Monotonic Extension

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1. Introduction

It is somewhat surprising that the problem of extension of a continuous monotonic function defined on a subset A of the Euclidean space R^n into the entire space with preservation of its properties has received scant attention. In this paper we tackle this problem for both cases of increasing and strictly increasing functions. The cases of decreasing and strictly decreasing functions can be readily reduced to the cases of increasing and strictly increasing functions. For the either case we find a property that is necessary and sufficient for the given function to be extendible with preserving the continuity and (strictly) increasing properties. The result concerning the case of (nonstrictly) increasing functions can be generalized into normally ordered uniform spaces.

Nachbin [5] studied the problem of extension of a continuous and (nonstrictly) increasing (*isotone* in his terms) functions defined on closed subsets of an arbitrary normally ordered topological space. He found a property (called further *Nachbin property*) that is necessary and sufficient for the existence of a continuous and increasing extension. Since both Nachbin property and the property found here for the (nonstrictly) increasing case are necessary and sufficient for the extendibility they are equivalent. Notice that a direct proof of their equivalence does not seem so simple.

Let us introduce some notation and concepts. For two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in R^n we write $x \leqslant y$ if $x_i \leqslant y_i$ for all $i = 1, \ldots, n$. $x \leq y$ if $x \leqslant y$ and $x \neq y$, and $x < y$ if $x_i < y_i$ for all $i = 1, \ldots, n$. We also write $x \geqslant y$, $x \geq y$, $x > y$ if $y \leqslant x$, $y \leq x$, $y < x$, respectively. Denote by e the vector in R^n all of whose components are 1, and e_k ($k = 1, \ldots, n$) the vector in R^n whose k-th component is 1 and all the other components are 0. A real function f defined on a subset D in R^n is said to be *increasing* if for any two points $x, y \in D$ such that $x \leqslant y$, $f(x) \leqslant f(y)$. And f is said to be *strictly increasing* if for any two points $x, y \in D$ such that $x \leq y$, $f(x) < f(y)$. A function $f' : D' \to R$ is an *extension* of function f if $D \subset D'$ and $f'(x) = f(x)$ for all $x \in D$.

We consider here the following two problems: Given a continuous, (strictly) increasing function $f : D \to R$, where D is a closed set in R^n, does there exist a continuous (strictly) increasing extension F of function f into the entire space R^n? Here, for either case we give necessary and sufficient conditions for the existence of such extensions.

Since, there exists an order preserving homeomorphism between R and $(0,1)$ there is no loss of generality in assuming that the range of function f, $f(D)$, is contained in $(0,1)$.

For a function $f : D \to R$, where D is an arbitrary set in R^n, we set

$$m(x) = \inf_{r>0} \sup\{f(z) : z \in D,\ z \leqslant x+re\} \text{ and } M(x) = \sup_{r>0} \inf\{f(z) : z \in D,\ z \geqslant x-re\},$$

with the agreement that $m(x) = \inf\{f(x) : x \in D\}$ and $M(x) = \sup\{f(x) : x \in D\}$, if $\{z \in D,\ z \leqslant x+re\} = \emptyset$ and $\{z \in D,\ z \geqslant x-re\} = \emptyset$ for some $r > 0$, respectively.

2. Extension of increasing functions

Theorem 1. *Let $D \subset R^n$ be a nonempty, closed set and $f : D \to R$ a continuous, increasing function. Then there exists a continuous, increasing function $F : R^n \to R$ such that $F(x) = f(x)$ for $x \in D$ if and only if function f satisfies the inequality*

$$m(x) \leqslant M(x) \text{ for all } x \in R^n. \tag{1}$$

Proof: If there exists a continuous, increasing extension F of function f, then obviously $m(x) \leqslant F(x+re)$ and $M(x) \geqslant F(x-re)$ for all $x \in R^n$ and all $r > 0$. Since, F is continuous it follows that $m(x) \leqslant F(x) \leqslant M(x)$ and hence $m(x) \leqslant M(x)$ for all $x \in R^n$.

We prove now that if the assumption $m(x) \leqslant M(x)$ is satisfied, then there exists an extension F as stated in the theorem.

Claim: $m(\cdot)$ is upper semicontinuous and $M(\cdot)$ is lower semicontinuous. Hence, the correspondence $x \mapsto [m(x), M(x)]$, $x \in R^2$ is lower hemicontinuous.

Proof: Fix $x_0 \in R^n$. Let $\varepsilon > 0$. By the definition of $m(\cdot)$ there exists a positive number r such that

$$\sup\{f(z) : z \in D, z \leqslant x_0 + 2re\} < m(x_0) + \varepsilon.$$

Since for each $x \in B_r(x_0)$ the inequality $z \leqslant x + re$ implies $z \leqslant x_0 + re$ we have $m(x) < m(x_0) + \varepsilon$ for each $x \in B_r(x_0)$. That is $m(\cdot)$ is upper semicontinuous. Lower semicontinuity of $M(\cdot)$ is proved similarly.

We extend first f into $D \cup K_1$, where $K_1 = [-1, 1]^n$. By Michael's selection theorem there exists a continuous function $g' : K_1 \to R$ such that $m(x) \leqslant g'(x) \leqslant M(x)$ for all $x \in K_1$. By the property (1), $g'(x) = f(x)$ for $x \in D \cap K_1$. Set

$$g(x) = \max\{g'(z) : z \in K_1,\ z \leqslant x\} \ \ for \ \ x \in K_1.$$

It is an easy matter to show that g is continuous and increasing. Moreover, since $M(\cdot)$ is increasing we have

$$m(x) \leqslant g(x) \leqslant M(x) \ \ for \ \ x \in K_1.$$

Indeed, we claim that the function $f_1 : D \cup K_1 \to R$ defined as $f(x)$ for $x \in D \setminus K_1$, and as $g(x)$ for $x \in K_1$ is continuous and increasing. Obviously, f_1 is continuous on $D \setminus K_1$. Let $x_0 \in K_1$ and $\{x_k\}$ be a sequence in $D \setminus K_1$ converging to x_0. Since D is assumed to be closed, $x_0 \in D$. Therefore $g'(x_0) = f(x_0)$. By the definition of function g, $g(x_0) = f(x_0)$. Since, f is continuous on D it follows that $f_1(x_k) = g(x_k) \to g(x_0) = f_1(x_0)$. Show that f_1 is increasing. Take $x \in K_1$ and $y \in D,\ y \leqslant x$. Then, $f_1(y) = f(y) \leqslant m(x) \leqslant f_1(x)$. Take $x \in K_1$ and $y \in D,\ y \geqslant x$. Then, $f_1(y) = M(y) \geqslant M(x) \geqslant f_1(x)$. Since, $f_1|_{K_1}$ and $f_1|_D$ are increasing it follows that f_1 is increasing. So we constructed a continuous and increasing extension f_1 of function f into $D_1 = D \cup K_1$.

Now we show that f_1 has the property

$$m_{f_1}(x) \leqslant M_{f_1}(x) \text{ for all } x \in R^n. \tag{2}$$

For any point x in R^n denote by $\hat{x}$ the point in K_1 that is the closest to x. We consider four cases:

Case 1: $x \in R^n \setminus [(K_1 + R^n_+) \cup (K_1 - R^n_+)]$. Clearly $m_{f_1}(x) = m_f(x)$ and $M_{f_1}(x) = M_f(x)$, and hence $m_{f_1}(x) \leqslant M_{f_1}(x)$.

Case 2: $x \in (K_1 + R^n_+) \setminus K_1$. By monotonicity and continuity of function f_1 we have $m_{f_1}(x) = \max\{f_1(\hat{x}), m_f(x)\}$. This and the inequalities $f_1(\hat{x}) \leqslant M_f(\hat{x}) \leqslant M_f(x)$ and $m_f(x) \leqslant M_f(x)$ and the equality $M_f(x) = M_{f_1}(x)$ imply $m_{f_1}(x) \leqslant M_{f_1}(x)$.

Case 3: $x \in (K_1 - R^n_+) \setminus K_1$. Again by monotonicity and continuity of function f^1 we have $M_f(x) = \min\{f_1(\hat{x}), M_f(x)\}$. This and the inequalities $f_1(\hat{x}) \geqslant m_f(\hat{x}) \geqslant m_f(x)$ and $M_f(x) \geqslant m_f(x)$, and the equality $m_f(x) = m_{f_1}(x)$ imply $m_{f_1}(x) \leqslant M_{f_1}(x)$.

Case 4: $x \in K_1$. We have $M_f(x) \leqslant f_1(x) \leqslant m_f(x)$ and hence $M_f(x) = f_1(x) = m(x)$. Obviously $M_{f_1}(x) \geqslant M_f(x)$ and $m_{f_1}(x) \leqslant m_f(x)$. Therefore $m_{f_1}(x) \leqslant M_{f_1}(x)$.

Since, $f_1 : D_1 \to R$, where $D_1 \subset R^n$ is a closed set, is continuous and possesses the property (2) by the above argument we can extend f_1 into $D_2 = D \cup K_2$, where $K_2 = [-2, 2]^n$. Proceeding in this manner we will obtain a continuous and increasing extension F of function f into the whole space R^n. $\square$

Corollary 1. *Let $D \subset R^n$ be a nonempty, compact set and $f : D \to R$ a continuous, increasing function. Then there exists a continuous, increasing function $F : R^n \to R$ such that $F(x) = f(x)$ for $x \in D$.*

Proof: It is easy to see that when D is nonempty and compact, functions M and m can be defined as

$$m(x) = \max\{f(z) : z \in D \cap L(x)\} \text{ and } M(x) = \min\{f(z) \,:\, z \in D \cap U(x)\} \text{ for } x \in R^n, \tag{3}$$

where $U(x) = x + R^n_-$ and $L(x) = x + R^n_+$.
By the monotonicity of f, obviously $m(x) \leqslant M(x)$ for all $x \in R^n$. Theorem 1 applies. $\square$

3. Extension of strictly increasing functions

We shall consider R^n with the square-norm $||x|| = \max\{|x_i|, \ i = 1, \ldots, n\}$. For a nonempty set $E \subset R^n$ and a point $x \in R^n$ the distance dist (x, E) between them is defined as dist $(x, E) = \inf\{||x - y|| : y \in E\}$. For a set $E \subset R^n$, $\mathring{E}$ will denote its interior.
Throughout this section K, possibly equipped with indexes, will denote a cube in R^n with the edges parallel to coordinate axes. A face of the cube $K = [a_i, b_i]^n$ in R^n is called a *lower (upper)* face if it contains the smallest (greatest) vertex $a = (a_1, \ldots, a_n)$ $(b = (b_1, \ldots, b_n))$. The word 'extension' will mean 'continuous strictly increasing extension'.
Further for $t \in R$ the interval (t, t) will mean the singleton $\{t\}$.

Proof of the following statement is straightforward.

Claim 1. *The supremum and infimum of a family of equicontinuous functions defined on a set $E \subset R^n$ is continuous.*

Claim 2. *Let $K \subset R^n$ be a cube as before and $F_1, F_2 : K \to R$ continuous, monotone functions such that $F_1(x) < F_2(x)$ for all $x \in K$, and $f : \partial K \to R$ continuous, strictly increasing function such that*

$$f(x) \in (F_1(x), F_2(x)) \ \text{ for } x \in \partial K.$$

Then there exists a continuous, strictly increasing extension F of function f into K such that

$$F(x) \in (F_1(x), F_2(x)) \text{ for all } x \in K.$$

Proof: Define $\bar{m},\ \bar{M} : K \to R$ as

$$\bar{m}(x) = \max\{f(z) : z \in \partial K,\ z \leqslant x\} \text{ and } \bar{M}(x) = \min\{f(z) : z \in \partial K,\ z \geqslant x\}.$$

Note that functions $\bar{m},$ and $\bar{M}$ are monotone and

$$\bar{m}|_{\partial K} = \bar{M}|_{\partial K} = f.$$

Moreover, $\bar{m}$ is upper semicontinuous and $\bar{M}$ is lower semicontinuous, $\bar{m}$ continuous on $(\underline{\partial} K) \cup \mathring{K}$ and $\bar{M}$ continuous on $(\overline{\partial} K) \cup \mathring{K}$, where $\underline{\partial} K$ is the union of the lower faces of K and $\overline{\partial} K$ is the union of upper faces of K. Set

$$m'(x) = \max\{\bar{m}(x), F_1(x)\}, \text{ and } M'(x) = \min\{\bar{M}(x), F_2(x)\}.$$

Functions m', and M' are monotone,

$$F_1(x) \leqslant m'(x) < F_2(x),\ F_1(x) < M'(x) \leqslant F_2(x) \text{ for all } x \in K. \tag{4}$$

Moreover

$$m'(x) < M'(x) \text{ for } x \in \mathring{K} \text{ and } m'(x) = M'(x) = f(x) \text{ for } x \in \partial K. \tag{5}$$

Set

$$F(x) = \Lambda(x) M'(x) + (1 - \Lambda(x)) m'(x) \text{ for } x \in K, \tag{6}$$

where $\Lambda : K \to R$ is defined as

$$\Lambda(x) = \begin{cases} 1 & \text{for} \qquad x \in \overline{\partial} K, \\ \frac{\operatorname{dist}(x, \underline{\partial} K)}{\operatorname{dist}(x, \underline{\partial} K) + \operatorname{dist}(x, \overline{\partial} K)} & \text{otherwise.} \end{cases}$$

It follows that F is continuous, $F|_{\partial K} = f$. Since Λ is strictly increasing on $\mathring{K}$, functions m', M' are monotone it follows that $F(x)$ is strictly increasing on $\mathring{K}$. This, continuity of F and strict monotony of $F|_{\partial K}$ imply that F is strictly increasing.
From the relations (4) and (6), $F(x) \in (F_1(x), F_2(x))$ for all $x \in \mathring{K}$. From the relations (5) and (6) we have $F(x) = f(x) \in (F_1(x), F_2(x))$ for $x \in \partial K$. Thus $F(x) \in (F_1(x), F_2(x))$ for all $x \in K$. □

Claim 3: *Let K be a cube in R^n and $G_1, G_2 : K \to R$ increasing functions such that*

$$G_1(x) < G_2(x) \textit{ for all } x \in K.$$

Moreover, let G_1 be upper semicontinuous, G_2 lower semicontinuous, and $f : C \to R$, where C is a closed subset (possibly empty) of K, be a continuous function. Then there exist continuous increasing functions $F_1, F_2 : K \to R$ such that

$$G_1(x) \leqslant F_1(x) < F_2(x) \leqslant G_2(x) \text{ for all } x \in K \tag{7}$$

and

$$F_1(x) < f(x) < F_2(x) \text{ for all } x \in C. \tag{8}$$

Basic Lemma. *Let $K = \Pi^n[a_i.b_i]^n$ be a cube in R^n, $C \subset K$ the union of a family (possibly empty) of faces of K, and $f : C \to R$ a continuous, strictly increasing function. Let $F_1, F_2 : K \to R$ be continuous, increasing functions such that $F_1(x) < F_2(x)$ for all $x \in K$ and $F_1(x) < f(x) < F_2(x)$ for all $x \in C$. Then there exists a continuous, strictly increasing extension F of function f into K such that*

$$f(x) \in (F_1(x), F_2(x)) \text{ for all } x \in K.$$

Proof. Assume without loss of generality $K = [0, 1]^n$. Arrange all the faces of K into a sequence $K^1, K^2, \dots, K^s$ so that each face comes before all faces of larger dimensions and faces of the same dimension are arranged arbitrarily with respect to each other. If $a \notin C$, where $K^1 = \{a\}$, then we set $f(a)$ to be any number in the interval $(F'_1(a), F'_2(a))$, where $F'_1(a) = F_1(a)$ if $\nexists z \in C$ such that $a \geqslant z$, and

$$F'_1(a) = \max\{\max\{f(b) : b \in C,\ b \leqslant a\},\ F_1(a)\} \text{ otherwise,}$$

and $F'_2(a) = F_2(a)$ if $\nexists z \in C$ such that $z \geqslant a$, and

$$F'_2(x) = \min\{\min\{f(b) : b \in C,\ b \geqslant a\},\ F_2(a)\} \text{ otherwise.}$$

Now assume function f is extended into all faces K^j for $j < i$. We denote this extension as f. For two faces K', $K'' \in \mathcal{F}$ of the same dimension we say K' is *below* K'' if there exists a nonnegative vector x such that $K' + x = K''$ and denote this as $K' \prec K''$. In this case we also say K'' is *above* K' and write $K'' \succ K'$. Denote by $\mathcal{F}$ the set of all faces of K, $K_i = C \cup (\cup_{1 \leqslant j < i} K^j)$, $i = 1, \dots, s$, and $\mathcal{F}_b(K^i) = \{K' \in \mathcal{F} : K' \subset K_i \text{ and } K' \prec K^i\}$ and $\mathcal{F}_a(K^i) = \{K' \in \mathcal{F} : K' \subset K_i \text{ and } K' \succ K^i\}$. For each $K' \in \mathcal{F}_b(K^i)$ ($K' \in \mathcal{F}_a(K^i)$) we denote by $e(K')$ the nonnegative vector such that $K^i - e(K') = K'$ ($K^i + e(K') = K'$). Define functions $F'_1, F'_2 : K^i \to R$ as

$$F'_1(x) = \max\{\max\{f(x - e(K')) : K' \in \mathcal{F}_b(K^i)\},\ F_1(x)\}$$

and

$$F'_2(x) = \min\{\min\{f(x + e(K')) : K' \in \mathcal{F}_a(K^i)\},\ F_2(x)\},$$

Functions F'_1 and F'_2 are continuous, increasing and $F'_1(x) < F'_2(x)$ for all $x \in K^i$, and $F'_1(x) < f(x) < F'_2(x)$ for all $x \in \partial K^i$. By Claim 2 there exists an extension of

function $f|_{\partial K^i}$ into K^i. So we have extended function f into $K_{i+1} = K_i \cup K^i$. This extension is continuous and strictly increasing.
The above inductive procedure extends function f into $K^s = K$. □

Theorem 2. *Let $D \subset R^n$ be a nonempty, closed set and $f : D \to R$ a continuous, strictly increasing function. Then there exists a continuous, strictly increasing function $F : R^n \to R$ such that $F(x) = f(x)$ for $x \in D$ if and only if function f satisfies the following condition:*

$$m(x) \leqslant M(x) \text{ for all } x \in D \text{ and } m(x) < M(x) \text{ for all } x \notin D. \tag{9}$$

Corollary 2. *Let $D \subset R^n$ be a nonempty, compact set, and $f : D \to R$ a continuous, strictly increasing function. Then there exists a continuous, strictly increasing function $F : R^n \to R$ such that $F(x) = f(x)$ for $x \in D$.*

Proof: As it was noted in the proof of Corollary 1, functions m and M can be equivalently defined by formulas (3). It is clear from formulas (3) that $m(x) = M(x)$ for $x \in D$ and $m(x) < M(x)$ for $x \notin D$. Theorem 2 applies. □

Formulas (3) may hold for some unbounded closed domains as well. However this alone is not sufficient for the assumption (9) of Theorem 2 to hold. If in addition the sets $D \cap L(x)$ and $D \cap U(x)$ have the compact sets of $\leqslant -$maximal and $\leqslant -$minimal elements, respectively, then the assumption (9) holds. Examples of the domains with this property are subsets of Z^n where $Z = \{0, \pm 1, \pm 2, \ldots\}$.

Corollary 3. *Let $f : D \to R$, where $D \subset Z^n$, be a strictly increasing function. Then there exists a continuous strictly increasing function $F : R^n \to R$ such that $F(x) = f(x)$ for all $x \in D$. In particular, for every strictly increasing function $f : D \to R$ there exists a strictly increasing extension of function f into Z^n.*

4. Applications to the Extension of Preferences

In this section we consider the question of extension of monotonic and strictly monotonic preorders given on closed subsets of R^n_+ into R^n_+.

Let S be a subset in real line R. A *gap* of S is a maximal nondegenerate interval in $R \setminus S$ that has upper and lower bounds in S.

Proposition 1. *Let $D \subset R^n_+$ be a nonempty closed set and $\succcurlyeq$ a rational continuous monotone preference on D. Then $\succcurlyeq$ is extendable into R^n_+ if and only if the following condition is satisfied:*

$$\text{for each } x, y \in D \text{ such that } x \succ y,\ \text{dist}\left(\{z \in D : y \succcurlyeq z\},\ x + R^n_+\right) > 0. \tag{10}$$

Proof: We assume without loss of generality that $0 \in D$ and assume $\succcurlyeq$ be extendable into R^n_+. Let $\dot{\succcurlyeq}$ be such an extension and F a continuous representation of $\dot{\succcurlyeq}$ such that $F(0) = 0$. Now let $x, y \in D$ be such that $x \succ y$. Then $F(x) > F(y) \geqslant F(0)$. By the continuity and monotony of F there exists $\alpha \in (0, 1)$ such that $F(\alpha x) = F(y)$. By the monotony of F for each $z \geqslant \alpha x,\ F(z) \geqslant F(\alpha x) > F(y)$. Hence for each $z \geqslant \alpha x$ we have $z \succ y$, which implies (10).

Assume condition (10) is satisfied. Let $f' : D \to R$ be an arbitrary continuous representation of $\succcurlyeq$ which exists by the Debreu-Eulenberg theorem [1]. By the Debreu Gap lemma there exists a continuous strictly increasing transformation $h : f'(D) \to R$ such that $h(f'(D))$ has only open gaps. Thus $f = h \circ f'$ is a continuous representation of $\succcurlyeq$ such that $f(D)$ has only open gaps, if any. Now we claim that f satisfies the assumption (1) of Theorem 1. Assume on the contrary $M(x) < m(x)$ for some $x \in R^n_+$. Obviously there exists a point $x_0 \in D$ such that $m(x) = f(x_0)$. Since $M(x) < f(x_0)$ there exists a sequence $y_k \in D$ such that $||y_k|| \to \infty$ and

$$\text{dist}\,(y_k, x + R^n_+) \to 0 \tag{11}$$

$f(y_k)$ strictly increasing converges to $M(x)$. Since $f(D)$ may have only open gaps there should exist $y \in D$ such that $f(x) > f(y) \geqslant M(x)$. From here $x \succ y$ and $y \succcurlyeq y_k$ for $k = 1, 2, \ldots$ By assumption (10), $\text{dist}\left(\{z \in D : y \succcurlyeq z\},\ x + R^n_+\right) > 0$ which contradicts to (11).
Now by Theorem 1 there exists an extension F of function f into R^n_+. The preference relation $\dot{\succcurlyeq}$ represented by function F is an extension of $\succcurlyeq$. □

Proposition 2. *Let $D \subset R^n_+$ be a nonempty closed set and $\succcurlyeq$ a rational continuous strictly monotone preference on D. Then $\succcurlyeq$ is extendable into R^n_+ if and only if the following condition is satisfied: for each $x \in D$ there exists a positive number δ such that*

$$\{x \in D : x \succcurlyeq z\} \cap \{z \in R^n_+ : z \geqslant x - \delta e\}, \tag{12}$$

is bounded.

Proof: Assume $\succcurlyeq$ be extendable into R^n_+. Let $\dot{\succcurlyeq}$ be such an extension and F its continuous representation. F is strictly increasing. Let $x \in D$. Then $F(x + e_i) > F(x),\ i = 1, \ldots, n$. By the continuity of F there exists $\delta > 0$ such that

$$F(z) > F(x) \text{ if } z \geqslant x + e_i - \delta e$$

for some $i = 1, \ldots, n$. This implies that set (12) is bounded.

Let $f : D \to R$ be an arbitrary continuous representation of $\succcurlyeq$ which exists by the Debreu-Eulenberg theorem [1]. f is strictly increasing. We show that f satisfies the assumptions (9) in Theorem 2. Take $x \in D$. Then $m(x) = f(x)$ and assumptions (12) imply that $M(x) \geqslant f(x)$. So $m(x) \leqslant M(x)$.
Take $x \in R^n_+ \setminus D$. Then $D_-(x) = \{z \in D : z \leqslant x\}$ is (possibly empty) compact. If $D_-(x) = \emptyset$ then $m(x) = -\infty$. If $D_+(x) = \{z \in D : z \geqslant x\} = \emptyset$ then $M(x) = \infty$. If $D_+(x) \neq \emptyset$ then (12) implies that $M(x) \in R$. Thus $m(x) < M(x)$.
If $D_-(x) \neq \emptyset$ then there exists $\bar{x} \in D_-(x)$ such that $f(\bar{x}) = m(x)$. Now assumption (12) implies $M(x) > f(\bar{x})$. Thus $M(x) > m(x)$ in this case.
By Theorem 2 there exists an extension F of function f into R^n_+. The preference relation represented by function F is the required one. $\square$

5. Applications to the G-P theory of temptation and self-control

Let Z be a compact, connected, metric space of alternatives and $\mathcal{K}$ be the set of nenempty closed subsets of Z. The individual is characterized by a rational preference relation $\succcurlyeq$ on $\mathcal{K}$.
Gul-Pesendorfer [2] models a decision-maker who must deal with temptations. This means that adding an alternative x to a choice problem $A \in \mathcal{K}$ may make the individual strictly worse off. If $A \succ A \cup \{x\}$ following Gul-Pesendorfer [2] we say that x is *more tempting* than $y \in A$. This will be denoted as $x_t y$. If adding an alternative makes the individual better off, i.e., $A \cup \{x\} \succ A$, then x will be chosen from $A \cup \{x\}$, and hence x is a *better choice* than $y \in A$. This will be denoted as $x_c y$.

Definition. A function $U : \mathcal{K} \to R$ is a TSU utility if there exist functions $v, w : X \to R$ and $u : w(X) \times v(X) \to R$ increasing in its first variable and decreasing in its second variable, such that $U(K) = u(\max_{x \in A} w(x), max_{x \in A} v(x))$. We write $U = (u, w, v)$. A preference relation $\succcurlyeq$ on $\mathcal{K}$ is called a TS preference if there exists a TSU such that representsi it.

Axiom A. The binary relations $\succ_c$ and $\succ_t$ are acyclic.

Temptation self-control preference and utility

Theorem A. *(i) Assume that the set of alternatives Z is a compact, connected, metric space. If preference relation $\succcurlyeq$ is a TS preference then it satisfies Axiom A.*
If preference relation $\succcurlyeq$ satifies Axiom A, is continuous, and locally nonsatiated at

each set $A \in \mathcal{K}$ that is not a satiation point of $\succcurlyeq$, then it is a TS preference.
(ii) Assume Z is a compact metric space. Then part (i) holds with Z replaced with $\mathcal{L}(Z)$, the set of all lotteries over Z.

Proof: (i) Proof of the first statement is simple and follows the lines of the proof of its finite version in G-P. We prove the second statement. So, assume that binary relations $\succ_c$ and $\succ_t$ are acyclic and that preference relation $\succcurlyeq$ is continuous, and locally nonsatiated at each set $A \in \mathcal{K}$ that is not a satiation point of $\succcurlyeq$. We show first that binary relations $\succ_c$ and $\succ_t$ have continuous utility representations. Show this for binary relation $\succ_c$. Let $x \succ_c y$. Then there exists $A \in \mathcal{K}$ such that $y \in A$ and $A \cup \{x\} \succ A$. By the continuity of $\succcurlyeq$ there exists $\delta > 0$ such that for each A' with $h(A', A) < \delta$ for each $x' \in B_\delta(x)$

$$A' \cup \{x\} \succ A'. \tag{13}$$

It follows that for each $x' \in B_\delta(x)$ and $y' \in B_\delta(y)$, $x' \succ_c y'$. That is $\succ_c$ is continuous. Similar proof goes through for $\succ_t$.

By the Corollary x in Bosi and Herden [3] there exist rational and continuous extensions $\dot{\succ}_c$ and $\dot{\succ}_t$ of $\succ_c$ and $\succ_t$, respectively. Now by Debreu theorem [1] there exist continuous functions w and v representing $\dot{\succ}_c$ and $\dot{\succ}_t$, respectively.

Now as w and v are continuous functions on the compact connected metric space Z their images are finite closed intervals, i.e., $w(Z) = I_w$ and $v(Z) = I_v$ are finite closed intervals.

Claim 1. $A \sim \{x\} \cup \{y\}$, if $x, y \in A$ are such that $w(x) = \max w(A)$ and $v(y) = \max v(A)$.

Proof. Let $B = \{x\} \cup \{y\}$ for x, y as in Claim 1. If $B = A$ there is nothing to prove. Assume $B \neq A$. By the definition of w and v it follows that

$$z \not\succ_c x \text{ and } z \not\succ_t y \text{ for each } z \in A \setminus B. \tag{14}$$

Let $F = \{z_1, \ldots, z_n\}$ be an arbitrary finite set in $A \setminus B$. By (14) we have $B \sim B \cup \{z_1\}$. By the same token $B \cup \{z_1\} \sim B \cup \{z_1, z_2\}$. Continuing in this way we get $B \cup \{z_1, \ldots, z_{n-1}\} \sim B \cup \{z_1, \ldots, z_n\}$. Thus

$$B \sim B \cup F \text{ for an arbitrary finite set } F \subset A \setminus B. \tag{15}$$

Now let $G = \{z_1, z_2, \ldots\}$ be an arbitrary countable dense set in $A \setminus B$. By (15) we have

$$B \sim B \cup G_n \text{ where } G_n = \{z_1, \ldots, z_n\}. \tag{16}$$

Obviously $B \cup G_n \to A$. By the continuity of $\succcurlyeq$ in the Hausdorff metric h this implies $B \sim A$.

Claim 2. $A \succcurlyeq B$ if $\max w(A) \geq \max w(B)$ and $\max v(A) \leq \max v(B)$.

Proof. Let $x \in \arg\max w(A)$ and $y \in \arg\max v(B)$. By the assumptions $w(x) \geq w(y)$ and $v(x) \leq v(y)$. The first inequality implies that $y \not\succ_c x$ and the second inequality implies that $x \not\succ_t y$. Hence $A \succcurlyeq A \cup \{y\}$ and $B \cup \{x\} \succcurlyeq B$ By Claim 1 we have $A \cup \{y\} \sim B \cup \{x\} \sim \{x\} \cup \{y\}$. Therefore

$$A \succcurlyeq A \cup \{y\} \sim B \cup \{x\} \succcurlyeq B.$$

□

Let $g : \mathcal{K} \to I_w \times (-I_v)$ be defined as $g(A) = (\max w(A), -\max v(A))$ and let $D = g(\mathcal{K}) \subset I_w \times (-I_v)$ be the image of mapping g. Function g is continuous by the uniform continuity of w and v on a compact metric space Z, and by the definition of the Hausdorff metric h. By the Blaschke's compactness theorem [6] $(\mathcal{K}, h)$ is a compact metric space. Since $\succcurlyeq$ is a continuous preference relation on $(\mathcal{K}, h)$ by the Debreu Theorem [1] there exists a continuous utility function $U : \mathcal{K} \to R$ representing $\succcurlyeq$. By Claim 2 $\max w(A) = \max w(B)$ and $\max v(A) = \max v(B)$ imply that $U(A) = U(B)$. Therefore, we can define function $f : D \to R$ by

$$f(\max w(A), -\max v(A)) = U(A). \tag{17}$$

By Claim 2 function f is increasing. Compactness of $(\mathcal{K}, h)$ and continuity of U and g imply that f is continuous. By Corollary 1 of Theorem 1 there exists a continuous increasing extension F of function f into R^2. Define $u : I_w \times I_v \to R$ by $u(a, b) = F(a, -b)$, and note that $U(A) = f(a, -b) = u(a, b)$ for $(a, b) = (\max w(A), -\max v(A))$. Hence (u, v, w) is a TSU representation. □

A Strict Representation

Let $C, T : \mathcal{K} \to \mathcal{K}$ two choice functions. that is $C(A) \subset A$, and $T(A) \subset A$ $C(A)$ and $T(A)$ are interpreted as the set of chosen alternatives and as the set of most tempting alternatives for each menu $A \in \mathcal{K}$, respectively.

A choice function $F : \mathcal{K} \to \mathcal{K}$ satisfies Houthakker's axiom if $x \in F(A) \cap B$ and $y \in A \cap F(B)$ imply $x \in F(B)$. It is well known that a choice function F satisfies Houthakker's axiom if and only if there exists a utility function on the set of alternatives Z such that F maximizes this funtion.

Property 1. Preference relation $\succcurlyeq$ is continuous and choice functions C, T are continuous and satisfy Houthakker's Axiom.

Property 2. $T(A \cup B) \cap B \neq \emptyset$ implies $A \cup B \succcurlyeq B$. If also $C(A \cup B) \cap B = \emptyset$ then $A \cup B \succ B$.

Property 3. $C(A \cup B) \cap A \neq \emptyset$ implies $A \succcurlyeq A \cup B$. If also $T(A \cup B) \cap A = \emptyset$ then $A \succ A \cup B$.

Axiom B. There exist choice functions C, T such that $(\succcurlyeq, C, T)$ satisfy properties 1-3.

Definition. Preference $\succcurlyeq$ has a strict TSU representation if there exist continuous functions $v : Z \to R,\ w : Z \to R,\ u : w(Z) \times v(Z) \to R$ with u is strictly increasing in its first variable and strictly decreasing in its second variable and such that function

$$U(A) = u(\max_{x \in A} w(x), \max_{y \in A} v)y))$$

represents $\succcurlyeq$.

Theorem B. *A continuous preference relation $\succcurlyeq$ on $(\mathcal{K}, h)$ satisfies Axiom B if and only if $\succcurlyeq$ has a strict TSU representation.*

Proof. Let (u, v, w) be a strict TSU representation for $\succcurlyeq$. Then it is easy to see that $\succcurlyeq$ is continuous and choice functions $C, T : \mathcal{K} \to \mathcal{K}$ defined as

$$C(A) = \{x \in A : w(x) \geq w(y),\ \forall y \in A\}$$

$$T(A) = \{x \in A : v(x) \geq v(y),\ \forall y \in A\}$$

are upper hemicontinuous. Show this for C. Indeed, let $A_k \to A$ in h metric and $x_k \in C(A_k)$ $(k \in N)$. Since Z is compact there exists a subsequence $x_{k(l)}$ $(l \in N)$ converging to some point x. Obviously $x \in A$. Wlg assume $x_k \to x$. We have $w(x_k) \geq w(y),\ \forall y \in A_k,\ k \in N$. Since w is uniformly continuous on Z by the definition of the Hausdorff metric h, it follows that $w(x) \geq w(y),\ \forall y \in A$. That is $x \in C(A)$. So, C is closed and hence uhc.

Strict increasing and decreasing properties of function u imply the other statements in properties 1-3. (That is H's Axiom and properties 2,3.) Assume now that properties 1-3 are satisfied for relation $\succcurlyeq$. Show that there exists a continuous function $w : Z \to R$ To this end we set $x \dot{\succcurlyeq} y$ for $x, y \in Z$ if there exists $A \in \mathcal{K}$ such that

$x, y \in A$ and $x \in C(A)$. By the Arrow theorem the Houthakker's Axiom implies that $\dot{\succcurlyeq}$ is a rational preference on Z rationalizing C. We show that $\dot{\succcurlyeq}$ is continuous on Z. Let $x_k \dot{\succcurlyeq} y_k$ $(k \in N)$ and $x_k \to x$ and $y_k \to y$. Then $\{x_k, y_k\} \xrightarrow{h} \{x, y\}$, $x_k \in C(\{x_k, y_k\})$ and upper hemicontinuity of C imply that $x \in C(\{x, y\})$. That is $x \dot{\succcurlyeq} y$. By the Debreu theorem [1] there exists a continuous utility function $w : Z \to R$ representing $\dot{\succcurlyeq}$. We show in the similar way that there exists a continuous function $v : Z \to R$ representing $\ddot{\succcurlyeq}$, where $\ddot{\succcurlyeq}$ is a rational preference rationalising the choice function T.

Claim 3. (i) If x, y are such that $w(x) = \max w(A)$ and $v(y) = \max v(A)$ then $A \sim \{x\} \cup \{y\}$.
(ii) If $\max w(A) \geq \max w(B)$ and $\max v(A) \leq \max v(B)$ then $A \succcurlyeq B$. If one of these inequalities is strict then $A \succ B$.

Proof. (i) Let $B = \{x, y\}$. By properties 2 and 3 $B \succcurlyeq A \cup B = A \succcurlyeq B$ and therefore $A \sim B$.
(ii) Property 2 implies that $A \cup B \succcurlyeq B$, and Property 3 implies that $A \succcurlyeq A \cup B$. Therefore $A \succcurlyeq B$. The strict version follows from the second parts of properties 2 and 3.
Since $\succcurlyeq$ is rational and $(\mathcal{K}, h)$ is second countable there exists a continuous utility function $U \colon \mathcal{K} \to \mathcal{K}$ representing $\succcurlyeq$. Set

$$D = \{(a, b) \in w(Z) \times v(Z) : (a, b) = (\max w(A), \max v(A)) \text{ for some } A \in \mathcal{K}\},$$

and define $f : D \to R$ by

$$f(a, b) = U(A)$$

for A such that $(\max w(A), -\max v(A)) = (a, b)$. By Claim 3 (ii) f is well defined and strictly increasing. Continuity of functions w and v imply that f is continuous. By Corollary 2 of Theorem 2 there exists a continuous strictly increasing extension F of function f into R^2. Set $u(a, -b) = F(a, b)$ and note that $U(A) = F(a, b) = u(a, -b)$ for set $A \in \mathcal{K}$ such that $(\max w(A), \max v(A)) = (a, -b)$. Hence (u, v, w) is a strict TSU representation of $\succcurlyeq$. $\square$

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