[\cdot|\mathcal{F}_t^i\right] = E_t^i[\cdot]$ denote agent i's expectation operator conditional on $\mathcal{F}_t^i$. Especially, when the game starts at time t with a given certain state denoted by y, their expectation operator is written as $E_t^{i,y}[\cdot]$. I may suppress the time parameter unless it causes any confusion. For convenience, I will use female pronouns for the lender, and male ones for the firm.

The firm has no capital, whereas the lender has one unit of the capital.[12] The firm has access to a production technology that requires one unit of the capital for production, whereas the lender does not. The firm and the lender make a loan contract of the capital if it provides both of the players with higher utility than their reservation utility, and share the outcome of the production. For simplicity, assume that there is no capital accumulation and no depreciation.[13] The technology, if invested, produces a predictable cash-flow (or income) process of the consumption good, denoted by $\{X_t\}_{t\geq 0}$, being characterized by the following stochastic differential equation (SDE): with constants $\mu \in \mathbb{R}, \sigma > 0$ given,

$$\mathrm{d}X_t = \mu \,\mathrm{d}t + \sigma \,\mathrm{d}B_t; \; X_0 = x > 0. \tag{2.1}$$

The values μ, σ are public information, but the realization of the cash flows is private information of the firm, except for $X_0 = x$.

Also, a costly, deterministic[14] disclosure technology is available in this economy. This technology reveals the firm's *current* true cash-flow level to the lender with perfect accuracy. The disclosure technology is used not only by the lender, but also voluntarily by the firm in this model, in contrast to the previous costly disclosure literature in which the lender alone discloses. Precisely, the firm makes a voluntary disclosure at a positive constant disclosure cost $C_X > 0$ in a state of restructuring (i.e., continuation of the contract beyond default), whereas the lender uses the disclosure technology at a positive constant cost $C_L > 0$ in a state of liquidation (i.e., termination of the contract). Note that I will define the two types of default in more detail below. The firm's disclosure cost C_X is dead-weight resource loss and causes the time path of the cash flows to decrease permanently relative to what it would otherwise be: $X_\tau = X_{\tau^-} - C_X$. On the other hand, the lender's cost is a utility cost, and includes not only a disclosure cost but also all other costs in liquidation procedures – called C_L a liquidation cost. The values C_X, C_L are public information. For convenience, in the following, the disclosure made by the lender is called (a part of) liquidation; I mean by "disclosure" only the voluntary disclosure made by the firm. Assume that the stochastic process of the disclosure, denoted by $\{d_t \in \{0, 1\}, t \in \{0, \infty\}\}$, is predictable. In particular, a disclosure can be undertaken only at the left-limit time $t^- := \lim_{s\uparrow t} s$ for $t > 0$: that is, if $d_{t^-} = 1$, a disclosure is made (if $d_{t^-} = 0$, no disclosure).

2.2 Contracts

The firm designs a contract, which prescribe a rule of payments, and makes a take-it-or-leave-it offer to the lender in order to raise one unit of the capital just before time 0.[15] If they do not reach agreement on the contracting, then they live in autarky from the time onwards forever. If both of the agents agree, then the contract is restructured ex post. Assume that there are no other renegotiation opportunities than restructuring in default, to make clear the strategic role of restructuring as compared to the one of liquidation. Let $\{S_t\}_{t\geq 0}$ denote the process of payments from the firm to the lender.

In addition, assume that this contracting is competitive in the sense that the contract promises the lender the maximum amount the firm would be willing to pay. Behind this, I implicitly assume that, when writing the contract, the firm exposes himself to the risk of losing a competition with outside firms. Being under such a competitive threat, the contract is designed so as to provide the lender with the firm's willing-to-pay, rather than with the lender's autarky utility.

A dynamic game follows after the contracting. Assume that time-t component game (for $t > 0$) evolves for a very short fine duration $\{t^-\} \cup [t, t + \mathrm{d}t)$ (or grid t). Note that this assumption will be verified later.[16] Mathematically, the left-limit time $t^- = \lim_{s\uparrow t} s$ is an isolated point and is attached to the time-t component game. However, from a game-theoretic viewpoint, $\{t^-\}$ stands for the last stage of the "previous" component game. In order to help understand the game structure intuitively, the description of time-t (≥ 0) component game starts after $\{t^-\}$: the grid-t cash flows are produced and the true realization is revealed only to the firm. The firm then makes a report of his current cash flow level, denoted by $\hat{X}$, at no cost, and transfers the reported amount to the lender. The report can be a lie. As usual in the costly-disclosure literature, the reports are unverifiable.[17] In particular, assume that the report is continuous, $\{\mathcal{F}_t\}_{t\geq 0}$-adjusted, and square integrable. There exists a predictable process $\hat{\mu}$ such that $\mathrm{d}\hat{X}_t = \hat{\mu}\,\mathrm{d}t + \sigma\,\mathrm{d}B_t$, $\hat{X}_0 = x$. Define $u := (\mu - \hat{\mu})/\sigma$ and $B_t^u := B_t - \int_0^t u_s\,\mathrm{d}s$ for each t. Let $\{\mathcal{F}_t^u\}_{t\geq 0}$ denote the filtration generated by B^u – call it a reported Brownian motion. That is,

$$\mathrm{d}\hat{X}_t = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B_t^u,\ \hat{X}_0 = x.$$

In other words, the firm can pretend as if his cash-flow process is $\{\mathcal{F}_t^u\}_{t\geq 0}$-adjusted. After the reported cash-flows are transferred, the lender seizes the promised payment out of them and returns the residual cash flows to the firm. If the reported cash flows are short of the promised payment, then the lender keeps the whole reported cash flows. This stage has started with $\{X_t, \hat{X}_t, S_t\}$. Notice that $X_0 = \hat{X}_0 = x, S_0 = s$ at time 0.

Next, the game reaches the left-limit time of the next-grid (call it time-t') component game. The firm decides whether or not to disclose the true current cash-flow level. On the one hand, suppose that the firm does not disclose. If the reported cash flow level is lower than the promised payment level, then the contract is terminated, that is, the lender verifies the truth by incurring the cost C_L, repossesses the capital and seizes the capital and all the true cash flows X from the firm. This is the liquidation that I define in this paper.[18] Assume that, even if the true cash flows turn out to be larger than the promised payment, the contract must be liquidated in order to penalize the lie-telling firm. Subsequently, the firm and the lender live in autarky from the time onwards forever: the firm receives the autarky utility $\bar{U}$, whereas the lender receives some utility from the seized cash flows, denoted by $g(\max\{X, 0\})$, where g is an increasing function and receives $\bar{V}$ additionally. If the firm makes the promised payment, then both of the agents consume the allocated goods, that is, the component game then moves on to the next stage.

On the other hand, suppose that the firm discloses. The firm is then given the right to ask for restructuring of the contract – call it re-contracting. The contract re-contracted at t' is called time-t' contract. After the lender is informed of the firm's current state, the income path is lowered discontinuously by the disclosure cost, and then moves to another contracting (i.e., re-contracting) stage. The re-contracting is also competitive. Suppose that the lender rejects the firm's newly announced contract denoted by $\{S_u, u \geq t'; S_{t'} = s'\}$. If $X_{t'-} - C_X < S_{t'-}$, then the firm misses (defaults on) the payment promise. The contract is terminated and liquidated. If $X_{t'-} - C_X \geq S_{t'-}$, then both of the agents consume the allocated goods. The component game then moves on to the next stage, taking $X_{t'} = \hat{X}_{t'} = X_{t'-} - C_X$ and $S_{t'} = S_{t'-}$ as given. If the lender accepts the re-contracting plan, the game goes on to the next stage under the new contract, taking $X_{t'} = \hat{X}_{t'} = X_{t'-} - C_X$ and $S_{t'} = s'$ as given, after the firm suffers reputation loss $R(X_{t'-}, S_{t'-}, S_{t'})$:

$$R(X_{t'-}, S_{t'-}, S_{t'}) := C_R \left[K + (S_{t'-} - S_{t'}) - (X_{t'-} - S_{t'-})\right] > 0 \tag{2.2}$$

where $C_R, K > 0$ are constants. This is a utility cost. $(S_{t'-} - S_{t'})$ represents the payment allowance at the re-contracting, and $(X_{t'-} - S_{t'-})$ represents the firm's consumption. Roughly speaking, the reputation loss is increasing in the payment allowance that is discounted by the consumption size of the firm. Assume that $K > 0$ is large enough to satisfy $R(X_{t'-}, S_{t'-}, S_{t'}) > 0$ for all t'^{-}. This technical assumption is imposed in order to prevent a large discount factor $(X_{t'-} - S_{t'-})$ from causing negative reputation loss. Note that this assumption will be satisfied in equilibrium numerically. The total costs of the disclosure and the reputation loss are called default costs in this paper. As I will discuss below, the existence of the reputation loss ensures that the accumulated default costs are finite for any finite time in equilibrium.

I focus on a particular form of the rule of payments: the payment $\{S_t\}$ is characterized as:

Assumption 2.1 *Unless a disclosure is made,*

$$\mathrm{d}S_t(\hat{X}_t, S_t) = \mu_S(\hat{X}_t, S_t)\,\mathrm{d}t + \sigma_S(\hat{X}_t, S_t)\,\mathrm{d}B_t^u; \quad S_0 = s \in \mathbb{R}. \tag{2.3}$$

The sets of the controls μ_S and σ_S are well-defined, given sets M_S and Σ_S, respectively. Assume that the process $\{S_t\}$ satisfying Eq.(2.3) exists. In other words, a contract is characterized by $\{s, \mu_S, \sigma_S\}$. Recall that B^u is the reported Brownian motion. That is,

$$\mathrm{d}S_t(\hat{X}_t, S_t) = \Big(\mu_S(\hat{X}_t, S_t) - u_t\Big)\,\mathrm{d}t + \sigma_S(\hat{X}_t, S_t)\,\mathrm{d}B_t; \quad S_0 = s \in \mathbb{R}.$$

The Markovian property of the payment rule may look restrictive. However, as I will show below, under some mild conditions, we can obtain as good performance with the optimal Markov control as with an arbitrary $\mathcal{F}_t^1$-adapted control in equilibrium. Accordingly, the Markovian property is not restrictive.

Finally, players' strategies are very general: with such contracts given, the firm's strategies of the contract announcements, the disclosures and the reports, and the lender's (re-)contracting strategy are assumed to depend on his or her information set $\mathcal{F}_t^i$ (for $i \in \{1, 2\}$).

3 Optimal contracting

3.1 Formal representation

This subsection formulates an optimal contracting problem as a control problem with the initial states $X_0 = x, S_0 = s$. For mathematical convenience, $S_0 = s$ is given here. A control for this system is then a quartet sequence: $v := \{\tau, \kappa, \mu_S, \sigma_S\}$ where (1) $\tau = \{\tau_1, \tau_2, \cdots, \tau_j, \cdots\}$ where

τ_j denotes the jth disclosure time and $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_j \leq \cdots$, (2) $\kappa = \{\kappa_1, \kappa_2, \cdots, \kappa_j, \cdots\}$ where $\kappa_j = \kappa_{\tau_j} := S_{\tau_{j-}} - S_{\tau_j}$ denotes a downward jump of the payment path at τ_j, (3) $\mu_S = \{\mu_0, \mu_1, \mu_2, \cdots, \mu_j, \cdots\}$ where $\mu_j = \mu_S(\tau_j) - u$ is the drift coefficient in the τ_j contract, and (4) $\sigma_S = \{\sigma_0, \sigma_1, \sigma_2, \cdots, \sigma_j, \cdots\}$ where $\sigma_j = \sigma_S(\tau_j)$ is the diffusion coefficient in the τ_j contract. In other words, this control is a combination of stochastic controls $\{\mu_S, \sigma_S\} \in M_S \times \Sigma_S$ and an impulse control $\{\tau, \kappa\} \in \mathcal{W}$ that is well-defined in $[0, \infty) \times \mathbb{R}$. Note that, as I will show below, disclosures are not undertaken continuously because of the default costs in equilibrium: $\tau_1 < \tau_2 < \cdots < \tau_j < \cdots$. For convenience, define $\tau_0 := 0$. With the combined control v given, the corresponding state process $Y_t^{(v)} := \begin{bmatrix} X_t^{(v)} & S_t^{(v)} \end{bmatrix}^\top$, where the superscript $\top$ of a vector and a matrix represents their transpose, is defined inductively by: for each natural number $j \in \mathbb{N} = \{1, 2, 3, ...\}$,

$$
\begin{aligned}
\mathrm{d}Y_t^{(v)} &= \begin{bmatrix} \mu \\ \mu_0 \end{bmatrix} \mathrm{d}t + \begin{bmatrix} \sigma \\ \sigma_0 \end{bmatrix} \mathrm{d}B_t \text{ for } \tau_0 \leq t \leq \tau_1^-;\ Y_0^{(v)} = y = \begin{bmatrix} x \\ s \end{bmatrix} \\
Y_{\tau_j}^{(v)} &= \begin{bmatrix} X_{\tau_j^-} - C_X \\ S_{\tau_j^-} - \kappa_j \end{bmatrix} \text{ for } t = \tau_j; \\
\mathrm{d}Y_t^{(v)} &= \begin{bmatrix} \mu \\ \mu_j \end{bmatrix} \mathrm{d}t + \begin{bmatrix} \sigma \\ \sigma_j \end{bmatrix} \mathrm{d}B_t \text{ for } \tau_j \leq t \leq \tau_{j+1}^-.
\end{aligned}
\tag{3.1}
$$

I may suppress the superscript (v) unless it causes any confusion. A contract is said to be feasible if the allocation satisfies the condition:

$$
S_t^{(v)} \geq 0, X_t^{(v)} \geq 0, X_t^{(v)} - S_t^{(v)} \geq 0 \text{ for all } t. \tag{3.2}
$$

Since the lender possesses log utility, the feasibility condition should be satisfied in the optimal contract. Define the explosion time of $Y_t^{(v)}$ as $\tau_\infty := \tau_\infty(\omega) = \lim_{R \to \infty} \left(\inf \left\{ t > 0; \left| Y_t^{(v)} \right| \geq R \right\} \right)$. Also, a contract can be terminated before the explosion when it does not give the autarky utility to either agent or both. Let $\mathcal{S} \subset \mathbb{R}$ denote the set of the state variables that promise no smaller than the autarky utility to the contracting agents – called a solvency region. Assume that there exists an open set $\mathcal{S}$ such that $Y_0 = y = [x, s]^\top \in \mathcal{S}$. Define another $\mathcal{F}_t$-stopping time $\tau_\mathcal{S}$ as $\tau_\mathcal{S} := \inf \left\{ t \in (0, \tau_\infty), Y_t^{(v)} \notin \mathcal{S} \right\}$. Call $\tau_\mathcal{S}$ liquidation time. Assume that we are given a set $\mathcal{V}$ of *admissible* combined controls $v = (\tau, \kappa, \mu_S, \sigma_S)$ such that a unique strong solution $Y_t^{(v)}$ of Eq.(3.1) exists, $\tau_\infty = \infty$, and $\lim_{j\to\infty}\tau_j = \tau_{\mathcal{S}}$ a.s. for all y. Define the firm's expected utility, taking as given $Y_0 = y$ and a quartet of the stochastic and impulse controls v:

$$J^{(v)}(y) := E^{e,y}\left[\begin{array}{l} \int_0^{\tau_{\mathcal{S}}} e^{-\delta t}\left[-\frac{1}{\alpha}\exp\{-\alpha(X_t - S_t)\}\right]\mathrm{d}t + e^{-\delta\tau_{\mathcal{S}}}\bar{U}\cdot\chi_{\{\tau_{\mathcal{S}}<\infty\}} \\ -\sum_{\tau_j\le\tau_{\mathcal{S}}} e^{-\delta\tau_j}C_R\left[K+\kappa_j-\left(X_{\tau_{j-}}-S_{\tau_{j-}}\right)\right] \end{array}\right] \tag{3.3}$$

where, for a condition $\mathcal{O}$, $\chi_{\{\mathcal{O}\}}$ is an indicator function of $\mathcal{O}$, i.e., it takes on 1 if $\mathcal{O}$ holds true (otherwise, 0). Assume that the right hand side of Eq.(3.3) is finite for all $Y_0 = y = [x,s]^\top \in \mathcal{S}, v \in \mathcal{V}$. Assume technically that:

$$E^{e,y}\left[\int_0^{\tau_{\mathcal{S}}} \max\left\{-\left(-\frac{\exp(-\alpha(X_t - S_t))}{\alpha}\right), 0\right\}\mathrm{d}t\right] < \infty \quad \text{for all } y\in\mathcal{S},\ v\in\mathcal{V},$$

$$E^{e,y}\left[\sum_{\tau_j<\tau_{\mathcal{S}}} \max\left\{-C_R\left[K+\kappa_j-\left(X_{\tau_{j-}}-S_{\tau_{j-}}\right)\right], 0\right\}\mathrm{d}t\right] < \infty \quad \text{for all } y\in\mathcal{S},\ v\in\mathcal{V}.$$

The expected liquidation utility is also finite. Now, our optimal contracting problem is written as: with $Y_0 = y \in \mathcal{S}$ as given,

$$U(y) \quad = \quad \sup_{v\in\mathcal{V}}\left\{J^{(v)}(y)\right\} = J^{(\hat{v})}(y) \tag{3.4}$$

subject to Eq.(3.1),(3.2). $U(y)$ denotes the firm's value function, and $\hat{v}\in\mathcal{V}$ is an optimal control.

3.2 Incentive compatibility

As usual in contract theory, I impose the incentive compatibility condition: I restrict the contract space to the set of the contracts that induce the firm to tell the truth. A contract is said to be incentive compatible if the firm is better off telling the truth than telling a lie for a.a $y\in\mathcal{S}$ for all t. Define a temporary incentive compatibility condition: for a.a. $(X_t, S_t)\in\mathcal{S}$ for each t,

$$\begin{array}{ll} E^e_{t-}\left[\begin{array}{l} \int_t^{\tau_{\mathcal{S}}} e^{-\delta u}\left[-\frac{1}{\alpha}\exp\{-\alpha(X_u - S(X_u, S_{u^-}))\}\right]\mathrm{d}u + e^{-\delta\tau_{\mathcal{S}}}\bar{U}\cdot\chi_{\{\tau_{\mathcal{S}}<\infty\}} \\ -\sum_{\tau_j\le\tau_{\mathcal{S}}} e^{-\delta\tau_j}C_R\left[K+\left(S_{\tau_j^-}-S_{\tau_j}\right)-\left(X_{\tau_j^-}-S_{\tau_j^-}\right)\right] \end{array}\right] \ge \\ E^e_{t-}\left[\begin{array}{l} \int_t^{\tau'_{\mathcal{S}}} e^{-\delta u}\left[-\frac{1}{\alpha}\exp\{-\alpha(X'_u - S(X'_u + I_u, S_{u^-}))\}\right]\mathrm{d}u + e^{-\delta\tau'_{\mathcal{S}}}\bar{U}\cdot\chi_{\{\tau'_{\mathcal{S}}<\infty\}} \\ -\sum_{\tau'_j\le\tau'_{\mathcal{S}}} e^{-\delta\tau'_j}C_R\left[K+\left(S'_{\tau'^-_j}-S'_{\tau'_j}\right)-\left(X'_{\tau'^-_j}-S'_{\tau'^-_j}\right)\right] \end{array}\right] \end{array} \tag{3.5}$$

such that $I_u = 0$ if $u > t$ (that is, the firm always tells the truth after time t), where $I := \hat{X} - X$ denotes the deviation of the report from the truth. The superscript x' of a variable x means that x' is the variable that is realized when the firm does not report the truth, whereas x is the one when he does. Accordingly, Eq.(3.5) means that, for each t, the firm is better off telling the truth than telling a lie only at the time-t component game. Note that, when the firm's reports deviate from the truth only at one time in Eq.(3.5), the one-time lie-telling influences the future state process and the future information set in a history-dependent way. Similarly to Fernandes and Phelan (2000),

Lemma 3.1 *The incentive compatibility condition holds true if, and only if, the temporary incentive compatibility condition holds true.*

Proof of Lemma 3.1: See Appendix.

With regard to the firm's disclosure strategy, define the following two disjoint, complementary sets: $D_t := \{Y_{t^-} \in \mathcal{S} : d_{t^-} = \tilde{d}(\mathcal{F}^1_{t^-}) = 1\}$ and $D^c_t := \{Y_{t^-} \in \mathcal{S} : d_{t^-} = \tilde{d}(\mathcal{F}^1_{t^-}) = 0\}$ where $\tilde{d}(\mathcal{F}^1_{t^-})$ represents the disclosure strategy. D_t denotes the set of Y_{t^-} that triggers disclosure at left-limit time t^- with the information set $\mathcal{F}^1_{t^-}$ given – call it disclosure region (of the cash-flow set); D^c_t is its complementary set – call it non-disclosure region. The temporary incentive compatibility condition is then characterized by the following two conditions: for a.a. $(X_t, S_t) \in \mathcal{S}$ for each t,

Condition 3.1 *When the truth is in the disclosure region (i.e., $(X_{t^-}, S_{t^-}) \in D_t$), the firm should not make any reports $\hat{X}_{t^-}$ that satisfy $\left(\hat{X}_{t^-}, S_{t^-}\right) \in D^c_t$.*

Condition 3.2 *When the truth is in the non-disclosure region (i.e., $(X_{t^-}, S_{t^-}) \in D^c_t$), the firm should not make any reports $\hat{X}_{t^-}$ that satisfy $\left(\hat{X}_{t^-}, S_{t^-}\right) \in D^c_t$ and $\hat{X}_{t^-} \neq X_{t^-}$.*

Suppose $\left(\hat{X}_{t^-}, S_{t^-}\right) \in D_t$. Since the truth is disclosed, the incentive-compatibility problem does not matter in the case. Thus, I must look only at the case that disclosures are not triggered, that is, $\left(\hat{X}_{t^-}, S_{t^-}\right) \in D^c_t$. We can classify this case further into two sub-categories: $(X_{t^-}, S_{t^-}) \in D_t$ and $(X_{t^-}, S_{t^-}) \in D^c_t$. They are corresponding to Condition 3.1 and Condition 3.2, respectively. Condition 3.1 implies that when the truth is in the disclosure region (i.e., $(X_{t^-}, S_{t^-}) \in D_t$), the firm should not make a lie-telling report in order to avoid disclosure. Condition 3.2 implies that the firm should make a truth-telling report even when the truth is in the non-disclosure region (i.e., $(X_{t^-}, S_{t^-}) \in D^c_t$). Since the reports are unverifiable, from the two conditions,

Lemma 3.2 *The incentive compatibility contracts are characterized as follows:*

(1) μ_S is a function only of S and $\sigma_S = \mathbf{0}$ where $\mathbf{0}$ denotes a zero vector,

(2) a disclosure occurs when, and only when, restructuring occurs,

(3) re-contracting provides the firm with no lower continuation utility (i.e., the remaining utility under future truth revelation) than when re-contracting is not undertaken.

Proof of Lemma 3.2: See Appendix.

In other words, first, the incentive compatible contract is deterministic. Still, the payment may be time-varying. Second, disclosure is necessary and sufficient for restructuring. Finally, restructuring should provide the firm with the payoffs enough to induce the firm to disclose voluntarily. For the lender, the restructuring has both its pros and cons. A lack of the firm's commitment is obviously averse to the lender. At the same time, however, restructuring is favorable to the lender because it induces the firm to reveal his true current cash flows to the lender at some intervals and of mitigating the firm's misbehavior. If the firm is not expected to be profitable in the future, restructuring could not provide sufficiently high future payoffs for either the firm or the lender. The firm is then enforced into liquidation. From Lemma 3.1 and Lemma 3.2,

Proposition 3.1 *Eq.*(3.3) *satisfies the incentive compatibility condition if (1) $\mu_S = \mu_S(S)$ and $\sigma_S = \mathbf{0}$, (2) a disclosure occurs when, and only when, restructuring occurs, and (3) restructuring provides the firm with no lower continuation utility than when restructuring is not undertaken.*

Impose $\mu_S = \mu_S(S)$ and $\sigma_S = \mathbf{0}$ henceforth. The contracts are characterized by $\{s, \mu_S(S)\}$.

3.3 Verification theorem

In this subsection, I show a verification theorem for the firm's optimization problem (3.4) by using the Hamilton-Jacobi-Bellman (HJB) equation of a Markov continuous control and the quasi-variational inequalities (QVI) of an impulse control. Call this method quasi-variational Hamilton-Jacobi-Bellman inequalities (HJBQVI). Define the generator of Y_t is:

$$L^{(\mu_S)}h(y) := -\delta h(y) + \mu \left.\frac{\partial h}{\partial X^{(v)}}\right|_{X^{(v)}=x} + \mu_S \left.\frac{\partial h}{\partial S^{(v)}}\right|_{S^{(v)}=s} + \frac{1}{2}\sigma^2 \left.\frac{\partial^2 h}{\partial (X^{(v)})^2}\right|_{X^{(v)}=x}$$

for each $\mu_S \in M_S$ and for a twice differentiable function h under the evolution Eq.(3.1). Also, define a restructuring operator:

$$\mathcal{M}h(y) := \sup_{\kappa \in \mathbb{R}} \left\{ h(y - \begin{bmatrix} C_X \\ \kappa \end{bmatrix}) - R^Y(y, \kappa); \quad y - \begin{bmatrix} C_X \\ \kappa \end{bmatrix} \in \mathcal{S} \right\}$$

where $R^Y(y, \kappa) := R(x, s, s - \kappa)$. In addition, for some $h(y)$, define a set:

$$G := \{y \in \mathcal{S}; h(y) > \mathcal{M}h(y)\}.$$

In other words, G stands for the continuation region for h, that is, the region the firm does not choose default strategically and keeps the promised payment.

Theorem 3.1 *Suppose that we can find a continuation utility function $u : \bar{\mathcal{S}} \mapsto \mathbb{R}$ such that*[19]

(i) $u \in C^1(\mathcal{S}) \cap C(\bar{\mathcal{S}})$,

(ii) $u \geq \mathcal{M}u$ on $\mathcal{S}$, and $u = \mathcal{M}u$ on ∂G almost surely,

Suppose that $Y_t^{(v)}$ spends 0 time on ∂G a.s., i.e.,

(iii) $E^{e,y}\left[\int_0^{\tau_\mathcal{S}} \chi_{\{Y_t^{(v)} \in \partial G\}}\, \mathrm{d}t\right] = 0 \; \forall\, y \in \mathcal{S},\, v \in \mathcal{V}$,

and suppose that

(iv) ∂G is locally the graph of a Lipschitz continuous function,

(v) $u \in C^2(S^0 \setminus \partial G)$ and the second order derivatives of u are locally bounded near ∂G,

(vi) $f(y) + L^{(\mu_S)}u(y) \leq 0 \; \forall\, \mu_S \in M_S,\, y \in S^0 \setminus \partial G$ where $f(y) := -\frac{\exp(-\alpha(x-s))}{\alpha}$,

(vii) $Y_{\tau_\mathcal{S}} \in \partial\mathcal{S}$ a.s. $P^{y,v}$ on $\{\tau_\mathcal{S} < \infty\}$, and $u(Y_t^{(v)}) \to \bar{U} \cdot \chi_{\{\tau_\mathcal{S} < \infty\}}$ as $t \to \tau_\mathcal{S}^-$ a.s. $P^{y,v}$ $\forall\, y \in \mathcal{S},\, v \in \mathcal{V}$,

(viii) the family $\{u^-(Y_\tau^{(v)})\}_{\tau \leq \tau_\mathcal{S}}$ is uniformly $P^{y,v}$-integrable $\forall\, y \in \mathcal{S},\, v \in \mathcal{V}$ where $u^-(y) := \max(-u(y), 0)$,

(ix) $E^{e,y}\left[|u(Y_\tau)| + \int_0^\tau \left\{\left|L^{(\mu_S)}u(Y_t)\right| + \left|\sigma^\top \nabla u(Y_t)\right|^2\right\} \mathrm{d}t\right] < \infty \; \forall\, y \in \mathcal{S}, v \in \mathcal{V}, \tau$, where $\nabla u(y) := \left[\frac{\partial u}{\partial y_i}\right]_{i=1}^n$. Then,

$$u(y) \geq U(y) \quad \forall\, y \in \mathcal{S}. \tag{3.6}$$

In addition, suppose that

(x) there exists a function $\hat{\mu}_S : G \mapsto \mathbb{R}$ such that $f + L^{(\hat{\mu}_S)}u(y) = 0 \; \forall\, y \in G$, and

(xi) $\hat{\kappa} = \hat{\kappa}(y) \in \arg\max_{\kappa} \left\{ u(y - \begin{bmatrix} C_X \\ \kappa \end{bmatrix}) - R^Y(y,\kappa) \right\}$ *exists for all* $y \in \mathcal{S}$.

Define an impulse control $\{\hat{\tau}, \hat{\kappa}\} = \{\hat{\tau}_1, \hat{\tau}_2, \cdots; \hat{\kappa}_1, \hat{\kappa}_2, \cdots\}$ *as: Put* $\tau_0 = 0$ *and inductively*

$$\hat{\tau}_{k+1} = \inf\{t > \hat{\tau}_k; Y^{(v_k)} \notin G\} \wedge \tau_{\mathcal{S}} \tag{3.7}$$

$$\hat{\kappa}_{k+1} = \hat{\kappa}(Y^{(v_k)}_{\hat{\tau}^-_{k+1}}) \quad \textit{if} \quad \hat{\tau}_{k+1} < \tau_{\mathcal{S}};\ k = 1, 2, \cdots \tag{3.8}$$

where $Y^{(\hat{v}_k)}$ *is defined as the result of applying the combined control*

$$\hat{v}_k := \{\hat{\mu}_S, \{\hat{\tau}_1, \cdots, \hat{\tau}_k; \hat{\kappa}_1, \cdots, \hat{\kappa}_k\}\}$$

Put $\hat{v} = \{\hat{\mu}_S, \hat{\tau}, \hat{\kappa}\}$. *Suppose*

(xii) $\hat{v} \in \mathcal{V}$, *and* $\{u(Y^{(\hat{v})})\}$ *is uniformly* $P^{y,\hat{v}}$*-integrable* $\forall\, y \in \mathcal{S}$. *Then*

$$u(y) = U(y) \quad \forall\, y \in \mathcal{S} \tag{3.9}$$

and $\hat{v} \in \mathcal{V}$ *is an optimal combined control.*

Proof of Theorem 3.1: See Appendix.

Let us look at the economic motivations of the conditions in Theorem 3.1. Conditions (i),(iv),(v) represent the continuous differentiability of the utility function. In general, the value function need not be differentiable everywhere – not even continuous. In that case, we may interpret the above equations in the sense of viscosity solutions (see Øksendal and Sulem (2005), Ch.9). However, I look for classical solutions here by imposing Conditions (i),(iv),(v), because they are a good fit for this economic analysis. Condition (ii) stands for the temporary incentive compatibility condition of Proposition 3.1, together with $\sigma_S = \mathbf{0}$. Without Condition (iii), the default costs could become infinite for some finite period, i.e., which is not an equilibrium. Also, restructuring is instantaneous, i.e., I ignore the periods of Chapter 11 bankruptcy protection, for simplicity. Conditions (vi),(x),(xi) characterize the optimal conditions. Condition (vii) regulates the terminal condition of the firm's payoffs. Suppose that this condition is violated. If $u(Y^{(v)}_{\tau^-_{\mathcal{S}}}) > \bar{U}$, there could exist some other renegotiation chances than restructuring. This is not an equilibrium. If $u(Y^{(v)}_{\tau^-_{\mathcal{S}}}) < \bar{U}$, the liquidation time should be earlier. It contradicts the definition of liquidation. Finally, Conditions (viii),(ix),(xii) ensure the integrability of the payoffs. The plausibility of these conditions will be checked numerically later.

From Theorem 3.1, the optimization problem can be rewritten into:

$$\max\left(\sup_{\mu_S}\left\{f(y)+L^{(\mu_S)}u(y)\right\},\mathcal{M}u(y)-u(y)\right)=0 \quad \text{for all } y\in\mathcal{S}. \tag{3.10}$$

Call this the HJBQVI equation. The HJB equation $f(y)+L^{(\hat{\mu}_S)}u(y)=0$ holds when the state is inside the continuation region G (i.e., $u(y)>\mathcal{M}u(y)$). When the state hits the boundary of G (i.e., $u(y)=\mathcal{M}u(y)$), restructuring is filed for. Since the continuation utility function u is C^1, "high contact (smooth fit)" conditions hold: in a default state (say, $y^d\in\partial G$), $u(y^d)=\mathcal{M}u(y^d)$ for continuity and $\frac{\partial u(y)}{\partial y}|_{y=y^d}=\frac{\partial \mathcal{M}u(y)}{\partial y}|_{y=y^d}$ for differentiability – call them value-matching and smooth-pasting conditions, respectively.

The continuation utility $u(y)$ and the optimal stochastic/impulse control $\hat{v}\in\mathcal{V}$ are characterized inductively by $\{\tau,k_\tau,\mu_S\}$ such that

$$\begin{aligned}
u(y) &= \max_{\{\tau,\kappa_\tau,\mu_S\}} E^{e,y}\left[\begin{array}{l}\int_0^\tau e^{-\delta t}\left\{-\frac{1}{\alpha}e^{-\alpha(X_t-S_t)}\right\}\mathrm{d}t\\ +e^{-\delta\tau}\left\{u\left(X_\tau,S_\tau\right)-C_R\left[K+\kappa_\tau-\left(X_{\tau^-}-S_{\tau^-}\right)\right]\right\}\end{array}\right]\\
s.t.\ \mathrm{d}X_t &= \mu\,\mathrm{d}t+\sigma\,\mathrm{d}B_t \quad \text{for} \quad 0\le t<\tau,\\
\mathrm{d}S_t &= \mu_S\,\mathrm{d}t \quad \text{for} \quad 0\le t<\tau,\\
X_\tau &= X_{\tau^-}-C_X \quad \text{for} \quad t=\tau,\\
u(y) &\ge \bar{U},\\
V(y) &:= E^{m,y}\left[\int_0^{\tau_S} e^{-\delta t}\ln\left[S_t\right]\mathrm{d}t+e^{-\delta\tau_S}\left\{\bar{V}+g(X_{\tau_S})-C_L\right\}\cdot\chi_{\{\tau_S<\infty\}}\right]\ge\bar{V}.
\end{aligned}$$

I have assumed that the payment rule is Markovian. With the Markov control, the HJB equation provides a very nice solution. Still, one might think that the assumption is too restrictive. However, I can show that we necessarily obtain as good performance with the optimal Markov control as performance with an arbitrary $\mathcal{F}_t^1$-adapted control if mild conditions are satisfied:

Theorem 3.2 *Let*

$$\begin{aligned}
U^m &:= \sup\{J^{(v)} : \textit{Markov control}\}\\
U^a &:= \sup\{J^{(v)} : \mathcal{F}_t^1\textit{-adapted control}\}
\end{aligned}$$

Suppose that there exists an optimal Markov control for the Markov control problem for all $y \in \mathcal{S}$ such that all the boundary points of $\mathcal{S}$ are regular with respect to $Y_t^{(\hat{v})}$ (i.e., for $y \in \partial\mathcal{S}$, $P^y[\tau_{\mathcal{S}} = 0] = 1$) and that U^m is a bounded function in $C^2(\mathcal{S}) \cap C(\bar{\mathcal{S}})$. Then

$$U^m = U^a \quad \textit{for all } y \in \mathcal{S}.$$

Proof of Theorem 3.2: See Appendix.

3.4 Characterization of the optimal contract

I examine the characteristics of the equilibria in detail. The solution procedures for the HJBQVI equation (3.10) are as follows. First, I find the optimal re-contracted payment rule in the QVI, taking as given a contract (s, μ_S) and $\{\tau^-, X_{\tau^-}, S_{\tau^-}\}$. Second, I solve the corresponding optimal stopping problem, taking as given the optimal re-contracted payment rule.

Fix a trio $\{\tau^-, X_{\tau^-}, S_{\tau^-}\}$. I examine the optimal payment allowance κ_τ in the restructuring operation $\mathcal{M}u(y_{\tau^-})$, that is, the optimization with respect to κ at given τ:

$$\max_{\kappa_\tau} \{u(X_\tau, S_\tau) - C_R[K + \kappa_\tau - (X_{\tau^-} - S_{\tau^-})]\}$$

The first-order condition with respect to κ_τ (or S_τ) is:

$$\frac{\partial u(X_\tau, S_\tau)}{\partial S_\tau} + C_R = 0. \tag{3.11}$$

Assume that, for each $\{\tau^-, X_{\tau^-}, S_{\tau^-}\}$, there exists some S_τ satisfying Eq.(3.11), denoted by S_τ^*. Let $\kappa_\tau^* := S_{\tau^-} - S_\tau^*$ denote the optimal payment allowance. For the sufficiency of the optimality, the second-order condition should be satisfied:

$$\left.\frac{\partial^2 u(X_\tau, S_\tau)}{\partial S_\tau^2}\right|_{S_\tau = S_\tau^*} < 0. \tag{3.12}$$

Define $Z_t = X_t - S_t$. With small abuse of language, let us redefine $U(Z) = U(X, S)$, $u(Z) = u(X, S)$, and $V(Z) = V(X, S)$. I can conjecture that the firm's program is rewritten as: with

$Z_0 = z = x - s$ given, I solve for the value function $U(z) = u(z)$ and the optimal control $\hat{v} \in \mathcal{V}$:

$$\begin{aligned}
u(z) &= \max_{\{\tau,\mu_S\}} E^{e,y}\left[\begin{array}{l} \int_0^\tau e^{-\delta t}\left\{-\frac{1}{\alpha}\exp\left[-\alpha Z_t\right]\right\}\mathrm{d}t \\ +e^{-\delta\tau}\left\{u\left(Z_\tau^*\right) - C_R\left[K + \kappa_\tau^* - Z_{\tau^-}\right]\right\} \end{array}\right] \qquad (3.13)\\
\text{s.t.}\quad \mathrm{d}Z_t &= (\mu - \mu_S)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t \quad \text{for} \quad 0 \le t < \tau,\\
Z_\tau^* &= Z_{\tau^-} + \kappa_\tau^* - C_X \quad \text{for} \quad t = \tau,\\
u(z) &\ge \bar{U} \quad \text{and} \quad V(z) \ge \bar{V}.
\end{aligned}$$

The continuation region G is redefined as $G_Z := \{Z \in \mathcal{Z}; Y \in G\}$ where $\mathcal{Z}$ denotes the set of $Z \in \mathbb{R}$ that is consistent with $Y \in \mathcal{S}$. $u(Z_t)$ is twice continuously differentiable with respect to Z_t. The first-order condition (3.11) and the second-order condition (3.12) are rewritten as:

$$u_z\left(Z_\tau^*\right) = C_R \text{ and } u_{zz}\left(Z_\tau^*\right) < 0. \qquad (3.14)$$

where $u_z := \frac{\mathrm{d}u}{\mathrm{d}Z}$ and $u_{zz} := \frac{\mathrm{d}^2 u}{\mathrm{d}Z^2}$. The original problem is now being reduced into an optimal stopping problem, that is, I solve for the first default time τ.

For further analytic investigation, I look at some characteristics of the value function. A restructuring time is expected to arrive based on probability distributions, called intearrival distributions. Because of the value-matching and the smooth-pasting condition, there is no jump of the utility at points of restructuring. I can use the same generator as above: $L^{(\mu_S)}u(z) = -\delta u(z) + (\mu - \mu_S)u_z(z) + \frac{1}{2}\sigma^2 u_{zz}(z)$. Impose an assumption:

Assumption 3.1 *The firm expects $\tau_S^0 \le \tau_S^1$ for any $z^0, z^1 \in \mathbb{R}$ satisfying $z^0 < z^1$, where τ_S^0 and τ_S^1 denote the liquidation times in case of $Z_0 = z^0$ and of $Z_0 = z^1$, respectively.*

In other words, when the firm's initial consumption level is higher, the default time is expected to arrive later. When the initial income is lower, the liquidation time is expected to arrive earlier. $\tau_S = \infty$ may also hold true. Impose another assumption:

Assumption 3.2 *There is a constant $L > 0$ such that $\left|\frac{z^1 - z^0}{u(z^1) - u(z^0)}\right| \le L$ for any $z^0, z^1 \in G_Z$ satisfying $u(z^1) \ne u(z^0)$.*

$\left|\frac{z^1 - z^0}{u(z^1) - u(z^0)}\right| < L$ means that the inverse of the slope of the value function is finite on G_Z. As I will show later, this is satisfied in equilibrium numerically. Under Assumption 3.2,

Lemma 3.3 *$u(z)$ is strictly increasing and strictly concave.*

Proof of Lemma 3.3: See Appendix.

As a result, the value function satisfies the usual characteristics of the utility function.

Now, I characterize the optimal contract explicitly. From Lemma 3.3, there exists a unique Z^* satisfying Eq.(3.14). Furthermore, I characterize G_Z more precisely:

Lemma 3.4 *G_Z is not bounded from above. If restructuring occurs in equilibrium, there are $b = \inf G_Z \in \mathbb{R}$ and $z^* \in G_Z = (b, \infty)$ such that*

(i) If $Z \leq b$, then the path immediately jumps to z^* unless liquidation occurs,

(ii) If for some t, Z is inside $G_Z = (b, \infty)$, the firm commits to the payment rule.

Proof of Lemma 3.4 : See Appendix.

Figure 1 illustrates the optimal contract structure. Given the deterministic payment rule, the consumption space Z is divided into two parts: non-default region (i.e., continuation region $G_Z = (b, +\infty)$) and default region. The default region is categorized further into two subregions: restructuring region and liquidation region. Inside the continuation region G_Z (i.e., $Z \in (b, +\infty)$), the firm commits to the contract. The evolution of the firm's continuation utility $u(Z)$ is characterized by the HJB equation: $\delta u(Z) = \left\{-\frac{1}{\alpha}\exp\left(-\alpha Z\right)\right\} + (\mu - \mu_S)u_z(Z) + \frac{\sigma^2}{2}u_{zz}(Z)$. The lower barrier b provides the firm with a "put-option" opportunity: i.e., when the firm's current cash flows become so low that his after-payment income Z is below b, he files for restructuring, so long as restructuring is possible (that is, the lender will accept the restructuring in a default state when believing that the defaulting firm will recover enough in the future). The consumption level then jumps to $z^* \in G_Z$. Notice that b and z^* are $\{\mathcal{F}_t\}_{t\geq 0}$-adjusted. Restructuring is always preferred to liquidation in a default state if the restructuring region exists, because it provides both of the players with higher utility than the autarky utility. That is, the restructuring region is located on the right side of the liquidation region. The costly restructuring reduces the whole income, and shrinks the restructuring region. After one or more restructurings, the restructuring region vanishes eventually, that is, the defaulting firm is then enforced into a liquidation. Since the uncontrolled cash-flow process is continuous here, restructuring necessarily occurs in advance of liquidation, unless the restructuring region vanishes. I will later modify the state process to have a jump term, so as to have a sudden liquidation beyond the restructuring region.

From the proof of Lemma 3.4, the following three by-products are obtained. First, the smooth-

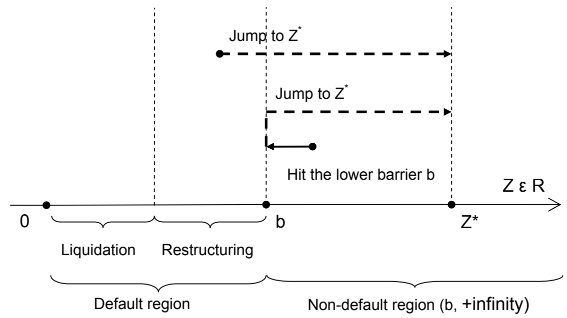

Figure 1: Impulse control

pasting condition is: for a boundary $b \in \partial G_Z$,

$$u_z(b) = u_z(z^*) + C_R = 2C_R \tag{3.15}$$

where $z^* := b + \kappa_\tau^* - C_X$ represents the firm's restructured allocation after the costly default. The last equality uses the result of Eq.(3.14). The value-matching condition is: for $b \in \partial G_Z$,

$$u(b) = u(z^*) - C_R(K + \kappa_\tau^* - b). \tag{3.16}$$

Second, without the reputation cost R^Y, the disclosure costs could become infinite for some finite periods of time. Autarky would then be an equilibrium from the initial point. This would violate Condition (iii). Thus, the existence of the reputation cost is justified in this model. Third,

Corollary 3.1 *The contract is necessarily feasible right after restructuring in equilibrium.*

This is because the payment allowance is higher than the disclosure cost, i.e., $\kappa_\tau^* > C_X$. z^* is necessarily inside G_Z. From this corollary, restructuring never occurs continuously. Accordingly, the very fine interval $\{t^-\} \cup [t, t + \mathrm{d}t)$ for each t can be interpreted as a component game. This also means that the probability of restructuring becomes zero shortly after restructuring. That is, based on this model, the short-term default probability is underestimated as compared to actual data. I will overcome this difficulty by introducing the jump term later.[20] From Lemma 3.4 and Corollary 3.1,

Proposition 3.2 *The optimal contract takes the form of a debt contract in the sense that, (1) the contract promises the lender deterministic payments, (2) if the firm defaults on the promised* *payment, the lender has the right to choose either restructuring or liquidation, and (3) the contract has priority for repayment in a liquidation.*

In this paper, the debt contract stipulates only deterministic payments, not constant ones. In fact, $\mu_S = 0$ (i.e., a constant coupon) may not be necessarily optimal. So far I have not solved for the optimal μ_S explicitly, since I have not obtained the explicit form of the value function. In the next section, I will find the optimal μ_S numerically.

4 Quantitative results

This section examines quantitatively the optimal contract and the equilibrium default behavior in structural relationships with underlying economic factors. I focus on a specific form of the contract in the following two respects. First, assume that the payment process is linear in S.

Assumption 4.1 *μ_S is a linear function of S.*

Lemma 4.1 $\mu_S \geq 0$

Proof of Lemma 4.1: Suppose that $\mu_S < 0$. The payment becomes negative with some positive probability. Since the lender has the log utility, the contract is not feasible. Hence, $\mu_S \geq 0$. ■

Correspondingly, Condition (ix) of Theorem 3.1 is modified to solve the optimization problem with a constraint $\mu_S \geq 0$. Second, assume that the first-order condition holds true at $t = 0$:

Assumption 4.2 *At the initial point of time $t = 0$ with $Z_0 = z = x - s$, $u_z(z) = C_R$.*

This assumption means that Eq.(3.14) holds true at time 0. Since this paper focuses on stationary (i.e., long-run) characteristics of the equilibria, I assume that the contract starts with some re-contracted state at the initial point. It looks as if the default occurs at 0^-.

4.1 Parametrization

Table 1 shows baseline parametrization. Specifically, I set the cash-flow volatility at a conventional level $\sigma = 25\%$. With regard to μ, Goldstein, Ju, and Leland (2001) calibrate a slightly negative μ, whereas Leland (1998) chooses $\mu = 1\%$. This paper follows the parameterization of Leland (1998). Next, the coefficient of absolute risk aversion is set at 0.7. In previous literature, there exist very few empirical studies of the CARA parameters, as compared to the CRRA parameters. As

Kimball and Mankiw (1989) discuss, the product of the CARA coefficient and the average wealth (or endowed income) (i.e., $\alpha \cdot z$ on average) can be interpreted as a good proxy for the CRRA coefficient. I choose the CARA parameter $\alpha = 0.7$ so that the value $\alpha \cdot z$, on average, ranges over the conventional region of the CRRA parameters from 1 to 50. Finally, as for estimates of the default costs (i.e., the disclosure cost and the reputation cost), there is a controversy in previous empirical literature. Warner (1977) estimates a bankruptcy cost at approximately 1.0% $\sim$ 5.3% of firm value, by using the data of US railroad firms in 1933-1955. However, in his paper, the costs are only direct bankruptcy costs such as legal fees. Altman (1984) estimates the sum of direct and indirect bankruptcy costs at about 11% $\sim$ 17% of firm value. In addition, the disclosure costs may be currently bigger after recent dramatic financial innovations and the Subprime crisis than before. Based on those observations, in this analysis, C_X ranges over 0.01 $\sim$ 0.51 widely relative to the resource levels. Also, I do a comparative static analysis in order to check robustness of this model.

Table 1: Baseline Parameters

Cash flows	Drift μ	1%
	Diffusion σ	25%
Disclosure cost	C_X	0.01 $\sim$ 0.51
Reputation cost	K	0.5
	C_R	0.1
Utility	Risk aversion α	0.7
	Time preference δ	0.03

4.2 Optimal contract

Following standard discussions of ordinary differential equations, try the functional form of u: with three parameters $A, C_1, C_2 \in \mathbb{R}$,

$$u(z) = A \cdot \exp(-\alpha z) + C_1 \cdot \exp(\nu_1 z) + C_2 \cdot \exp(\nu_2 z) \tag{4.1}$$

where $\{\nu_1, \nu_2\}$ $(\nu_1 \leq \nu_2)$ are the roots of the equation $\frac{\sigma^2}{2}\nu^2 + (\mu - \mu_S)\nu - \delta = 0$. This numerical method is based on Eq.(3.14), Eq.(3.15), Eq.(3.16) and Eq.(4.1) as follows:

$$\{\hat{b}, \hat{z}^*, \hat{\kappa}^*\} = \arg \max_{\{b, z^*, \kappa^*\}} \left(- \left[\begin{array}{l} \{u_z(z^*) - C_R\}^2 + \{u_z(b) - 2C_R\}^2 \\ + \{u(b) - (u(z^*) - C_R(K + \kappa^* - b))\}^2 \end{array} \right] \right).$$

From quantitative results, I find that there exists an open set $\mathcal{S}$ such that $Y_0 = y = [x, s]^\top \in \mathcal{S}$. All the high-level assumptions Condition (i)-Condition (xii) of Theorem 3.1 are relevant numerically. In particular, $u_z(\hat{z}^*) > 0$, $u_{zz}(\hat{z}^*) < 0$, $z_0 = \hat{z}^* > 0$, and $0 < \hat{b} < \hat{z}^*$. Also, the inverse of the slope of the utility is finite numerically. From Figure 2, the optimal $\hat{\mu}_S = 0$. That is, the zero drift is desirable to the firm because of high payoffs during the continuation periods and low default probability. There exists a stationary equilibrium such that $z^*(\tau_j) = \hat{z}^*$, $b_j = \hat{b}$, and $\kappa^*(\tau_j) = \hat{\kappa}^*$ are constants over time beyond default. In consequence, the optimal contract takes the form of a debt contract with fixed constant coupons.

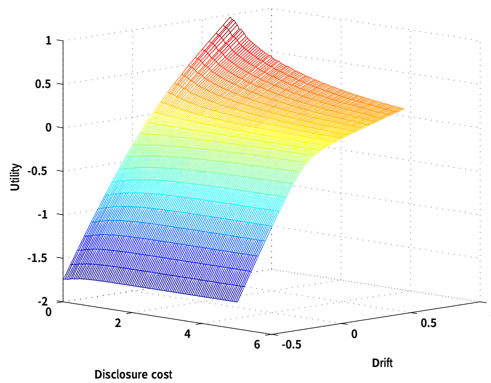

Figure 2: Effect of the drift coefficient μ on the firm's utility u

4.3 Equilibrium default behavior

A default time is expected to arrive based on an endogenously formed probability distribution. The Laplace transform of the default probability distribution $Q_0 := \text{Prob}[Z_\tau = \hat{b} | Z_0 = \hat{z}^* \in G_Z \setminus \partial G_Z] = \exp\left\{\nu_1(\hat{z}^* - \hat{b})\right\}$ stands for the probability that the firm files for restructuring (i.e., Z reaches $\hat{b}$ at a stopping time τ) at least once after the initial state starts with $Z_0 = \hat{z}^* \in G_Z \setminus \partial G_Z$ (Harrison (1985)). It is useful for analyzing endogenous default behavior because it has an explicit solution here. Note that Assumption 3.1 is satisfied.

I examine the case $\hat{z}^* \in G_Z \setminus \partial G_Z$. In Figure 3, the drift coefficient takes on the values of the range $(-50\%, +70\%)$, with everything else remaining the same. In Figure 4, the diffusion coefficient takes on the values of the range $(5\%, 50\%)$, with everything else remaining the same. From the results, the default probability is decreasing (increasing) in the disclosure cost when the cost is low (high) relative to expected cash flows. Note that the expected cash flows are measured by Sharpe

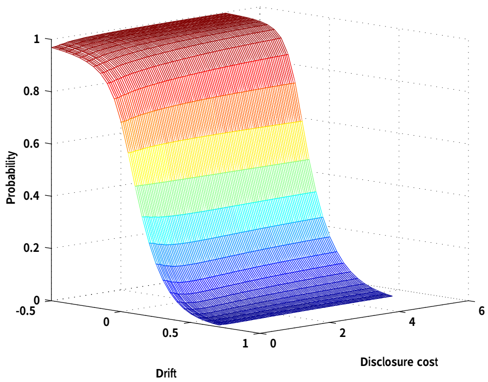

Figure 3: Effect of μ on Q_0

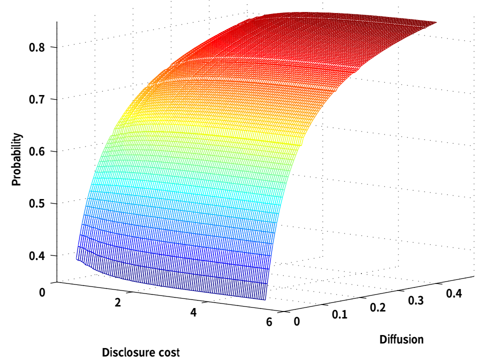

Figure 4: Effect of σ on Q_0

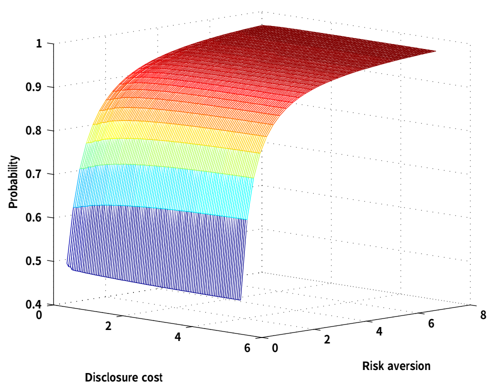

Figure 5: Effect of α on Q_0

ratios $\frac{\mu}{\sigma}$. When the disclosure cost is relatively low, restructuring occurs as frequently as possible in order to make the firm reveal the truth, because the disclosure is a convenient method of truth revelation. The default probability is decreasing in the cost. On the other hand, when the cost is high, restructuring occurs as infrequently as possible in order to minimize expected dead-weight loss of the costs dynamically, because the large cost shrinks the whole income excessively and increases the default probability. This result shows a non-monotonic relationship between the disclosure cost and the default probability. Also, the default probability is increasing in risk aversion (Figure 5).

5 Concluding remarks

This paper studied the role of strategic restructuring in optimally designed defaultable long-term loans and securities in a continuous-time environment with the agency problem caused by costly information disclosure. For future research, I will extend this model into a model of valuation of loans and securities.

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Appendices

A Supplementary remark on voluntary information disclosure

In the main text, I have focused on voluntary information disclosure. The reasoning for that is as follows. The defaulting firm has an incentive to request restructuring by disclosing his current bad condition at the cost C_X voluntarily, in order to enjoy payoffs from future production. In addition, if the lender believes that the defaulting firm will recover enough in the future and she will receive higher payoffs in the restructuring than when liquidated, then she accepts the restructuring offer by the firm. Such voluntary corporate disclosures are prevalent in practice, and have been studied a lot in accounting literature (e.g., Dye (2001)). Following the line of research, I stressed the role of the disclosure by the firm.

Although the cost is incurred directly by the firm, it is also loss for the lender from a dynamic viewpoint, because it is dead-weight loss and shrinks the whole income. In addition, the lender compensates, in equilibrium, for the firm's disclosure cost by giving some payment allowance to the firm, in order to induce the firm's voluntary disclosure. Meanwhile, if the defaulting firm is expected to have a poor future, then the contract is terminated (i.e., liquidated) in the default state. Since the firm walks away from the contract after the liquidation, he has no incentive to disclose voluntarily. That is why I assumed that the lender discloses at the cost C_L.

In practice, the lender should bear some costs during a restructuring process. Also, the firm should incur some direct costs in a liquidation in actual legal procedures. However, I omit those realistic factors from this model in order to confine attention to the discussion of how such voluntary corporate disclosure influences equilibrium default behavior and optimal allocations. Note that, in order to save the disclosure costs, the defaulting firm should disclose voluntarily only when he expects that the request of restructuring will be accepted in a bad condition. If the defaulting firm is expected to have a poor future, equilibrium should be a liquidation. Accordingly, without loss of generality, I can assume that the lender does not disclose except in a liquidation.

B Proofs of theorems

B.1 Proof of Lemma 3.1

Suppose that the temporary incentive compatibility condition holds true for all t. Then, by induction, the incentive compatibility condition necessarily holds true. Conversely, suppose that the temporary incentive compatibility condition does not hold true in some state that occurs almost surely at some point of time. When the game reaches this state at this point of time, the firm is better off telling a lie. That contradicts the incentive compatibility condition. ■

B.2 Proof of Lemma 3.2

First, noting that the reports are unverifiable, in order to satisfy Condition 3.2, the payment rule should not depend on B, that is, $\sigma_S = \mathbf{0}$ and μ_S is independent of the report. Otherwise, when the truth is in the non-disclosure region, the firm would make a lie-telling report that minimizes the payment. Second, by construction, the restructuring of the contract needs disclosure. Also, disclosure is costly for the firm. As usual in contract theory, when the firm is indifferent between two actions, the firm will choose the one that is better to the lender. The firm chooses disclosure only when the lender accepts a file for restructuring in equilibrium. Third, in order to satisfy Condition 3.1, when the truth is in the disclosure region, disclosure should provide the firm with no lower continuation utility than when he would not undertake disclosure. Otherwise, the firm would pretend as if the cash flows would be in the non-disclosure region when it is in the disclosure region. The firm's re-contracted utility should be no less than the utility when restructuring is not undertaken. ■

B.3 Proof of Theorem 3.1

First, look at Dynkin's formula.

Lemma B.1 *$L^{(\mu_S)}u(y)$ exists for a.a. y with respect to the Green measure $G_S^{Y^{(v)}}(y,\cdot)$, and with $Y = Y^{(v)}$, the Dynkin formula holds true for u:*

$$E^{e,y}[u(Y_\tau)] - E^{e,y}[u(Y_{\tau'})] = -E^{e,y}\left[\int_\tau^{\tau'} L^{(\mu_S)}u(Y_t)\,\mathrm{d}t\right]$$

for all bounded stopping times τ, τ' with $\tau \le \tau' \le \inf\{t > 0; |Y_t| \ge R\}$ for some $R < \infty$.

Note that $G^{Y^{(v)}}_{\mathcal{S}}(y,H) := E^{e,y}\left[\int_0^{\tau_{\mathcal{S}}} \chi_{\{Y^{(v)}\in H\}}\,\mathrm{d}t\right]$ for $H \in \mathcal{S}$.

Proof of Lemma B.1: From Theorem 2.1 of Øksendal and Sulem (2005), by Condition (i),(iv), and (v), we can assume $u \in C^2(\mathcal{S}) \cap C(\bar{\mathcal{S}})$. Condition (ii) stands for the temporary incentive compatibility condition of Proposition 3.1. By Condition (iii), directly from Lemma 1 of Brekke and Øksendal (1991), the result is obtained. ■

Now, prove Theorem 3.1. From Proposition 3.1 and Condition (ii)(iii), restructuring occurs only on ∂G and $u = \mathcal{M}u$ holds true there. For $R > 0$, put

$$T_R = R \wedge \inf\{t > 0; |Y^{(v)}_t| \geq R\}$$

and set

$$\theta_{k+1} = \theta^{(R)}_{k+1} = \tau_k \vee (\tau_{k+1} \wedge T_R), \quad Y_t = Y^{(v)}_t$$

Then, by Lemma B.1 and Condition (vi), for each $k = 0, 1, \cdots$,

$$\begin{aligned} E^{e,y}[u(Y_{\tau_k})] - E^{e,y}[u(Y_{\theta^-_{k+1}})] &= -E^{e,y}\left[\int_{\tau_k}^{\theta_{k+1}} L^{(\mu_S)}u(Y_t)\,\mathrm{d}t\right] \\ &\geq E^{e,y}\left[\int_{\tau_k}^{\theta_{k+1}} f(Y_t)\,\mathrm{d}t\right] \end{aligned}$$

Letting $R \to \infty$, we obtain, by using Condition (viii),(ix) and Fatou's lemma,

$$E^{e,y}[u(Y_{\tau_k})] - E^{e,y}[u(Y_{\tau^-_{k+1}})] \geq E^{e,y}\left[\int_{\tau_k}^{\tau_{k+1}} f(Y_t)\,\mathrm{d}t\right]$$

Summing up this from $k = 0$ to $k = m$, we obtain:

$$u(y) + \sum_{k=1}^{m} E^{e,y}[u(Y_{\tau_k}) - u(Y_{\tau^-_k})] - E^{e,y}[u(Y_{\tau^-_{m+1}})] \geq E^{e,y}\left[\int_0^{\tau_{m+1}} f(Y_t)\,\mathrm{d}t\right]$$

Now, using Condition (vii),

$$\begin{aligned} u(Y_{\tau_k}) &= u(Y_{\tau^-_k} - \begin{bmatrix} C_X \\ \kappa_k \end{bmatrix}) \leq \mathcal{M}u(Y_{\tau^-_k}) + R^Y(Y_{\tau^-_k}, \kappa_k) \quad \text{if } \tau_k < \tau_{\mathcal{S}} \\ u(Y_{\tau_k}) &= u(Y_{\tau^-_k}) = \bar{U} \cdot \chi_{\{\tau_{\mathcal{S}} < \tau_\infty\}} \quad \text{if } \tau_k = \tau_{\mathcal{S}} \end{aligned}$$

and therefore

$$
\begin{aligned}
& u(y) + \sum_{k=1}^{m} E^{e,y}[(\mathcal{M}u(Y_{\tau_k^-}) - u(Y_{\tau_k^-})) \cdot \chi_{\{\tau_k < \tau_{\mathcal{S}}\}}] \\
\geq \quad & E^{e,y}\left[\int_0^{\tau_{m+1}} f(Y_t)\,\mathrm{d}t + u(Y_{\tau_{m+1}^-}) - \sum_{k=1}^{m} R^Y(Y_{\tau_k^-}, \kappa_k)\right]
\end{aligned}
$$

By Condition (ii),

$$
\mathcal{M}u(Y_{\tau_k^-}) - u(Y_{\tau_k^-}) \leq 0
$$

and hence

$$
u(y) \geq E^{e,y}\left[\int_0^{\tau_{m+1}} f(Y_t)\,\mathrm{d}t + u(Y_{\tau_{m+1}^-}) - \sum_{k=1}^{m} R^Y(Y_{\tau_k^-}, \kappa_k)\right]
$$

Letting $m \to N$ ($N \leq \infty$),

$$
u(y) \geq E^{e,y}\left[\int_0^{\tau_{\mathcal{S}}} f(Y_t)\,\mathrm{d}t + \bar{U} \cdot \chi_{\{\tau_{\mathcal{S}} < \tau_\infty\}} - \sum_{k=1}^{N} R^Y(Y_{\tau_k^-}, \kappa_k)\right]
$$

Hence, $u(y) \geq J^{(v)}(y)$ as claimed in Eq.(3.6).

Next, assume that the equality in Condition (x) also holds true. Define $(\hat{\tau}, \hat{\kappa})$ by Eq.(3.7) and Eq.(3.8) and define $\hat{v} = (\hat{\mu}_S, \hat{\tau}, \hat{\kappa})$. By Condition (x), we obtain equalities in the above inequalities. That is, for m,

$$
u(y) = E^{e,y}\left[\int_0^{\tau_{m+1}} f(Y_t)\,\mathrm{d}t + u(Y_{\tau_{m+1}^-}) - \sum_{k=1}^{m} R^Y(Y_{\tau_k^-}, \kappa_k)\right]
$$

Letting $m \to N$ ($N \leq \infty$), $u(y) = J^{(\hat{v})}(y)$. Combining this with Eq.(3.6), we obtain

$$
u(y) \geq \sup_{v \in \mathcal{V}} J^{(v)}(y) \geq J^{(\hat{v})}(y) = u(y)
$$

Hence, $u(y) = U(y)$ and $\hat{v}$ is optimal. ■

B.4 Proof of Theorem 3.2

Let u be a bounded function in $C^2(\mathcal{S}) \cap C(\bar{\mathcal{S}})$ satisfying

$$f + L^{(\mu_S)}u(y) \leq 0 \quad \text{for all } y \in \mathcal{S}, v \in \mathcal{V} \tag{B.1}$$

$$u(y) = \bar{U} \quad \text{for all } y \in \partial\mathcal{S} \tag{B.2}$$

where μ_S represents $\mathcal{F}_t^1$-adapted control. Let the combined controls induced by the $\mathcal{F}_t^1$-adapted control be denoted by v^a. The payment process is given by $\mathrm{d}S_t = \mu_S \,\mathrm{d}t$ with $S_0 = s$. From Lemma B.1, Dynkin's formula holds true for the Ito process that is controlled by v^a (Lemma 7.8 of Øksendal (2003), p.112). Hence,

$$u(y) \geq J^{(v^a)}(y) \tag{B.3}$$

However, by Theorem 3.1, $u(y) = U^m(y)$ satisfies Eq.(B.1) and Eq.(B.2). By Eq.(B.3), we obtain $U^m(y) \geq U^a(y)$ and Theorem 3.2 follows. ■

B.5 Proof of Lemma 3.3

First, prove the monotonicity of the value function. From Theorem 3.1, by Ito's formula,

$$\mathrm{d}\Big(e^{-\delta t}u(Z_t)\Big) = e^{-\delta t}\left(-\delta u(Z_t) + (\mu - \mu_S)u_z(Z_t) + \frac{1}{2}u_{zz}(Z_t)\sigma^2\right)\mathrm{d}t + \Sigma(Z_t)\,\mathrm{d}B_t$$

where $\Sigma(Z_t) := e^{-\delta t}u_z(Z_t)\sigma$. For any $z^0, z^1 \in G_Z$ with $z^0 < z^1$, let Z^0 and Z^1 denote the process Z with $Z(0) = z^0$ and $Z(0) = z^1$, respectively. Define

$$\begin{aligned}
\Delta_z(t) &:= Z_t^1 - Z_t^0, \\
\Delta_u(t) &:= u(Z_t^1) - u(Z_t^0), \\
\Delta_\Sigma(t) &:= \Sigma(Z_t^1) - \Sigma(Z_t^0).
\end{aligned}$$

Then,

$$\mathrm{d}\Big(e^{-\delta t}\Delta_u\Big) = e^{-\delta t}\left\{\begin{array}{l} \left(-\delta u(Z^1) + (\mu - \mu_S)u_z(Z^1) + \frac{1}{2}u_{zz}(Z^1)\sigma^2\right) \\ -\left(-\delta u(Z^0) + (\mu - \mu_S)u_z(Z^0) + \frac{1}{2}u_{zz}(Z^0)\sigma^2\right) \end{array}\right\}\mathrm{d}t + \Delta_\Sigma\,\mathrm{d}B.$$

Since the HJB equation $f(y) + L^{(\hat{\mu}_S)}u(y) = 0$ holds,

$$\mathrm{d}\Big(e^{-\delta t}\Delta_u\Big) = -e^{-\delta t}\left(f(Z^1) - f(Z^0)\right)\mathrm{d}t + \Delta_\Sigma\,\mathrm{d}B. \tag{B.4}$$

Since $f(z)$ is strictly concave, there is an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted, strictly positive process ζ_t such that

$$f(Z_t^1) = f(Z_t^0) + \zeta_t + f'(Z_t^1)\Delta_z(t).$$

Therefore, Eq.(B.4) can be rewritten into:

$$\mathrm{d}\Big(e^{-\delta t}\Delta_u(t)\Big) = -e^{-\delta t}\left(\zeta_t + f'(Z_t^1)\Delta_z(t)\right)\mathrm{d}t + \Delta_\Sigma(t)\,\mathrm{d}B_t.$$

Next, define an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted, strictly positive process $\{\mathcal{E}_t\}_{t\geq 0}$: for all $t \geq 0$,

$$\frac{\mathrm{d}\mathcal{E}_t}{\mathcal{E}_t} := \begin{cases} \frac{f'(Z_t^1)\Delta_z(t)}{\Delta_u(t)}\,\mathrm{d}t \quad \text{when } \Delta_u(t) \neq 0 \\ 0 \quad \text{when } \Delta_u(t) = 0 \end{cases} ;\quad \mathcal{E}_0 = 1.$$

Therefore,

$$\begin{aligned} \mathrm{d}\Big(\mathcal{E}_t e^{-\delta t}\Delta_u(t)\Big) &= \mathrm{d}\mathcal{E}_t\left(e^{-\delta t}\Delta_u(t)\right) + \mathrm{d}\Big(e^{-\delta t}\Delta_u(t)\Big)\,\mathcal{E}_t \\ &= -e^{-\delta t}\left(\mathcal{E}_t\zeta_t\right)\mathrm{d}t + \mathcal{E}_t\Delta_\Sigma(t)\,\mathrm{d}B_t \end{aligned}$$

By taking conditional expectation, for a stopping time τ_n,

$$e^{-\delta t}\mathcal{E}_t\Delta_u(t) = E_t^e\left[e^{-\delta\tau_n}\mathcal{E}_{\tau_n}\Delta_u(\tau_n) + \int_t^{\tau_n} e^{-\delta s}\left(\mathcal{E}_s\zeta_s\right)\mathrm{d}s\right]$$

Since $Z \geq 0$ by the feasibility condition, $0 \leq f' \leq 1$. By assumptions, using the method of successive approximations, $E^e\left[\sup_t \Delta_u(t)^2\right] < \infty$. Also, it is clear that, by the assumption of $\left|\frac{\Delta_z(t)}{\Delta_u(t)}\right| < L$, there is a unique solution $\mathcal{E}$ such that $E^e\left[\sup_t(\mathcal{E}_t)^2\right] < \infty$. Therefore, $E^e\left[\sup_t\left|\mathcal{E}_t e^{-\delta t}\Delta_u(t)\right|\right] < \infty$. Letting $n \to \infty$, and using dominated convergence,

$$\Delta_u(0) = E_0^e\left[e^{-\delta(\tau_S^0\wedge\tau_S^1)}\mathcal{E}_{\tau_S^0\wedge\tau_S^1}\Delta_u(\tau_S^0\wedge\tau_S^1) + \int_0^{\tau_S^0\wedge\tau_S^1} e^{-\delta s}\left(\mathcal{E}_s\zeta_s\right)\mathrm{d}s\right]$$

where τ_S^0 and τ_S^1 denote the liquidation times of Z^0 and Z^1, respectively. By assumptions, the transversality condition holds true: i.e., when liquidation never occurs, $\lim_{T\to\infty} e^{-\delta T} u(Z_T) = 0$. From this transversality condition and Assumption 3.1, by $\mathcal{E} > 0$ and $\zeta > 0$, $\Delta_u(0) > 0$. Hence, $u(z)$ is strictly increasing.

Next, examine the concavity of the value function. For any $z^0, z^1 \in G_Z$ where $Z^0(0) = z^0, Z^1(0) = z^1$ and $z^0 < z^1$, and for $\phi \in (0,1) \subset \mathbb{R}_+$, define

$$\begin{aligned}
\Delta_u^\phi(t) &:= u\left(\phi Z_t^1 + (1-\phi) Z_t^0\right) - \left(\phi u(Z_t^1) + (1-\phi) u(Z_t^0)\right), \\
\Delta_\Sigma^\phi(t) &:= \Sigma\left(\phi Z_t^1 + (1-\phi) Z_t^0\right) - \left(\phi \Sigma(Z_t^1) + (1-\phi) \Sigma(Z_t^0)\right).
\end{aligned}$$

Since f is strictly concave, by following the same procedures as above, $\Delta_u^\phi(0) > 0$. That is, the strict concavity of $u(z)$ is proved. ■

B.6 Proof of Lemma 3.4

Suppose that there exists the least upper bound $\sup G_Z$, denoted by c. That is, restructuring is accepted when the firm is in sufficiently good shape at $Z \geq c$. The firm verifies his true state by using the costly disclosure. Since $u = \mathcal{M}u$ holds true at $c = \sup G_Z$, the payment allowance should be strictly higher than the cost, because of the reputation loss. Every disclosure would lead to a strictly higher level of the re-contracted consumption, which must be in G_Z. Thus, c is not an upper bound. This contradicts.

Next, suppose that restructuring occurs in equilibrium. There exists $b = \inf G_Z$. At the boundary b, the smooth-pasting condition is: for a boundary $b \in \partial G_Z$,

$$u_z(b) = u_z(z^*) + C_R = 2C_R$$

where $z^* := b + \kappa_\tau^* - C_X$ represents the firm's restructured allocation after the costly default. Note that the last equality uses the result of Eq.(3.14). The value-matching condition is: for the same $b \in \partial G_Z$,

$$u(b) = u(z^*) - C_R\left(K + \kappa_\tau^* - b\right)$$

Hence, $z^* > b$. From Lemma 3.3, since u is strictly concave, such b is unique for $u = \mathcal{M}u$. z^* is also uniquely determined. In addition, since u is strictly increasing, for all z such that $z > b$, $z \in G_Z$. That is, z^* is inside in G_Z, and so $G_Z = (b, \infty)$. ■

Notes

[1]Duffie and Lando (2001) and Yu (2005) are in line with the literature on structural credit risk, which studies endogenous default under some exogenously given, sufficiently complete security structure of equity and debt (e.g., Leland (1994)) It also studies strategic debt services and debt renegotiation (e.g., Anderson and Sundaresan (1996), Broadie, et al. (2007), Fan and Sundaresan (2000), Mella-Barral and Perraudin (1997)).

[2]The paper of Chen (2003) applies the full-information strategic-renegotiation framework of Fan and Sundaresan (2000) to an informational asymmetry environment, by (i) restricting contracts *a priori* to having a debt contract providing constant coupons and (ii) taking as given constant loss rates and constant bargaining rates in default. My paper, by contrast, characterizes optimal competitive contracts and endogenous loss rates explicitly in more general economic settings, without imposing those restrictions, and provides deeper insights into optimal capital structure and equilibrium strategic default. Also, He (2010) extends a Leland (1994)-type structural default model to a moral hazard (hidden entrepreneurial effort) problem.

[3]In practice, debt restructuring can take the form either of a cut in principal, a lengthening of maturity, or a reduction in interest payments. This paper focuses only on the form of a reduction in interest payments.

[4]In previous literature, the finite-period CSV models have been successful in capturing the role of monitoring and auditing in strategically defaultable loans and securities, and have been often used for examining empirically default costs in aggregate economies. For example, see Bernanke, Gertler, and Gilchrist (1999), Levin, Natalucci, and Zakrajšek (2004).

[5]Contrary to Townsend (1979)'s model, the firm has the right to demand disclosure by incurring the costs. This twist simplifies the outcome function form in the contract, in the sense that the less informed lender designs a contract ex ante whereas the fully informed firm undertakes all the ex post actions.

[6]In his model, due to a lack of inter-temporal links across stages, the equilibrium disclosure strategy is static, in that only a current shock triggers a disclosure in a history-independent way. Also, Nakamura (2006) extends Wang's model to have two-state Markov chain shocks, which are restrictive. In contrast, this paper generalizes the Markov shock process to have a continuum of states.

[7]For example, see Biais, et al. (2007), Cvitanić, Wan, and Zhang (2006, 2009), Cvitanić and Zhang (2007), DeMarzo and Sannikov (2006), and He (2010).

[8]Let me mention one caveat: in general, we must be careful when constructing a continuous-time game. A discrete-time dynamic model can form an extensive-form game in a straightforward way by defining some relevant time interval of each stage (e.g., day, week, month, year, etc.) and the timing of events during a component game. In contrast, a continuous-time game has no natural notion of a "previous" stage before a point of time (Fudenberg and Tirole (1985)). Thus, it often faces some difficulty with extensive-form interpretations. Against this problem, this paper shows that $\{t_-\} \cup [t, t + \triangle)$ is a relevant grid of an infinitesimal component game at a point of time t.

[9]The control problem in Eq.(1.1) is a combination of an impulse control and a continuous Markov control.

[10]CARA utility has no wealth effect. Using the firm's utility, I characterize level-stationary equilibrium allocations under the optimal contract explicitly. On the other hand, in order to draw pricing implications conveniently, I assume the lender's log utility. The utility forms could be modified to take more general forms like HARA-type one. However, these simple forms are good enough to derive interesting results.

[11]In practice, the outside options are not necessarily exogenous. Rather, they might be state-dependent and influenced by strategic disclosure behavior. However, for simplicity, this model assumes the exogenous autarky utility.

[12]For simplicity, I omit corporate net-worth problems in this model.

[13]Another paper of mine modifies the model to have capital accumulation.

[14]This technology is deterministic in the sense that, when demanded, it occurs with probability one.

[15]Much corporate finance literature assumes that a lender is a principal and a borrower is an agent. In this current paper, the relationship is the opposite. I am considering the following situation: a firm writes a security (probably, with the assistance of an underwriter) in order to raise funds in markets, and an investor buys it.

[16]As Simon and Stinchcombe (1989) discuss, there is no natural notion of the "previous" stage before a point of time t in a continuous-time game. In other words, generally, there may not exist a sequence of the discrete-time games that would converge to the continuous-time game (with some relevant topology) as the discrete-time grid goes to zero. In this paper, by contrast, there exists.

This is because, due to default costs, information flows occur only discontinuously in equilibrium. As I will verify below in Corollary 3.1, for each t, I can define the very fine time grid $\{t^-\} \cup [t, t + \mathrm{d}t)$ during which a component game is played.

[17]There are no verifiable reports and no cheap talk here.

[18]The liquidation procedures are almost the same as in the previous costly disclosure models.

[19] With regard to mathematical notations, for any set $\mathcal{O}$, $\partial\mathcal{O}$ is the boundary of the set $\mathcal{O}$. $\bar{\mathcal{O}}$ is the closure of the set. $\mathcal{O}^0$ is the interior of the set. In addition, $C(\mathcal{O})$ denotes the continuous functions from $\mathcal{O}$ to $\mathcal{S}$. $C^k(\mathcal{O})$ denotes the functions in $C(\mathcal{O})$ with continuous derivatives up to order k. Also, $P^{y,v}$ denotes the probability law of the stochastic process $Y^{(v)}(t)$ starting at $Y_0^{(v)} = y$.

[20] Note that Duffie and Lando (2001) overcome this difficulty by incorporating incomplete information of the cash flows.