Generic Impossibility of Arrow's Impossibility Theorem

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Abstract

Bossert and Suzumura (2009) showed that the assignment of a quasi-transitive Arrovian collective choice rule F (not necessary reflexive and complete) to the corresponding set of decisive coalitions $\mathcal{V}_F$ defines a surjective map $\rho : \mathbb{CR}^{QT} \to \mathbb{F} \subseteq 2^{\mathcal{I}}$, where $\mathbb{CR}^{QT}$ is the set of all quasi-transitive Arrovian collective choice rules and $\mathbb{F}$ the set of all filters in $\mathcal{I}$. One major objective in the present paper is to determine the inverse image of the set of all ultrafilters $\mathbb{UF} \subseteq \mathbb{F}$ under ρ to be $\rho^{-1}(\mathbb{UF}) = \mathbb{CR}^{QT,SP}$, that is, the subset of $\mathbb{CR}^{QT}$ consisting of those satisfying the so-called strong preference property, which is also precisely the set of all Arrovian collective choice rules lying within $\mathbb{CR}^{QT}$ that admit dictators. Another major objective is to show that in the presence of infinitely many alternatives the set of Arrovian collective choice rules which fall into Arrow's impossibility theorem is "negligible" in the totality of quasi-transitive Arrovian collective choice rules, i.e., $\mathbb{CR}^{QT,SP}$ is nowhere dense in $\mathbb{CR}^{QT}$, where relevant spaces are equipped with suitable topologies.

1 Introduction

Arrow's seminal work on social welfare functions spawned a celebrated theorem, known in the literature as Arrow's impossibility theorem (Arrow, 1963), which states that the set of seemingly innocuous criteria imposed on social welfare functions (universal domain, unanimity, independence of irrelevant alternatives, and no dictatorship) is inconsistent provided that there are at least three alternatives and that society consists of finitely many individuals. Alternatively stated, in a finite society with at least three alternatives, if a social welfare function satisfies universal domain, unanimity, and independence of irrelevant alternatives conditions, then there must exist a dictator. On the other hand, Fishburn (1970) demonstrated that the same set of Arrow's conditions becomes consistent when society admits infinitely many individuals, which in turn entails the possibility of the existence of a social welfare function satisfying universal domain, unanimity, and independence of irrelevant alternatives, but yet free from dictatorship.

Kirman and Sondermann (1972) considered as a model of large society a measure space of agents in which each individual is negligible, and hence a dictator is inherently nonexistent. However, they argued that dictatorship in a different but meaningful sense still persists even in society with a continuum of agents, namely there may be an infinitesimal group of individuals who can dictate an Arrovian social welfare function. Moreover, Kirman and Sondermann (1972) demonstrated that the original set of individuals T can be topologically completed by the set of ultrafilters $\mathbb{UF}$ equipped with the Stone topology, in which the original "visible" dictators, if any, are embedded as the subset of fixed (trivial) ultrafilters, whereas the remaining free ultrafilters represent invisible dictators who are to be thought of as "idealized" dictators in the sense that they can be approximated by visible dictators.

Armstrong (1980) introduced the notion of coalition algebra $\mathcal{T}$, an algebra of subsets of T that is to be interpreted as a collection of admissible coalitions of individuals such as observable coalitions, and also introduced an ideal $\mathcal{N}$ of negligible coalitions consisting of a coalition that cannot at least in one situation dictate the Arrovian social welfare function under consideration. Note that filters and ultrafilters can be defined as objects associated with $\mathcal{T}$ instead of T and also that when $\mathcal{T} = 2^T$, the power set of T , and $\mathcal{N} = \{\varnothing\}$, Armstrong's formalism reduces to that of Kirman and Sondermann (1972). Furthermore, Armstrong (1980) constructed an injective function σ from $\mathbb{UF}$ to the set of all Arrovian social welfare functions, which can be viewed as a "section" of a function ρ in the sense that $\rho \circ \sigma = id$ holds where ρ assigns each Arrovian social welfare function to the corresponding ultrafilter of decisive coalitions. Armstrong (1985), moreover, identified the image of σ to be the set of Arrovian social welfare functions satisfying the so-called relational monotonicity.

Hansson (1976) took full advantage of mathematical structures of decisive coalitions (filters and ultrafilters) and obtained a series of results that we focus on in the present paper. To be more specific, Hansson (1976) proved among other things that each Arrovian social welfare function F gives rise to an ultrafilter, which is an ultrafilter of decisive coalitions associated with F , and conversely, given an ultrafilter $\mathcal{V}$ there is an Arrovian social welfare function $F_{\mathcal{V}}$ whose associated ultrafilter of decisive coalitions is precisely the one given at the outset. Noting that some of the properties of ultrafilters are debatable when viewed as properties of decisive coalitions, Hansson weakened transitivity of social preferences to quasi-transitivity and yet obtained analogous results concerning the relationship between quasi-transitive Arrovian social welfare functions and filters instead of ultrafilters. Another noteworthy result is that when the cardinal number α of the set of individuals T is infinite and there are finitely many alternatives, the cardinal number of the set of Arrovian social welfare functions whose associated ultrafilters are free (i.e., invisible dictators) is $2^{2^{\alpha}}$ and is equal to the cardinal number of the set of all Arrovian social welfare functions. This indicates that the set of non-dictatorial Arrovian social welfare functions becomes quite large in the presence of a relatively small number of alternatives.

Bossert and Suzumura (2009) sharpened the earlier results of Hansson (1976) by demonstrating that those results remain valid in the absence of reflexivity and completeness. To be more specific, they proved in the terminology here adopted that the assignment of a quasi-transitive Arrovian collective choice rule F (not necessary reflexive and complete) to the corresponding set of decisive coalitions $\mathcal{V}_F$ defines a surjective map $\rho : \mathbb{CR}^{QT} \to \mathbb{F} \subseteq 2^{\mathcal{T}}$ where $\mathbb{CR}^{QT}$ is the set of all quasi-transitive Arrovian collective choice rules and $\mathbb{F}$

the set of all filters in $\mathcal{T}$.

Our first goal in the present paper is to determine the inverse image of the set of all ultrafilters $\mathbb{UF} \subseteq \mathbb{F}$ under ρ , which turns out to be $\rho^{-1}(\mathbb{UF}) = \mathbb{CR}^{QT,SP} \subseteq \mathbb{CR}^{QT}$: the subset of $\mathbb{CR}^{QT}$ consisting of those satisfying the so-called strong preference property, i.e., those $F \in \mathbb{CR}^{QT}$ which transform strict individual preferences into strict social preferences. Note that $\mathbb{CR}^{QT,SP}$ is precisely the set of Arrovian collective choice rules lying within $\mathbb{CR}^{QT}$ that admit dictators.

Motivated by the work of Armstrong (1985) which identified the image of a "section" of the restriction ρ' of ρ to $\mathbb{CR}^{T,C} \subseteq \mathbb{CR}^{QT,SP}$, where $\mathbb{CR}^{T,C}$ is the set of all complete transitive Arrovian social choice rules which are nothing but the so-called Arrovian social welfare functions, our second goal is to find some useful properties of two sections $\sigma, \sigma^a : \mathbb{F} \to \mathbb{CR}^{QT}$ of ρ , where σ is a natural extension of the one constructed by Kirman and Sondermann (1972), Hansson (1976), Armstrong (1980) and others, and σ^a the one considered by Bossert and Suzumura (2009). We scrutinize the internal structure of $\mathbb{CR}^{QT}$ since $\mathbb{CR}^{QT}$ plays the role of a background space in our later discussions.

We endow $\mathbb{F}$ with a T_0 -topology in a manner analogous to that in which we equip $\mathbb{UF}$ with the Stone topology, and we endow $\mathbb{CR}^{QT}$ with a compact Hausdorff topology in a natural manner.

Our final goal is to show that in the presence of infinitely many alternatives the set of Arrovian collective choice rules that fall into Arrow's impossibility theorem is in some sense "negligible" in the totality of quasi-transitive

Arrovian collective choice rules, i.e., $\mathbb{CR}^{QT,SP}$ is nowhere dense in $\mathbb{CR}^{QT}$. We also show that in the presence of infinitely many alternatives the set of the so-called Arrovian social welfare functions $\mathbb{CR}^{T,C}$ is negligible in the totality of quasi-transitive Arrovian collective choice rules that give rise to ultrafilters of decisive coalitions, i.e., $\mathbb{CR}^{T,C}$ is nowhere dense in $\mathbb{CR}^{QT,SP}$.

We also discuss social choice problems in conjunction with "large society" and put some of the known results in our perspective.

Section 2 introduces notations and definitions, and Section 3 discusses, with the newly added notions of coalition algebras and ideals of negligible coalitions , properties of decisive coalitions, and the validity of some of the related results known in the literature to date including the so-called Field Expansion Lemma. Section 4 investigates properties of ρ which assigns each $F \in \mathbb{CR}^{QT}$ to a unique filter $\mathcal{V} \in \mathbb{F}$. We are particularly concerned with determining the inverse image $\rho^{-1}(\mathbb{UF}) \subseteq \mathbb{CR}^{QT}$ along with restating Arrow's classical impossibility theorem in our framework. In Section 5, inspired by the work of Armstrong (1985) which identifies the set of precisely dictatorial Arrovian social welfare functions to be the image set of the restriction $\sigma' : \mathbb{UF} \to \mathbb{CR}^{QT,SP}$, we identify the image set of $\sigma^{a\prime} : \mathbb{UF} \to \mathbb{CR}^{QT,SP}$ and explicitly construct a bijection $\widetilde{\Phi}$ between them. Section 6 contains our main assertions that when there are infinitely many alternatives, $\mathbb{CR}^{QT,SP}$ becomes nowhere dense in $\mathbb{CR}^{QT}$, and $\mathbb{CR}^{T,C}$ becomes nowhere dense in $\mathbb{CR}^{QT,SP}$. Section 5 discusses society $(T, \mathcal{T}, \mathcal{N})$ with and without atoms.

2 Notations and Definitions

Throughout the paper we adhere to the following standard logic notations:

1. $\forall$ abbreviates "for all"
2. $\exists$ abbreviates "there exist"
3. $\wedge$ abbreviates "and"
4. $\vee$ abbreviates "or"
5. $\neg$ abbreviates "not"
6. $\Rightarrow$ abbreviates "imply"
7. $\Leftrightarrow$ abbreviates "if and only if"

We often omit these notations for brevity whenever meaning is clear from context.

We denote the set of all social alternatives by $\mathbb{A}$. A binary relation $\succsim$ on $\mathbb{A}$ is said to be a **preference relation**. We denote the set of all preference relations on $\mathbb{A}$ by $\mathcal{P}$.

The **asymmetric factor** $\succ$ and **symmetric factor** $\sim$ of $\succsim$ are defined by the following identities:

Asymmetric Factor $x \succ y \equiv x \succsim y \wedge \neg y \succsim x$

Symmetric Factor $x \sim y \equiv x \succsim y \wedge y \succsim x$

The following properties of $\succsim$ will be relevant in the sequel:

Transitivity $x \succsim y \wedge y \succsim z \Rightarrow x \succsim z$

Completeness $x \succsim y \vee y \succsim x$

Asymmetricity $x \succsim y \Rightarrow \neg y \succsim x$

Quasi-transitivity $x \succ y \wedge y \succ z \Rightarrow x \succ z$

Quasi-asymmetricity $x \succ y \Rightarrow \neg y \succ x$

Negative Quasi-transitivity $\neg x \succ y \wedge \neg y \succ z \Rightarrow \neg x \succ z$

A transitive and complete preference relation $\succsim$ on $\mathbb{A}$ is said to be an **ordering** (complete or total preorder).

Remark 1 *Our definition of completeness clearly subsumes reflexivity while some authors use an alternative definition of completeness, that is ,* $x \neq y \Rightarrow x \succsim y \vee y \succsim x$ *. In the discussions of the works of those authors who use the latter definition we say "reflexive and complete" to refer to "complete" in our sense.*

Remark 2 *It is important to note that the definition of quasi-transitivity adopted here is different from the one considered by Sen (1969), Schick (1969), and Hansson (1976) in that our definition, as in Bossert and Suzumura (2009), assumes neither reflexivity nor completeness.*

We denote the set of all orderings by $\mathcal{P}^{T,C}$.

Lemma 3 *If* $\succsim \in \mathcal{P}$ *is transitive, then it is quasi-transitive.*

Proof. Suppose $x \succ y \wedge y \succ z$. We claim $x \succ z$. By transitivity, $x \succsim z$. Suppose $z \succsim x$. Then since $x \succsim y$ we obtain by transitivity that $z \succsim y$. Since this contradicts to $y \succ z$, we deduce $\neg z \succsim x$, and hence $x \succ z$. ■

Lemma 4 *If* $\succsim \in \mathcal{P}$ *is transitive and complete, i.e.* $\succsim \in \mathcal{P}^{T,C}$ *, then it is negative quasi-transitive.*

Proof. Suppose $\neg x \succ y \wedge \neg y \succ z$. Since $\succsim$ is complete, we obtain $y \succsim x \wedge z \succsim y$, which implies by transitivity that $z \succsim x$. Thus $\neg x \succ z$ as desired. ■

Remark 5 *The preceding two lemmata show that both quasi-transitivity and negative quasi-transitivity are weaker than completeness and transitivity together.*

Lemma 6 *If* $\succsim \in \mathcal{P}$ *is negative quasi-transitive, then it is quasi-transitive.*

Proof. Suppose $x \succ y \wedge y \succ z$. By negative quasi-transitivity, $y \succ z$ implies that $y \succ x$ or $x \succ z$. Since $\succ$ is asymmetric, $x \succ y$ implies $\neg y \succ x$. Hence $x \succ z$. ■

Lemma 7 *If* $\succsim$ *is asymmetric, then* $x \succsim y \Leftrightarrow x \succ y$ *.*

Proof. Suppose $x \succsim y$. Then by asymmetry $\neg y \succsim x$ and thus $x \succsim y \wedge \neg y \succsim x \Leftrightarrow x \succ y$ while the other implication holds trivially. ■

Lemma 8 *If* $\succsim \in \mathcal{P}$ *is complete, then it is quasi-asymmetric.*

Proof. Let $\succsim \in \mathcal{P}$ be complete. Then $x \succ y \Leftrightarrow \neg y \succsim x$. Now suppose $x \succ y$. We must show $\neg y \succ x$. By way of contradiction, suppose $y \succ x$. Then $\neg x \succsim y$, which leads to a contradiction. ■

Since a preference relation $\succsim \in \mathcal{P}$ is nothing but a subset of $\mathbb{A} \times \mathbb{A}$, we can consider its characteristic function

$$R : \mathbb{A}\times\mathbb{A}\rightarrow \{0,1\}$$

defined by

$$\begin{aligned} x \succsim y \quad &\Leftrightarrow \quad R(x,y) = 1 \\ \neg x \succsim y \quad &\Leftrightarrow \quad R(x,y) = 0 \; . \end{aligned}$$

Note that R obviously possesses the following properties:

Property 1 $x \succ y \Leftrightarrow (R(x,y) = 1 \wedge R(y,x) = 0)$

In order to minimize notational complexity we introduce the following shorthand notation:

$$R(x,y) = 1^{+} \text{ abbreviates } R(x,y) = 1 \wedge R(y,x) = 0$$

Property 2 $x \sim y \Leftrightarrow (R(x,y) = R(y,x) = 1)$

Property 3 $(x \succsim y \vee y \succsim x) \Leftrightarrow (R(x,y) + R(y,x) \geq 1)$

Property 4 $[(x \succsim y \wedge y \succsim z) \Rightarrow x \succsim z]$

$$\Leftrightarrow [(R(x,y) = 1 \wedge R(y,z) = 1) \Rightarrow R(x,z) = 1]$$

Property 5 $[(x \succ y \wedge y \succ z) \Rightarrow x \succ z]$

$$\Leftrightarrow \left[\left(R(x,y) = 1^{+} \wedge R(y,z) = 1^{+}\right) \Rightarrow R(x,z) = 1^{+}\right]$$

We have seen that to each preference relation $\succsim$ a certain $\{0,1\}$-valued function R , namely its characteristic function, can be associated. On the other hand, it is obvious that to each $\{0,1\}$-valued function R we can associate a preference relation $\succsim$ defined by

$$x \succsim y \Leftrightarrow R(x,y) \geq 0 \ .$$

In our remaining discussion we do not distinguish between preference relations $\succsim$ on $\mathbb{A}$ and their associated characteristic functions R and use the same symbol $\mathcal{P}$ to refer to the totality of $\succsim$ and of R .

It is a simple exercise to confirm that the following two properties of R

$$\begin{aligned} R(x,y) + R(y,x) &\geq 1 \text{ and} \\ (R(x,y) = 1 \wedge R(y,z) = 1) &\Rightarrow R(x,z) = 1 \end{aligned}$$

completely characterize orderings on $\mathbb{A}$. In fact, the preference relation $\succsim$ associated with a $\{0, 1\}$-valued function R satisfying the above two properties can easily be seen to be an ordering.

In order to distinguish aggregated preference relations of society from those of individuals we call the former **social preference relations** and the latter **individual preference relations**. Likewise we call aggregated orderings of society **social orderings** and orderings of individuals **individual orderings** or **orderings** for short. In the sequel we assume that all the individual preference relations are orderings while social preference relations may or may not be social orderings.

Definition 9 *A Boolean algebra is a non-empty set $\mathcal{T}$ with three operations*

$$\cup \ , \ \cap \ , \ and \ -$$

satisfying the following axioms: For $U, V, \ldots \in \mathcal{T}$

1. $U \cup V = V \cup U, \quad U \cap V = V \cap U,$
2. $U \cup (V \cup W) = (U \cup V) \cup W, \quad U \cap (V \cap W) = (U \cap V) \cap W,$
3. $(U \cap V) \cup V = V \quad (U \cup V) \cap V = V,$
4. $U \cap (V \cup W) = (U \cap V) \cup (U \cap W) \quad U \cup (V \cap W) = (U \cup V) \cap (U \cup W)\,,$
5. $(U \cap -U) \cup V = V \quad (U \cup -U) \cap V = V\ .$

The element $U \cap -U$, which does not depend on the choice of $U \in \mathcal{T}$, is said to be the **zero element** of $\mathcal{T}$ and is denoted by $\curlywedge$. The element $U \cup -U$, which does not depend on the choice of $U \in \mathcal{T}$, is said to be the **unit element** of $\mathcal{T}$ and is denoted by $\curlyvee$. Note that when $\mathcal{T}$ is a Boolean algebra of subsets of T , $-U$ is the complement of U , $\varnothing$ the zero element of $\mathcal{T}$, and T the unit element of $\mathcal{T}$.

Definition 10 *A non-empty subset $\mathcal{N}$ of a Boolean algebra $\mathcal{T}$ is said to be an ideal provided*

1. if $V, V' \in \mathcal{N}$, then $V \cup V' \in \mathcal{N}$;

2. if $V' \in \mathcal{N}$ and $V \subseteq V'$, then $V \in \mathcal{N}$, where $V \subseteq V'$ if and only if $V \cap V' = V$ if and only if $V \cup V' = V'$.

Definition 11 *In the sequel T denotes a set of individuals, which may be finite or infinite, and we consider a triple $(T, \mathcal{T}, \mathcal{N})$ where $\mathcal{T}$ is a Boolean algebra of subsets of T and $\mathcal{N}$ an ideal in $\mathcal{T}$. Following Armstrong (1980) we call $\mathcal{T}$ a **coalition algebra** and each element $V \in \mathcal{T}$ a **coalition**. $\mathcal{N}$ is to be thought of as an ideal of **negligible coalitions** in $\mathcal{T}$.*

Remark 12 *If $\mathcal{T} = 2^T$ (the set of all subsets of a finite set T) and $\mathcal{N} = \{\varnothing\}$, the above definition reduces to the one considered by Arrow (1963) and if $\mathcal{T}$ is a σ-algebra of subsets of T and $\mathcal{N}$ the set of all λ -null sets, where λ is a countably additive finite measure on $\mathcal{T}$, it reduces to the one considered by Kirman and Sondermann (1980).*

Definition 13 *A* ***preference profile*** *is a function* $\widehat{R} : T \to \mathcal{P}^{T,C}$ *. We require* $\widehat{R}(x,y) : T \to \{0,1\}$ *to be* $\mathcal{T}$*-measurable for each pair* $(x,y) \in \mathbb{A} \times \mathbb{A}$ *.*

Let Γ be the set of all preference profiles, i.e., $\Gamma = \left\{\widehat{R}\right\}$ and let $\Delta \subseteq \Gamma$.

Definition 14 *A* ***collective choice rule*** *is a function* $F : \Delta \to \mathcal{P}$ *.*

Remark 15 *A similar but slightly different construction identifying* $\precsim$ *with its indicator function (characteristic function) was adopted by Torres (2005) under the assumption that what we refer to as collective choice rules always generate social orderings.*

We introduce the following axioms on collective choice rules that are prevalent in the literature. These axioms are also known as Arrow's conditions.

In what follows, whenever a property $P(t)$ holds for almost all $t \in V \in \mathcal{T}$, i.e. $\{t \in V : \neg P(t)\} \in \mathcal{N}$, we simply write $P(V)$ for brevity unless confusion may result.

Unristricted Domain, UD

$$\Delta = \Gamma$$

Independence of Irrelevant Alternatives, IIA

$$\widehat{R}(T)(x,y) = \widehat{R}'(T)(x,y) \Rightarrow F\left(\widehat{R}\right)(x,y) = F\left(\widehat{R}'\right)(x,y)$$

Unanimity, U

$$\forall (x,y)\ \forall \widehat{R}\ \left[\widehat{R}(T)(x,y) = 1^{+} \Rightarrow F\left(\widehat{R}\right)(x,y) = 1^{+}\right]$$

Remark 16 *Some authors use the following alternative form of condition IIA:*

Independence of Irrelevant Alternatives, IIA'

$$\forall z, w \in \{x,y\}$$
$$\left\{\begin{array}{c} \left[\widehat{R}(t)(z,w) = 1 \Leftrightarrow \widehat{R}'(t)(z,w) = 1\right] \text{ for almost all } t \in T \\ \Rightarrow \left[F\left(\widehat{R}\right)(z,w) = 1 \Leftrightarrow F\left(\widehat{R}'\right)(z,w) = 1\right] \end{array}\right\}.$$

The following lemma shows that our version of independence of irrelevant alternatives condition follows from the above alternative version. It is a simple exercise to verify that **IIA'** and **IIA** are indeed equivalent.

Lemma 17 ***IIA'*** *implies* ***IIA****.*

Proof. Suppose $\widehat{R}(T)(x,y) = \widehat{R}'(T)(x,y)$. Then either

$$\begin{aligned} \widehat{R}(t)(x,y) &= \widehat{R}'(t)(x,y) \\ &= 1 \text{ or } 0 \end{aligned}$$

for almost all $t \in T$. Considering the truth values of $\Leftrightarrow$ we obtain

$$\widehat{R}(t)(x,y) = 1 \Leftrightarrow \widehat{R'}(t)(x,y) = 1 \text{ for almost all } t \in \mathcal{T}$$

and as a consequence of **IIA'** we obtain

$$F\left(\widehat{R}\right)(x,y) = 1 \Leftrightarrow F\left(\widehat{R'}\right)(x,y) = 1 \ ,$$

which is equivalent to

$$F\left(\widehat{R}\right)(x,y) = 0 \Leftrightarrow F\left(\widehat{R'}\right)(x,y) = 0 \ .$$

Consequently,

$$F\left(\widehat{R}\right)(x,y) = F\left(\widehat{R'}\right)(x,y)$$

as desired. ■

3 Properties of Decisive Coalitions

For the rest of this paper, we assume that $\mathbb{A}$ contains at least two elements so that there exists at least one pair of distinct alternatives $x, y \in \mathbb{A}$.

Definition 18 *For distinct $x, y \in \mathbb{A}$, a coalition $V \in \mathcal{T}$ is said to be (x,y)-* ***decisive*** *if*

$$\forall \widehat{R} \ \left[\widehat{R}(V)(x,y) = 1^{+} \Rightarrow F\left(\widehat{R}\right)(x,y) = 1^{+}\right] \ .$$

Definition 19 *A coalition $V \in \mathcal{T}$ is said to be* ***decisive*** *if it is (x, y)-decisive for all distinct $x, y \in \mathbb{A}$.*

Lemma 20 *Every coalition $V \in \mathcal{T}$ which contains a (x, y)-decisive coalition $V' \in \mathcal{T}$ is itself (x, y)-decisive.*

Proof. It follows from Definition 18. ■

Proposition 21 *Suppose F satisfies* ***UD*** *. Then if V is decisive, V cannot be negligible.*

Proof. Assume the contrary to the conclusion, suppose $V \in \mathcal{N}$ while V is decisive. This would mean that for all distinct x, y

$$\forall \widehat{R} \left[\widehat{R}(V)(x, y) = 1^+ \Rightarrow F\left(\widehat{R}\right)(x, y) = 1^+ \right] .$$

Note that $\widehat{R}(V)(x, y) = 1^+ \Leftrightarrow \left\{ t \in V : \widehat{R}(V)(x, y) \neq 1^+ \right\} \in \mathcal{N}$ so that with $V \in \mathcal{N}$, $\widehat{R}(V)(x, y) = 1^+$ follows trivially. Consequently, for all distinct x and y , $\forall \widehat{R}\ F\left(\widehat{R}\right)(x, y) = 1^+$. Recalling that $\Gamma \neq \varnothing$, choose any profile $\widehat{R} \in \Gamma$ (let $\widehat{R} = \varnothing$ if $T = \varnothing$) and let x , y be any pair of distinct alternatives. We then have $F\left(\widehat{R}\right)(x, y) = F\left(\widehat{R}\right)(y, x) = 1^+$, which means that $F\left(\widehat{R}\right)(x, y) = 1 \wedge F\left(\widehat{R}\right)(y, x) = 0$ and $F\left(\widehat{R}\right)(y, x) = 1 \wedge F\left(\widehat{R}\right)(x, y) = 0$. This leads to a contradiction. ■

Corollary 22 *If F satisfies* ***UD*** *and* ***U****, T cannot be negligible.*

Lemma 23 *Suppose $|\mathbb{A}| \geq 3$ and that F is quasi-transitive and satisfies* ***UD****,* ***U****, and* ***IIA****. Then a coalition $V \in \mathcal{T}$ is (x, y)-decisive if and only if*

$$\forall \widehat{R} \ \left\{ \left[\widehat{R}\,(V)\,(x,y) = 1^{+} \wedge \widehat{R}\,(V^{c})\,(y,x) = 1^{+} \right] \Rightarrow F\left(\widehat{R}\right)(x,y) = 1^{+} \right\} \ .$$

Proof. Since "only if" direction of the assertion trivially holds, it suffices to prove the assertion only for "if" direction. To this end, suppose $\widehat{R}\,(V)\,(x,y) = 1^{+}$, where x, y are distinct. Let $z \notin \{x, y\}$ and modify $\widehat{R}$ without altering the preferences over x and y to get $\widehat{R}'$ such that

$$\begin{aligned} \widehat{R}'\,(V)\,(x,z) &= \widehat{R}'\,(V)\,(z,y) = 1^{+} \text{ and} \\ \widehat{R}'\,(V^{c})\,(y,z) &= \widehat{R}'\,(V^{c})\,(x,z) = 1^{+} \ . \end{aligned}$$

This can be carried out since $|\mathbb{A}| \geq 3$ and F satisfies **UD**. Then by assumption, $F\left(\widehat{R}'\right)(z,y) = 1^{+}$ and by **U,** $F\left(\widehat{R}'\right)(x,z) = 1^{+}$. Now by quasi-transitivity, we obtain $F\left(\widehat{R}'\right)(x,y) = 1^{+}$ and by **IIA**, we deduce

$$F\left(\widehat{R}\right)(x,y) = F\left(\widehat{R}'\right)(x,y) = 1^{+}$$

as desired. ■

Armstrong (1980, p.65) introduced the notion of $\mathcal{N}$- competitive social welfare functions. In our context a similar notion can be defined as follows:

Definition 24 *A collective choice rule F is said to be $\mathcal{N}$-* ***competitive*** *pro-*

vided that $\forall N \in \mathcal{N}\, \exists x \neq y\, \exists \widehat{R}\ \left[\widehat{R}(N)(x,y) = 1^{+} \wedge F\left(\widehat{R}\right)(x,y) \neq 1^{+}\right]$ *or alternatively stated, no null set* N *is decisive.*

Proposition 21 asserts that with our definition of decisiveness every collective choice rule F is $\mathcal{N}$- **competitive** provided that $|\mathbb{A}| \geq 2$ and F satisfies **UD**.

Lemma 25 *Suppose* $|\mathbb{A}| \geq 3$ *and that* F *is quasi-transitive and satisfies* ***UD***, ***U***, *and* ***IIA***. *Then a coalition* $V \in \mathcal{T}$ *is* (x,y)*-decisive if and only if*

$$\exists \widehat{R}\ \left[\widehat{R}(V)(x,y) = 1^{+} \wedge \widehat{R}(V^{c})(y,x) = 1^{+} \wedge F\left(\widehat{R}\right)(x,y) = 1^{+}\right] .$$

Proof. Suppose that $\widehat{R}$ satisfies $\widehat{R}(V)(x,y) = 1^{+} \wedge \widehat{R}(V^{c})(y,x) = 1^{+} \wedge F\left(\widehat{R}\right)(x,y) = 1^{+}$ and let $\widehat{R}'$ be an arbitrary preference profile such that $\widehat{R}'(V)(x,y) = 1^{+} \wedge \widehat{R}'(V^{c})(y,x) = 1^{+}$. Note that we have $\widehat{R}(T)(x,y) = \widehat{R}'(T)(x,y)$ and $\widehat{R}(T)(y,x) = \widehat{R}'(T)(y,x)$. Then by **IIA** we obtain $F\left(\widehat{R}'\right)(x,y) = F\left(\widehat{R}\right)(x,y) = 1$ and $F\left(\widehat{R}'\right)(y,x) = F\left(\widehat{R}\right)(y,x) = 0$, and hence $F\left(\widehat{R}'\right)(x,y) = 1^{+}$. Now we can appeal to Lemma 23 to establish our claim for "only if " implication. On the other hand, suppose that V is (x,y)-decisive. Choose $\widehat{R}$ such that $\widehat{R}(V)(x,y) = 1^{+} \wedge \widehat{R}(V^{c})(y,x) = 1^{+}$. Then by **UD** and Lemma 23 we obtain $F\left(\widehat{R}\right)(x,y) = 1^{+}$. This proves "if " implication of our claim. ■

Proposition 26 *Suppose* F *satisfies* ***UD***. *Then if* $V \in \mathcal{T}$ *is* (x,y)*-decisive,* V^{c} *cannot be* (y,x)*-decisive.*

Proof. Suppose V is (x, y)-decisive. Assume the contrary to the consequence, that V^c is (y, x)-decisive. By **UD** we may choose $\widehat{R}$ such that $\widehat{R}(V^c)(y, x) = 1^+$ and $\widehat{R}(V)(x, y) = 1^+$. By assumption on V^c we deduce that $F\left(\widehat{R}\right)(y, x) = 1^+$. On the other hand, by assumption on V we deduce that $F\left(\widehat{R}\right)(x, y) = 1^+$, which leads to a contradiction. ■

Corollary 27 *Suppose F satisfies **UD**. Then if $V \in \mathcal{T}$ is decisive , V^c cannot be decisive.*

Lemma 28 *Suppose $|\mathbb{A}| \geq 3$ and that F is quasi-transitive and satisfies **UD**, **U**, and **IIA**. Then if $V \in \mathcal{T}$ is (x, y)-decisive, V is (x, w)-decisive for each $w \notin \{x, y\}$.*

Proof. By **UD**, we may choose $\widehat{R}$ such that $\widehat{R}(V)(x, y) = \widehat{R}(V)(y, w) = 1^+$ and $\widehat{R}(V^c)(y, w) = \widehat{R}(V^c)(w, x) = 1^+$. By assumption on V we have $F\left(\widehat{R}\right)(x, y) = 1^+$, and by **U** we obtain $F\left(\widehat{R}\right)(y, w) = 1^+$. Then by quasi-transitivity $F\left(\widehat{R}\right)(x, w) = 1^+$. Furthermore, since $\widehat{R}(V)(x, w) = 1^+$ and $\widehat{R}(V^c)(w, x) = 1^+$, by Lemma 25 V is (x, w)-decisive. ■

The following proposition is a version of Sen's field expansion lemma (Sen, 1995, p.4), which is crucial to the rest of our arguments and was proved by Bossert and Suzumura (2009, p.139) for the case that T is finite, $\mathcal{T} = 2^T$, and $\mathcal{N} = \{\varnothing\}$ while F is assumed be neither reflexive nor complete. We present our version of proof even though it is merely a minor modification of the one appearing in Bossert and Suzumura (2009, p.139) since it is short and demonstrates that Bossert and Suzumura's basic arguments are still valid in the broader context of coalition algebras and that our formalism apparently renders these by now standard arguments more transparent.

Proposition 29 (Field Expansion Lemma) *Suppose* $|\mathbb{A}| \geq 3$ *and that* F *is quasi-transitive and satisfies* ***UD****,* ***U****, and* ***IIA****. Then if* $V \in \mathcal{T}$ *is* (x, y)*-decisive , V is decisive.*

Proof. Suppose V is (x, y)-decisive. Then by Lemma 28 , V is (u, v)-decisive for $u = x$ and $v \neq x$ as well. First, we claim that V is (u, v)-decisive for an arbitrary u and $v \neq x$ whenever u, v are distinct. To this end, consider $\widehat{R}$ such that $\widehat{R}(V)(u, x) = \widehat{R}(V)(x, v) = 1^+$ and $\widehat{R}(V^c)(v, u) = \widehat{R}(V^c)(u, x) = 1^+$. By assumption on V we have $F\left(\widehat{R}\right)(x, v) = 1^+$, and by **U** we obtain $F\left(\widehat{R}\right)(u, x) = 1^+$. Then by quasi-transitivity $F\left(\widehat{R}\right)(u, v) = 1^+$. Furthermore, since $\widehat{R}(V)(u, v) = 1^+$ and $\widehat{R}(V^c)(v, u) = 1^+$, by Lemma 25 V is (u, v)-decisive as desired. Second, we claim that V is (u, x)-decisive for an arbitrary u whenever u, x are distinct. To this end, choose $v \notin \{u, x\}$ so that V is (u, v)-decisive by the earlier claim and consider $\widehat{R}$ such that $\widehat{R}(V)(u, v) = \widehat{R}(V)(v, x) = 1^+$ and $\widehat{R}(V^c)(v, x) = \widehat{R}(V^c)(x, u) = 1^+$. Then the by now familiar argument shows that V is (u, x)-decisive and this completes our proof. ■

4 Structures of Decisive Coalitions

The following definition of a filter on algebras $\mathcal{T}$ is equivalent to the one appearing in Sikorski (1969). Note that when $\mathcal{T} = 2^T$ this definition reduces to the usual one as in Bourbaki (1966, p.57).

Definition 30 *Let $\mathcal{T}$ be a Boolean algebra. A* ***filter*** *on $\mathcal{T}$ is a subset $\mathcal{V} \subseteq \mathcal{T}$ which has the following properties:*

1. $\curlyvee \in \mathcal{V}$.
2. $\curlywedge \notin \mathcal{V}$.
3. If $V_1, V_2 \in \mathcal{V}$, then $V_1 \cap V_2 \in \mathcal{V}$.
4. If $V_1 \in \mathcal{V}, V_2 \in \mathcal{T}$, and $V_1 \subseteq V_2$, then $V_2 \in \mathcal{V}$.

Remark 31 *There is no filter on $\mathcal{T} = \{\varnothing\}$. If $\mathcal{N}$ is a proper ideal of $\mathcal{T}$, $\mathcal{V} = \{-V : V \in \mathcal{N}\}$ is a filter on $\mathcal{T}$. Our definition of a filter $\mathcal{V}$ assumes that it is nonempty. Some authors call such a filter a proper filter.*

We supply proofs of the results in this section only for $\mathcal{N} = \{\varnothing\}$ since the same line of arguments works as well for $\mathcal{N} \neq \{\varnothing\}$ by simply replacing $\mathcal{T}$ with its quotient Boolean algebra $\mathcal{T} / \mathcal{N}$ in which each element $\langle V \rangle$ is now an equivalence class of $V \in \mathcal{T}$ and in which the notion of decisiveness naturally descends from $\mathcal{T}$ [see Armstrong (1980, p.60) for general discussions about quotient Boolean algebras].

Definition 32 *A filter $\mathcal{V}$ on an algebra $\mathcal{T}$ is said to be an* ***ultrafilter*** *if it is a maximal element in the ordered set of all filters on $\mathcal{T}$.*

Definition 33 *A corrective choice rule F is said to be* ***Arrovian*** *whenever it satisfies* ***UD***, ***U***, *and* ***IIA***.

Definition 34 *For an Arrovian corrective choice rule F , $\mathcal{V}_F$ denotes the set of all decisive coalitions associated with F . We obtain a map*

$$\rho : \mathbb{CR} \to 2^{\mathcal{T}}$$

defined by $F \mapsto \mathcal{V}_F$.

Definition 35 *We denote the set of all filters on $\mathcal{T}$ by $\mathbb{F}$ and the set of all ultrafilters on $\mathcal{T}$ by $\mathbb{UF}$.*

Definition 36 *We denote the set of all Arrovian collective choice rules by $\mathbb{CR}$, the subset of all transitive and complete (hence reflexive) Arrovian collective choice rules by $\mathbb{CR}^{T,C} \subseteq \mathbb{CR}$, the set of all quasi-transitive and complete Arrovian collective choice rules by $\mathbb{CR}^{QT,C} \subseteq \mathbb{CR}$, and the set of all quasi-transitive Arrovian collective choice rules by $\mathbb{CR}^{QT} \subseteq \mathbb{CR}$.*

As noted in Bossert and Suzumura (2009), Hansson (1976) proved in the terminology adopted here that the restriction of ρ defines a surjective map

$$\rho : \mathbb{CR}^{T,C} \twoheadrightarrow \mathbb{UF} \subseteq 2^{\mathcal{T}}$$

and

$$\rho : \mathbb{CR}^{QT,C} \twoheadrightarrow \mathbb{F} \subseteq 2^{\mathcal{T}} ,$$

where T may be finite or infinite, $\mathcal{T} = 2^T$, and $\mathcal{N} = \{\varnothing\}$.

Bossert and Suzumura (2009) proved the following theorem for the special instance that T is finite, $\mathcal{T} = 2^T$, and $\mathcal{N} = \{\varnothing\}$, which sharpened the earlier results obtained by Hansson (1976) in the sense that analogous results remain valid in the absence of completeness of F (and hence in the absence of reflexivity of F as well).

Theorem 37 *Suppose* $|\mathbb{A}| \geq 3$ *and* $F \in \mathbb{CR}^{QT}$ *. Then* $\mathcal{V}_F \in \mathbb{F}$ *.*

Proof. By Proposition 21 we know that $\varnothing \notin \mathcal{V}$ and by definition it follows that if $V \in \mathcal{V}$ and $V \subseteq V'$, then $V' \in \mathcal{V}$. Suppose $V, V' \in \mathcal{V}$. We claim that $V \cap V' \in \mathcal{V}$. For distinct alternatives x , y , and z , consider $\widehat{R}$ such that $\widehat{R}(V \setminus V')(y,z) = \widehat{R}(V \setminus V')(z,x) = 1^+$, $\widehat{R}(V \cap V')(z,x) = \widehat{R}(V \cap V')(x,y) = 1^+$, and $\widehat{R}(V' \setminus V)(x,y) = \widehat{R}(V' \setminus V)(y,z) = 1^+$. Then since $\widehat{R}(V)(z,x) = 1^+$, $F\left(\widehat{R}\right)(z,x) = 1^+$ and since $\widehat{R}(V')(x,y) = 1^+$, $F\left(\widehat{R}\right)(x,y) = 1^+$. Now by quasi-transitivity we deduce that $F\left(\widehat{R}\right)(z,y) = 1^+$. Note that $\widehat{R}(V \cap V')(z,y) = 1^+$ and $\widehat{R}((V \cap V')^c)(y,z) = 1^+$. Then by Lemma 25 we conclude that $V \cap V' \in \mathcal{V}$. ■

Thus the restriction of ρ defines a map

$$\rho : \mathbb{CR}^{QT} \to \mathbb{F} \subseteq 2^{\mathcal{T}}$$

which was shown to be surjective by Bossert and Suzumura (2009) in a similar situation. For each $\mathcal{V} \in \mathbb{F}$, we define collective choice rules $\sigma(\mathcal{V})$ and $\sigma^a(\mathcal{V})$

as follows:

$$\begin{aligned}\sigma\left(\mathcal{V}\right)\left(\widehat{R}\right)(x,y) &= 1\Longleftrightarrow\left\{t:\widehat{R}\left(t\right)(x,y)=1\right\}\in\mathcal{V}\\ \sigma^{a}\left(\mathcal{V}\right)\left(\widehat{R}\right)(x,y) &= 1\Longleftrightarrow\left\{t:\widehat{R}\left(t\right)(x,y)=1^{+}\right\}\in\mathcal{V}\end{aligned}$$

We will show in the following sequence of lemmata that both $\sigma\left(\mathcal{V}\right)$ and $\sigma^{a}\left(\mathcal{V}\right)$ are Arrovian collective choice rules which are transitive and hence by Lemma 3 quasi-transitive. We thus obtain maps $\sigma:\mathbb{F}\rightarrow\mathbb{CR}^{QT}$ and $\sigma^{a}:\mathbb{F}\rightarrow\mathbb{CR}^{QT}$

.

Lemma 38 $\sigma^{a}\left(\mathcal{V}\right)\left(\widehat{R}\right)(x,y)=1^{+}\Longleftrightarrow\left\{t:\widehat{R}\left(t\right)(x,y)=1^{+}\right\}\in\mathcal{V}$

Proof. Note that $\sigma^{a}\left(\mathcal{V}\right)\left(\widehat{R}\right)(y,x)=1\Longleftrightarrow\left\{t:\widehat{R}\left(t\right)(y,x)=1^{+}\right\}\in\mathcal{V}$. Thus $\sigma^{a}\left(\mathcal{V}\right)$ is asymmetric, i.e., $\sigma^{a}\left(\mathcal{V}\right)\left(\widehat{R}\right)(x,y)=1\Rightarrow\sigma^{a}\left(\mathcal{V}\right)\left(\widehat{R}\right)(y,x)=0$. This implies that $\sigma^{a}\left(\mathcal{V}\right)\left(\widehat{R}\right)(x,y)=1\Leftrightarrow\sigma^{a}\left(\mathcal{V}\right)\left(\widehat{R}\right)(x,y)=1^{+}$ and the result follows. ■

Lemma 39 $\sigma\left(\mathcal{V}\right)\left(\widehat{R}\right)(x,y)=1^{+}\Longleftarrow\left\{t:\widehat{R}\left(t\right)(x,y)=1^{+}\right\}\in\mathcal{V}$

Proof. Suppose $\left\{t:\widehat{R}\left(t\right)(y,x)=0\right\}=\left\{t:\widehat{R}\left(t\right)(y,x)=1\right\}^{c}\in\mathcal{V}$ and $\left\{t:\widehat{R}\left(t\right)(x,y)=1\right\}\in\mathcal{V}$, i.e., $\left\{t:\widehat{R}\left(t\right)(x,y)=1^{+}\right\}\in\mathcal{V}$. Then since $\mathcal{V}$ is a filter, $\left\{t:\widehat{R}\left(t\right)(y,x)=1\right\}\notin\mathcal{V}$ and hence $\sigma\left(\mathcal{V}\right)\left(\widehat{R}\right)(y,x)=0$. Thus $\sigma\left(\mathcal{V}\right)\left(\widehat{R}\right)(x,y)=1^{+}$. ■

Lemma 40 $\sigma\left(\mathcal{V}\right)$ *and* $\sigma^{a}\left(\mathcal{V}\right)$ *satisfy* ***UD*****,** ***U*****, and** ***IIA***.

Proof. We only need to verify **U** since the others follow trivially. Suppose $\widehat{R}(T)(x,y) = 1^+$. Since $T \in \mathcal{V}$, we obtain $\left\{t : \widehat{R}(t)(x,y) = 1^+\right\} = T \in \mathcal{V}$, and hence by Lemma 38 and Lemma 39 $\sigma^a(\mathcal{V})\left(\widehat{R}\right)(x,y) = 1^+$ and $\sigma(\mathcal{V})\left(\widehat{R}\right)(x,y) = 1^+$. ■

Lemma 41 $\sigma(\mathcal{V})$ *and* $\sigma^a(\mathcal{V})$ *are transitive and hence quasi-transitive.*

Proof. Suppose $\left\{t : \widehat{R}(t)(x,y) = 1\right\} \in \mathcal{V}$ and $\left\{t : \widehat{R}(t)(y,z) = 1\right\} \in \mathcal{V}$. Since

$$\begin{aligned} & \left\{t : \widehat{R}(t)(x,y) = 1 \wedge \widehat{R}(t)(y,z) = 1\right\} \\ = & \left\{t : \widehat{R}(t)(x,y) = 1\right\} \cap \left\{t : \widehat{R}(t)(y,z) = 1\right\} \\ \in & \mathcal{V} \end{aligned}$$

and $\left\{t : \widehat{R}(t)(x,y) = 1 \wedge \widehat{R}(t)(y,z) = 1\right\} \subseteq \left\{t : \widehat{R}(t)(x,z) = 1\right\}$, we obtain $\left\{t : \widehat{R}(t)(x,z) = 1\right\} \in \mathcal{V}$ as desired. σ^a can be treated in a similar manner keeping in mind that $\widehat{R}(t)$ is quasi-transitive. ■

We next demonstrate that σ^a is a section of $\rho : \mathbb{CR}^{QT} \to \mathbb{F}$ when it is viewed as a bundle, i.e., σ^a has the property that $\rho \circ \sigma^a = id_{\mathbb{F}}$, where id denotes the identity map on the space appearing in the subscript.

Proposition 42 *Suppose* $|\mathbb{A}| \geq 3$ *. Then* $\rho \circ \sigma^a = id_{\mathbb{F}}$

Proof. Suppose $V \in \mathcal{V}$ and $\widehat{R}(V)(x,y) = 1^+$. Then $\left\{t : \widehat{R}(t)(x,y) = 1^+\right\} \in \mathcal{V}$ and consequently, $\sigma^a(\mathcal{V})\left(\widehat{R}\right)(x,y) = 1^+$. Thus $V \in \mathcal{V}_{\sigma^a(\mathcal{V})} = \rho \circ \sigma^a(\mathcal{V})$.

This proves $\mathcal{V} \subseteq \rho \circ \sigma^a(\mathcal{V})$. Now let $V \in \mathcal{V}_{\sigma^a(\mathcal{V})} = \rho \circ \sigma^a(\mathcal{V})$. By Lemma 25, we can choose $x \neq y$ and $\widehat{R}$,such that $\widehat{R}(V)(x,y) = 1^+$, $\widehat{R}(V^c)(y,x) = 1^+$ and $\sigma^a(\mathcal{V})\left(\widehat{R}\right)(x,y) = 1^+$. Then $V = \left\{t : \widehat{R}(t)(x,y) = 1^+\right\}$ and hence by Lemma 39, $V \in \mathcal{V}$. This proves the other inclusion to complete the proof.

■

Corollary 43 *Suppose* $|\mathbb{A}| \geq 3$ *. Then the map* $\sigma^a : \mathbb{F} \to \mathbb{CR}^{QT}$ *is injective, and the map* $\rho : \mathbb{CR}^{QT} \to \mathbb{F}$ *is surjective.*

Remark 44 *The map* σ^a *is exactly the same as that constructed in Bossert and Suzumura (2009) for showing that in our terminology* $\rho : \mathbb{CR}^{QT} \to \mathbb{F}$ *is surjective. Our proof presented above is indeed verbatim to theirs. It is interesting to note that* $\sigma : \mathbb{F} \to \mathbb{CR}^{QT}$ *may not be a section of* $\rho : \mathbb{CR}^{QT} \to \mathbb{F}$ *. This is because a similar assertion in Lemma 38 may not hold in general for* σ *.*

Our next goal is to determine the inverse image $\rho^{-1}(\mathbb{UF})$ under the surjective map $\rho : \mathbb{CR}^{QT} \to \mathbb{F}$. When $T =$ a finite set, $\mathcal{T} = 2^T$, and $\mathcal{N} = \{\varnothing\}$, the set $\rho^{-1}(\mathbb{UF})$ turns out to be exactly the set of quasi-transitive Arrovian collective choice rules which are dictatorial, i.e., which admit a dictator.

We start with stating a lemma providing a necessary and sufficient condition for a filter to be an ultrafilter.

Lemma 45 *Let* $\mathcal{V}$ *be a filter on a Boolean algebra* $\mathcal{T}$ *. Then* $\mathcal{V}$ *is an ultrafilter if and only if* $\forall V \in \mathcal{T}\ V \in \mathcal{V}$ *or* $-V \in \mathcal{V}$ *.*

Proof. Let $\mathcal{F}$ be a filter containing $\mathcal{V}$ and suppose there is $V \in \mathcal{F}$ such that $V \notin \mathcal{V}$. Then by assumption $-V \in \mathcal{V} \subseteq \mathcal{F}$ and it follows that $V \cap -V = \curlywedge \in \mathcal{F}$. But this is absurd and thus $\mathcal{V}$ is an ultrafilter. Next, suppose $\mathcal{V}$ is an ultrafilter and there exists $V \in \mathcal{F}$ such that $V \notin \mathcal{V}$ and $-V \notin \mathcal{V}$. Define $\mathcal{F} = \{U \in \mathcal{T} : U \cup V \in \mathcal{V}\}$. A simple exercise shows that F is a filter properly containing $\mathcal{V}$, which is absurd. ■

We introduce the following condition which requires that strict individual preferences over distinct alternatives x, y result in a strict social preference over x, y .

$\{x, y\}$-Strict Preference, $\{x, y\}$-SP Let x, y be distinct alternatives. Then F is said to satisfy $\{x, y\}$-**Strict Preference**, or $\{x, y\}$-**SP** for short, if $\left[\widehat{R}(t)(x, y) = 1^{+} \vee \widehat{R}(t)(y, x) = 1^{+}\right]$ for almost all $t \in T$
$\Rightarrow \left[F\left(\widehat{R}\right)(x, y) = 1^{+} \vee F\left(\widehat{R}\right)(y, x) = 1^{+}\right]$.

Strict Preference, SP F is said to satisfy **Strict Preference**, or **SP** for short, if it satisfies $\{x, y\}$-**SP** for all distinct alternatives x and y .

Definition 46 *We denote the set of all quasi-transitive Arrovian collective choice rules which satisfy **SP** by* $\mathbb{CR}^{QT,SP} \subseteq \mathbb{CR}$

Remark 47 *Geanakoplos (2005) and Úbeda (2003) showed that when F is transitive and complete, **SP** follows from **UD**, **U**, and **IIA**. Thus we obtain the following inclusion:*

$$\mathbb{CR}^{T,C} \subseteq \mathbb{CR}^{QT,SP}.$$

The assertion in the next proposition is called *Equivalent Subsets* in Sen (1986). We prove it under somewhat weaker hypotheses than usual.

Proposition 48 (Equivalent Subsets) *Suppose* $|\mathbb{A}| \geq 3$ *and* $F \in \mathbb{CR}^{QT}$ *. Let* $V, V' \in \mathcal{T}$ *where* $V \in \mathcal{V}_F$ *and* $V' \subseteq V$ *. Then if* F *satisfies* $\{x, y\}$ *-**SP** for some distinct* x *and* y *, either* $V' \in \mathcal{V}_F$ *or* $V \setminus V' \in \mathcal{V}_F$ *.*

Proof. Let $x, y, z \in \mathbb{A}$ be distinct alternatives where F satisfies $\{y, z\}$ -**SP**. Consider $\widehat{R}$ such that $\widehat{R}(V')(x,y) = \widehat{R}(V')(y,z) = 1^+$, $\widehat{R}(V \setminus V')(z,x) = \widehat{R}(V \setminus V')(x,y) = 1^+$ and $\widehat{R}(V^c)(y,z) = \widehat{R}(V^c)(z,x) = 1^+$. Since V is decisive and $\widehat{R}(V)(x,y) = 1^+$, we deduce $F\left(\widehat{R}\right)(x,y) = 1^+$. By assumption either $F\left(\widehat{R}\right)(z,y) = 1^+$ or $F\left(\widehat{R}\right)(y,z) = 1^+$. First, suppose $F\left(\widehat{R}\right)(z,y) = 1^+$. Then since

$$\widehat{R}(V \setminus V')(z,y) = 1^+ \text{ and } \widehat{R}\left((V \setminus V')^c\right)(y,z) = 1^+ ,$$

by Lemma 25 $V \setminus V'$ is (z,y)-decisive. Second, suppose $F\left(\widehat{R}\right)(y,z) = 1^+$. Then by quasi-transitivity, $F\left(\widehat{R}\right)(x,z) = 1^+$. Note that $\widehat{R}(V')(x,z) = 1^+$ and $\widehat{R}(V'^c)(z,x) = 1^+$. Consequently, V' is (x,z)-decisive. Thus by virtue of Proposition 29, either V' or $V \setminus V'$ is decisive. ■

Corollary 49 *Suppose* $|\mathbb{A}| \geq 3$ *and that* $F \in \mathbb{CR}^{QT}$ *. Then if* F *satisfies* $\{x, y\}$ *-**SP** for some distinct* x *and* y *,* $\mathcal{V}_F \in \mathbb{UF}$ *.*

Proof. Let $V = T$ in Proposition 48 and apply Lemma 45. ■

Proposition 50 *Let $V, V' \in \mathcal{T}$ where $V \in \mathcal{V}_F$ and $V' \subseteq V$. Then if either $V' \in \mathcal{V}_F$ or $V \setminus V' \in \mathcal{V}_F$, F satisfies* ***SP****.*

Proof. Let $x, y \in \mathbb{A}$ be distinct alternatives where for λ -almost all t , $\widehat{R}(t)(x,y) = 1^+$ or $\widehat{R}(t)(y,x) = 1^+$. Define $V' = \left\{t \in V : \widehat{R}(t)(x,y) = 1^+\right\}$. Then if V' is decisive, $F\left(\widehat{R}\right)(x,y) = 1^+$ and if $V \setminus V'$ is decisive, $F\left(\widehat{R}\right)(y,x) = 1^+$. ■

Corollary 51 *Suppose $|\mathbb{A}| \geq 3$ and $F \in \mathbb{CR}^{QT}$. Let $V, V' \in \mathcal{T}$ where $V \in \mathcal{V}_F$ and $V' \subseteq V$. Then F satisfies* ***SP*** *if and only if either $V' \in \mathcal{V}_F$ or $V \setminus V' \in \mathcal{V}_F$.*

Proof. Simply combine Proposition 48 and Proposition 50. ■

Corollary 52 *Suppose $|\mathbb{A}| \geq 3$ and $F \in \mathbb{CR}^{QT}$. Then F satisfies* ***SP*** *if and only if $\mathcal{V}_F \in \mathbb{UF}$.*

Proof. Let $V = T$ in Corollary 51 and apply Lemma 45. ■

We can summarize the results obtained thus far as the following theorem:

Theorem 53 *Suppose $|\mathbb{A}| \geq 3$ and that $F \in \mathbb{CR}^{QT}$. Then the following are equivalent.*

(1) F satisfies $\{x, y\}$-**SP** for some distinct x and y .

(2) $\mathcal{V}_F \in \mathbb{UF}$.

(3) F satisfies **SP**.

Proof. (1) $\Rightarrow$ (2) follows from Corollary 49 and (2) $\Leftrightarrow$ (3) follows from Corollary 50, while (3) $\Rightarrow$ (1) follows from the definition of **SP**. ■

Remark 54 *It is interesting to note that under the assumptions of the proceeding theorem, if strict individual preferences over a* ***particular*** *pair of distinct alternatives x, y result in a strict social preference over those x, y , then strict individual preferences over* ***any*** *pair of distinct alternatives x, y result in a strict social preference over those x, y .*

Corollary 55 *Suppose $|\mathbb{A}| \geq 3$. Then $\rho^{-1}(\mathbb{UF}) = \mathbb{CR}^{QT,SP}$.*

We next show that the restriction $\sigma' : \mathbb{UF} \to \mathbb{CR}^{QT}$ indeed defines a section of the restriction $\rho' : \mathbb{CR}^{QT,SP} \to \mathbb{UF}$. We already showed in Lemma 41 that $\sigma'(\mathcal{V})$ is transitive for all $\mathcal{V} \in \mathbb{UF}$. We will show in the following lemma that $\sigma'(\mathcal{V})$ is complete for all $\mathcal{V} \in \mathbb{UF}$ and combining this with the inclusion $\mathbb{CR}^{T,C} \subseteq \mathbb{CR}^{QT,SP}$, we deduce that $\sigma'(\mathbb{UF}) \subseteq \mathbb{CR}^{QT,SP}$.

Lemma 56 *$\sigma'(\mathcal{V})$ is complete for all $\mathcal{V} \in \mathbb{UF}$.*

Proof. Let $x \neq y$. Then by Theorem 45, either $\left\{t : \widehat{R}(t)(x,y) = 1\right\} \in \mathcal{V}$ or $\left\{t : \widehat{R}(t)(x,y) = 0\right\} = \left\{t : \widehat{R}(t)(x,y) = 1\right\}^c \in \mathcal{V}$. Then since $\left\{t : \widehat{R}(t)(x,y) = 0\right\} \subseteq \left\{t : \widehat{R}(t)(y,x) = 1\right\} \in \mathcal{V}$, we conclude that $F_{\mathcal{V}}\left(\widehat{R}\right)(x,y) = 1$ or $F_{\mathcal{V}}\left(\widehat{R}\right)(y,x) = 1$. ■

We next prove that a similar assertion in Lemma 38 holds for σ' .

Lemma 57 *For each* $\mathcal{V} \in \mathbb{UF}$,

$$\sigma'(\mathcal{V})\left(\widehat{R}\right)(x,y) = 1^{+} \Leftrightarrow \left\{t : \widehat{R}(t)(x,y) = 1^{+}\right\} \in \mathcal{V} \ .$$

Proof. Negating $\sigma'(\mathcal{V})\left(\widehat{R}\right)(y,x) = 1 \Leftrightarrow \left\{t : \widehat{R}(t)(y,x) = 1\right\} \in \mathcal{V}$, we obtain by Theorem 45 that $\sigma'(\mathcal{V})\left(\widehat{R}\right)(y,x) = 0 \Leftrightarrow \left\{t : \widehat{R}(t)(y,x) = 0\right\} \in \mathcal{V}$. Combining the last expression with

$$\sigma'(\mathcal{V})\left(\widehat{R}\right)(x,y) = 1 \Leftrightarrow \left\{t : \widehat{R}(t)(x,y) = 1\right\} \in \mathcal{V}$$

yields

$$\begin{aligned} \sigma'(\mathcal{V})\left(\widehat{R}\right)(x,y) &= 1^{+} \\ &\Leftrightarrow \left\{t : \widehat{R}(t)(x,y) = 1\right\} \in \mathcal{V} \wedge \left\{t : \widehat{R}(t)(y,x) = 0\right\} \in \mathcal{V} \\ &\Leftrightarrow \left\{t : \widehat{R}(t)(x,y) = 1\right\} \cap \left\{t : \widehat{R}(t)(y,x) = 0\right\} \in \mathcal{V} \\ &\Leftrightarrow \left\{t : \widehat{R}(t)(x,y) = 1^{+}\right\} \ . \end{aligned}$$

■

With Lemma 57 and replacing σ^{a} by σ' , the proof of Proposition 42 works word by word to obtain the following proposition:

Proposition 58 $\rho' \circ \sigma' = id_{\mathbb{UF}}$

Corollary 59 *The map* $\sigma' : \mathbb{UF} \rightarrow \mathbb{CR}^{QT,SP}$ *is injective, and the map* $\rho' : \mathbb{CR}^{QT,SP} \rightarrow \mathbb{UF}$ *is surjective.*

It is well-known that given an ultrafilter $\mathcal{V} \in \mathbb{UF}$, there exists a complete, transitive Arrovian collective choice rule $F_{\mathcal{V}}$ whose ultrafilter $\mathcal{V}_{F_{\mathcal{V}}}$ of decisive coalitions coincide with $\mathcal{V}$. The above assertion simply reiterates this fact in our terminology, and the essential arguments in our proof are all familiar in the literature, such as Kirman and Sondermann (1972, Theorem 1, p.269), Armstrong (1980, Proposition 3.1, p.62), and Torres (2005, Lemma 27, p.933).

Definition 60 *If a filter $\mathcal{V} \in \mathbb{F}$ arises as $\mathcal{V} = \{V \in \mathcal{T} : V_0 \subseteq V\}$ for some $\curlywedge \neq V_0 \in \mathcal{T}$, $\mathcal{V}$ is said to be the* ***principal filter generated by*** *$V_0 \in \mathcal{T}$ and is denoted by $\mathcal{V}(V_0)$.*

Definition 61 *Given a $F \in \mathbb{CR}$, a feasible individual $t \in T$, i.e., $t \in T$ such that $\{t\} \in \mathcal{T}$, is said to be a* ***dictator*** *if $\{t\} \in \mathcal{V}_F$. Note that when $\mathcal{T} = 2^T$, the above definition reduces to the usual one.*

Proposition 62 *Suppose $|\mathbb{A}| \geq 3$ and $F \in \mathbb{CR}^{QT}$. Then a feasible individual $t \in T$ is a dictator if and only if $\mathcal{V}_F = \mathcal{V}(\{t\})$.*

Proof. Suppose that $t \in T$ is a dictator for F satisfying the above conditions and that $V \in \mathcal{V}_F$. Let $x \neq y$. Then by Lemma 25, there exist $\widehat{R}$ such that $\widehat{R}(V)(x,y) = 1^+$, $\widehat{R}(V^c)(y,x) = 1^+$, and $F\left(\widehat{R}\right)(x,y) = 1^+$. If $t \notin V$ then $t \in V^c$ and since $\{t\}$ is not negligible, $\widehat{R}(t)(y,x) = 1^+$. Then $F\left(\widehat{R}\right)(y,x) = 1^+$ must hold, which leads to a contradiction and thus we conclude that $t \in V$ which shows that $\mathcal{V}_F \subseteq \mathcal{V}(\{t\})$. Now suppose $V \in \mathcal{V}(\{t\})$, i.e. $t \in V$, and

$\widehat{R}(V)(x,y) = 1^+$. Then since $\{t\}$ is not negligible, $\widehat{R}(t)(x,y) = 1^+$ and hence $F\left(\widehat{R}\right)(x,y) = 1^+$. This shows that $V \in \mathcal{V}_F$ and thus $\mathcal{V}(\{t\}) \subseteq \mathcal{V}_F$ as desired. Now suppose $\mathcal{V}_F = \mathcal{V}(\{t\})$. Then t clearly is a dictator. ■

Suppose that $\mathcal{T}$ is rich enough to make each individual $t \in T$ feasible and that $\mathcal{N} = \{\varnothing\}$. Then for each $t \in T$, the principal filter $\mathcal{V}(\{t\})$ will be an ultrafilter, i.e., $\mathcal{V}(\{t\}) \in \mathbb{UF}$. Moreover, the assignment $t \mapsto \mathcal{V}(\{t\})$ defines an embedding $i : T \to \mathbb{UF}$ such that the image $i(T)$ is dense in $\mathbb{UF}$ with respect to the Stone topology. This can be seen as follows: Let $[V]$, $V \in \mathcal{T}$, be a basic neighborhood in the Stone topology, i.e., $[V] = \{\mathcal{V} \in \mathbb{UF} : V \in \mathcal{V}\}$. Choose a point $t \in V$. Then $V \in \mathcal{V}(\{t\})$ since $\{t\} \subseteq V$. Thus $\mathcal{V}(\{t\}) \in [V]$ and this establishes our earlier claim.

When T is a finite set, every ultrafilter arises as $\mathcal{V}(\{t\})$ for some $t \in T$ and hence $i(T) = \mathbb{UF}$. When T is an infinite set, however, $\mathbb{UF} \backslash i(T)$ may be nonempty. Recall from Corollary 55 that every ultrafilter $\mathcal{V}$ is an ultrafilter of decisive coalitions of some $F \in \mathbb{CR}$. In light of the above discussions we may call an ultrafilter in $i(T)$ a **visible dictator** (for some $F \in \mathbb{CR}$) and that in $\mathbb{UF} \backslash i(T)$ an **invisible dictator** . Note that invisible dictators are in some sense the limits of visible dictators as reasoned out by Kirman and Sondermann (1972, p.272).

We can also describe the difference between visible and invisible dictators in terms of an intersection property of ultrafilters. We say an ultrafilter $\mathcal{V}$ is **fixed** if $\bigcap_{V \in \mathcal{V}} V \neq \varnothing$ and is **free** otherwise. Indeed, if $\mathcal{V}$ is fixed, there will be a $t \in T$ such that for all $V \in \mathcal{V}$, $t \in V$, i.e., $V \in \mathcal{V}(\{t\})$. Thus

$\mathcal{V} \subseteq \mathcal{V}(\{t\})$ but since $\mathcal{V}$ is maximal by assumption, we obtain $\mathcal{V} = \mathcal{V}(\{t\})$. It follows that the fixed ultrafilters are precisely the visible dictators and the free ultrafilters the invisible ones.

In the case where $\mathcal{N} \neq \{\varnothing\}$, we can naturally extend the above notions of visible and invisible dictators in the following manner: An ultrafilter $\mathcal{V}$ in $\mathcal{T}/\mathcal{N}$ is said to be a visible dictator if it equals the principal filter generated by the equivalence class of some $\{t\}$, $t \in T$, that is , $\mathcal{V} = \mathcal{V}(\langle\{t\}\rangle)$ for some $t \in T$, and an ultrafilter not in this form is said to be an invisible dictator.

Remark 63 *Note that the above definition of visible and invisible dictators generalizes the usual ones appearing in the literature, such as Kirman and Sondermann (1972), to the case that negligible coalitions are incorporated.*

If T is finite , $\mathcal{T} = 2^T$, and $\mathcal{N} = \{\varnothing\}$, then it follows from Lemma 45 that every ultrafilter $\mathcal{V} \in \mathbb{UF}$ is of the form $\mathcal{V}(\{t\})$ for some $t \in T$. The following theorem is a version of the celebrated impossibility theorem by Arrow (1963).

Theorem 64 *Suppose $|\mathbb{A}| \geq 3$. If T is finite, $\mathcal{T} = 2^T$, and $\mathcal{N} = \{\varnothing\}$, then $\mathbb{CR}^{QT,SP}$ is exactly the set of $F \in \mathbb{CR}^{QT}$ such that Arrow's impossibility theorem holds.*

Proof. Suppose $F \in \mathbb{CR}^{QT}$ admits a dictator. Then by Proposition 62, $\mathcal{V}_F = \mathcal{V}(\{t\})$ for some $t \in T$, which is a member of $\mathbb{UF}$. Then by Corollary 55 , $F \in \mathbb{CR}^{QT,SP}$. On the other hand, if $F \in \mathbb{CR}^{QT,SP}$, $\mathcal{V}_F \in \mathbb{UF}$, but as noted above, $\mathcal{V}_F = \mathcal{V}(\{t\})$ for some $t \in T$. Thus t is a dictator. ■

5 The Images of $\sigma^{a\prime}$ and σ' .

Armstrong (1980, 1985) considered the problem of identifying the image set of σ' , which is contained in $\mathbb{CR}^{T,C}$ and which corresponds bijectively to $\mathbb{UF}$ under σ' and hence under ρ' . This set is interesting because when $\mathbb{UF}$ consists only of visible dictators, a collective choice rule F in such a set has the property that if one alternative x is strictly socially preferred to another alternative y in a situation $\widehat{R}$, i.e., $F\left(\widehat{R}\right)(x,y) = 1^+$, then there is a dictator $t \in T$ whose preference over x and y agrees with the society, i.e., $\widehat{R}(t)(x,y) = 1^+$. Armstrong called such collective choice rules *precisely dictatorial Arrovian social welfare functions.* Armstrong (1985) introduced the notion of monotonicity of collective choice rules $F \in \mathbb{CR}^{T,C} \subseteq \mathbb{CR}^{QT,SP}$: F is said to satisfy *relational monotonicity* provided that the asymmetric factor of F as a relation on $\mathbb{A}$ enlarges when the asymmetric factor of $\widehat{R}(t)$ for each t enlarges as a relation on $\mathbb{A}$. He demonstrated that the set of precisely dictatorial Arrovian social welfare functions coincides with the subset of $\mathbb{CR}^{T,C}$ which consists of those F that satisfies relational monotonicity.

Corollary 52 and Lemma 41 together imply that $\sigma^a(\mathbb{UF}) \subseteq \mathbb{CR}^{QT,SP}$ provided $|\mathbb{A}| \geq 3$. In this section we identify the image set of the restriction $\sigma^{a\prime} : \mathbb{UF} \to \mathbb{CR}^{QT,SP}$ and find a relationship between this set and the image set of σ'.

Relational Monotonicity, RM F is said to satisfy **Relational Monotonicity**, or **RM** for short, if for each pair of preference profiles $\widehat{R}$ and $\widehat{R}'$

, F satisfies the condition that $\left[\widehat{R}(t)(x,y) = 1^{+} \Rightarrow \widehat{R'}(t)(x,y) = 1^{+}\right]$ for almost all t implies $\left[F\left(\widehat{R}\right)(x,y) = 1^{+} \Rightarrow F\left(\widehat{R'}\right)(x,y) = 1^{+}\right]$.

Definition 65 *We denote the set of all negative quasi-transitive and asymmetric Arrovian collective choice rules which satisfy **SP** and **RM** by* $\mathbb{CR}^{NQT,SP,A,RM} \subseteq \mathbb{CR}$ *. Note by virtue of Lemma 6 we have the inclusion* $\mathbb{CR}^{NQT,SP,A,RM} \subseteq \mathbb{CR}^{QT,SP}$ *. The next theorem shows that the image set of* $\sigma^{a\prime}$ *coincides with* $\mathbb{CR}^{NQT,SP,A,RM}$.

Lemma 66 *If* $F = \sigma^{a} \circ \rho(F)$ *,* F *is asymmetric.*

Proof. By definition we have

$$\begin{aligned}
\sigma^{a}(\mathcal{V})\left(\widehat{R}\right)(x,y) &= 1 \Leftrightarrow \left\{t : \widehat{R}(t)(x,y) = 1^{+}\right\} \in \mathcal{V} \\
\sigma^{a}(\mathcal{V})\left(\widehat{R}\right)(y,x) &= 1 \Leftrightarrow \left\{t : \widehat{R}(t)(y,x) = 1^{+}\right\} \in \mathcal{V} .
\end{aligned}$$

Since $\left\{t : \widehat{R}(t)(x,y) = 1^{+} \wedge \widehat{R}(t)(y,x) = 1^{+}\right\} = \varnothing$, the conclusion follows.
■

Lemma 67 *If* $F = \sigma^{a\prime} \circ \rho'(F)$ *,* F *is negative quasi-transitive.*

Proof. Suppose $F\left(\widehat{R}\right)(x,y) \neq 1^{+} \wedge F\left(\widehat{R}\right)(y,z) \neq 1^{+}$. Since F is asymmetric by Lemma 66 , Lemma 7 implies that $F\left(\widehat{R}\right)(x,y) \neq 1 \wedge F\left(\widehat{R}\right)(y,z) \neq 1$ and hence $\left\{t : \widehat{R}(t)(x,y) \neq 1^{+}\right\} \in \mathcal{V}_F \wedge \left\{t : \widehat{R}(t)(y,z) \neq 1^{+}\right\} \in \mathcal{V}_F$ since $\mathcal{V}_F \in \mathbb{UF}$. Thus by Lemma 4 $\left\{t : \widehat{R}(t)(x,z) \neq 1^{+}\right\} \in \mathcal{V}_F$, which implies $F\left(\widehat{R}\right)(x,z) \neq 1$ and hence $F\left(\widehat{R}\right)(x,z) \neq 1^{+}$ since F is asymmetric and since Lemma 7 holds. ■

Lemma 68 *If* $F = \sigma^{a'} \circ \rho'(F)$, F *satisfies* ***RM*** *.*

Proof. Suppose $F\left(\widehat{R}\right)(x,y) = 1^+$ and $\widehat{R}(t)(x,y) = 1^+ \Rightarrow \widehat{R}'(t)(x,y) = 1^+$. Then $\left\{t : \widehat{R}(t)(x,y) = 1^+\right\} \in \mathcal{V}_F$ and

$$\left\{t : \widehat{R}(t)(x,y) = 1^+\right\} \subseteq \left\{t : \widehat{R}'(t)(x,y) = 1^+\right\}$$

and consequently $\left\{t : \widehat{R}'(t)(x,y) = 1^+\right\} \in \mathcal{V}_F$, which implies $F\left(\widehat{R}'\right)(x,y) = 1$ and hence by Lemma 7 $F\left(\widehat{R}'\right)(x,y) = 1^+$. Thus F satisfies **RM** . ■

Given $\widehat{R} \in \Gamma$ and a negative quasi-transitive F , we define $\widehat{R}' : T \to \mathcal{P}$ by

$$\begin{aligned} \widehat{R}'(t)(x,y) &= 1 \\ &\Leftrightarrow \widehat{R}(t)(x,y) = 1 \\ \wedge &\quad \left[\widehat{R}(t)(y,x) = 1^+ \vee \widehat{R}(t)(x,y) = 1^+ \vee F\left(\widehat{R}\right)(x,y) \neq 1^+\right] . \end{aligned}$$

Lemma 69 $\widehat{R}'(t)$ *is complete for all* $t \in T$ *.*

Proof. Suppose $\widehat{R}'(t)(x,y) = 0$. Then $\widehat{R}(t)(x,y) = 0$ or $\widehat{R}(t)(x,y) = 1 \wedge \widehat{R}(t)(y,x) = 1 \wedge F\left(\widehat{R}\right)(x,y) = 1^+$. Note that $\widehat{R}(t)(x,y) = 0$ implies $\widehat{R}(t)(y,x) = 1^+$, which in turn implies $\widehat{R}'(t)(y,x) = 1$. On the other hand, since the asymmetric component of F is asymmetric, $F\left(\widehat{R}\right)(x,y) = 1^+$ implies $F\left(\widehat{R}\right)(y,x) \neq 1^+$. Thus $\widehat{R}(t)(x,y) = 1 \wedge \widehat{R}(t)(y,x) = 1 \wedge F\left(\widehat{R}\right)(x,y) = 1^+$ implies $\widehat{R}'(t)(y,x) = 1$ as well. ■

Lemma 70 $\widehat{R}'(t)$ *is transitive for all* $t \in T$ *.*

Proof. Suppose $\widehat{R}'(t)(x,y) = 1 \wedge \widehat{R}'(t)(y,z) = 1$. Assume the contrary: $\widehat{R}'(t)(x,z) = 0$. Note that $\widehat{R}'(t)(x,z) = 0$ implies $\widehat{R}(t)(x,z) = 0$ or $\widehat{R}(t)(z,x) \neq 1^+ \wedge \widehat{R}(t)(x,z) \neq 1^+ \wedge F\left(\widehat{R}\right)(x,z) = 1^+$. If $\widehat{R}(t)(x,z) = 0$ then by transitivity $\widehat{R}(t)(x,y) = 0 \vee \widehat{R}(t)(y,z) = 0$, which implies $\widehat{R}'(t)(x,y) = 0 \vee \widehat{R}'(t)(y,z) = 0$, a contradiction. Now suppose the second alternative holds. By negative quasi-transitivity of F , we obtain $F\left(\widehat{R}\right)(x,y) = 1^+ \vee F\left(\widehat{R}\right)(y,z) = 1^+$. Note that $\widehat{R}'(t)(x,y) = 1$ implies $\widehat{R}(t)(x,y) = 1$, which in turn implies $\widehat{R}(t)(y,x) \neq 1^+$, and similarly $\widehat{R}'(t)(y,z) = 1$ implies $\widehat{R}(t)(z,y) \neq 1^+$. Moreover, by Lemma 4 , $\widehat{R}(t)(x,z) \neq 1^+ \wedge \widehat{R}(t)(z,y) \neq 1^+$ implies $\widehat{R}(t)(x,y) \neq 1^+$, and similarly $\widehat{R}(t)(y,x) \neq 1^+ \wedge \widehat{R}(t)(x,z) \neq 1^+$ implies $\widehat{R}(t)(y,z) \neq 1^+$. We thus obtain $\widehat{R}(t)(y,x) \neq 1^+ \wedge \widehat{R}(t)(x,y) \neq 1^+ \wedge F\left(\widehat{R}\right)(x,y) = 1^+$, which implies $\widehat{R}'(t)(x,y) = 0$, or $\widehat{R}(t)(z,y) \neq 1^+ \wedge \widehat{R}(t)(y,z) \neq 1^+ \wedge F\left(\widehat{R}\right)(y,z) = 1^+$, which implies $\widehat{R}'(t)(y,z) = 0$. These contradict the initial premise. ■

Theorem 71 *Suppose* $|\mathbb{A}| \geq 3$ *and let* $F \in \mathbb{CR}^{QT,SP}$. *Then* $F = \sigma^{a'} \circ \rho'(F)$ *if and only if* $F \in \mathbb{CR}^{NQT,SP,A,RM} \subseteq \mathbb{CR}^{QT,SP}$.

Proof. The above lemmata prove that if $F = \sigma^{a'} \circ \rho'(F)$ then $F \in \mathbb{CR}^{NQT,SP,A,RM}$. For the converse, suppose $F \in \mathbb{CR}^{NQT,SP,A,RM}$ and $F \neq \sigma^{a'} \circ \rho'(F)$. First, note that for every $x \neq y$ and $\widehat{R}$, $\sigma^{a'} \circ \rho'(F)\left(\widehat{R}\right)(x,y)$ implies $F\left(\widehat{R}\right)(x,y) = 1^+$, which is on account of Lemma 7 equivalent to $F\left(\widehat{R}\right)(x,y) = 1$. Thus if $F \neq \sigma^{a'} \circ \rho'(F)$, then there must exist $x \neq y$ and $\widehat{R}$ such that $F\left(\widehat{R}\right)(x,y) = 1^+ \wedge \left\{t : \widehat{R}(t)(x,y) = 1^+\right\} \notin \mathcal{V}_F$

and consequently $\left\{t : \widehat{R}(t)(x,y) \neq 1^+\right\} \in \mathcal{V}_F$ since $\mathcal{V}_F \in \mathbb{UF}$. Now suppose $\left\{t : \widehat{R}(t)(y,x) = 1^+\right\} \in \mathcal{V}_F$. Then it would be true that $F\left(\widehat{R}\right)(y,x) = 1^+$, which is impossible. Thus $\left\{t : \widehat{R}(t)(y,x) \neq 1^+\right\} \in \mathcal{V}_F$ and we obtain $V = \left\{t : \widehat{R}(t)(x,y) \neq 1^+ \wedge \widehat{R}(t)(y,x) \neq 1^+\right\} \in \mathcal{V}_F$. Observe that $\widehat{R}'$ satisfies

$$
\begin{aligned}
\widehat{R}'(t)(y,x) &= 1^+ \\
&\Leftrightarrow \widehat{R}(t)(y,x) = 1^+ \\
\vee & \quad \left[\widehat{R}(t)(y,x) \neq 1^+ \wedge \widehat{R}(t)(x,y) \neq 1^+ \wedge F\left(\widehat{R}\right)(x,y) = 1^+\right] .
\end{aligned}
$$

Since F satisfies **RM,** $F\left(\widehat{R}\right)(x,y) = 1^+$ implies $F\left(\widehat{R}'\right)(x,y) = 1^+$. On the other hand, $\widehat{R}'(V)(y,x) = 1^+$ which implies $F\left(\widehat{R}'\right)(y,x) = 1^+$. We thus reached a contradiction and hence we deduce that $F = \sigma^{a'} \circ \rho'(F)$. ■

We next investigate the relationship between the image set of σ' and that of $\sigma^{a'}$. Note that there is a map $\Phi : \mathcal{P}^{T,C} \rightarrow \mathcal{P}$ given by $\succsim \rightarrow \succsim'$, where $x \succsim' y \Leftrightarrow x \succ y$, and a map $\Psi : \mathcal{P}^{NQT,A} \rightarrow \mathcal{P}$ given by $\succsim' \rightarrow \succsim$, where $x \succsim y \Leftrightarrow \neg y \succsim' x$. By virtue of Lemma 4 and Lemma 8, for each $\succsim \in \mathcal{P}^{T,C}$, $\Phi(\succsim)$ is asymmetric and negative quasi-transitive, i.e., $\Phi : \mathcal{P}^{T,C} \rightarrow \mathcal{P}^{NQT,A}$ where $\mathcal{P}^{NQT,A}$ consists of asymmetric and negative quasi-transitive preference relations. The following lemma shows that for each $\succsim' \in \mathcal{P}^{NQT,A}$, $\Psi(\succsim')$ is transitive and complete and hence we have $\Psi : \mathcal{P}^{NQT,A} \rightarrow \mathcal{P}^{T,C}$.

Lemma 72 *For each* $\succsim' \in \mathcal{P}^{NQT,A}$, $\Psi(\succsim') \in \mathcal{P}^{T,C}$.

Proof. Suppose $\neg x \succsim y$. Then $y \succsim' x$ and hence $\neg x \succsim' y$ since $\succsim'$ is asymmetric. Now suppose $x \succsim y \wedge y \succsim z$. Then $\neg y \succsim' x \wedge \neg z \succsim' y$. Since $\succsim'$ is asymmetric, $\neg y \succ' x \wedge \neg z \succ' y$, which implies $\neg z \succ' x$ since $\succsim'$ is negative quasi-transitive. Thus $\neg z \succsim' x$ and hence $x \succsim z$. Therefore, $\succsim$ is transitive. ■

We claim that $\Phi : \mathcal{P}^{T,C} \to \mathcal{P}^{NQT,A}$ is a bijection and $\Psi : \mathcal{P}^{NQT,A} \to \mathcal{P}^{T,C}$ is the inverse of Φ .

Proposition 73 $\Phi : \mathcal{P}^{T,C} \to \mathcal{P}^{NQT,A}$ *is a bijection and* $\Phi^{-1} = \Psi$.

Proof. We have $x\Psi \circ \Phi\left(\succsim\right) y \Leftrightarrow \neg y\Phi\left(\succsim\right) x \Leftrightarrow \neg y \succ x \Leftrightarrow x \succsim y$ on one hand and $x\Phi \circ \Psi\left(\succsim'\right) y \Leftrightarrow x\Psi\left(\succsim'\right) y \wedge \neg y\Psi\left(\succsim'\right) x \Leftrightarrow \neg y\Psi\left(\succsim'\right) x \Leftrightarrow x \succsim' y$ on the other hand. ■

Note that $\succsim$ and $\Phi\left(\succsim\right)$ have the same asymmetric component. Since **SP** and **RM** depend only on asymmetric components, Φ induces a bijection

$$\widetilde{\Phi} : \mathbb{CR}^{T,C,RM} \to \mathbb{CR}^{NQT,SP,A,RM}$$

by $F \longmapsto \Phi \circ F$. It is a simple matter to verify that σ' and $\sigma^{a\prime}$ are related by

$$\widetilde{\Phi} \circ \sigma' = \sigma^{a\prime}.$$

6 Topological Structures of $\mathbb{CR}^{QT}$ and Density Theorems

We construct a topology on $\mathbb{F}$ in an analogous manner to that used to construct the Stone topology on $\mathbb{UF}$. In the sequel (V) denotes the set of all filters containing $V \in \mathcal{T}$ and $[V]$ the set of all ultrafilters containing $V \in \mathcal{T}$. Define $\alpha = \{(V) : V \in \mathcal{T}\}$ and $\beta = \{[V] : V \in \mathcal{T}\}$. We will show in the next proposition that α defines a topology $\mathcal{O}$ on $\mathbb{F}$.

Proposition 74 *α is a basis for a topology $\mathcal{O}$ on $\mathbb{F}$.*

Proof. Suppose $\mathcal{V} \in \mathbb{F}$. We must show that there is an element in α which contains $\mathcal{V}$. Since $T \in \mathcal{V}$ we obtain $\mathcal{V} \in (T)$, where $T \in \mathcal{T}$. We next show that for (V_1) , $(V_2) \in \alpha$, each point in the intersection $(V_1) \cap (V_2)$ belongs to an element in α . This assertion clearly holds by the following argument:

$$\begin{aligned} \mathcal{V} &\in (V_1) \cap (V_2) \Leftrightarrow \mathcal{V} \in (V_1) \wedge \mathcal{V} \in (V_2) \\ &\Leftrightarrow V_1 \in \mathcal{V} \wedge V_2 \in \mathcal{V} \\ &\Leftrightarrow V_1 \cap V_2 \in \mathcal{V} \\ &\Leftrightarrow \mathcal{V} \in (V_1 \cap V_2) , \end{aligned}$$

which shows that $(V_1) \cap (V_2) = (V_1 \cap V_2)$. ■

Proposition 75 *$\mathcal{O}$ is T_0 .*

Proof. Suppose $\mathcal{V}_1 \neq \mathcal{V}_2 \in \mathbb{F}$. Then there must be a $V \in \mathcal{T}$ such that $V \in \mathcal{V}_1 \wedge V \notin \mathcal{V}_2$ or $V \notin \mathcal{V}_1 \wedge V \in \mathcal{V}_2$. Then $\mathcal{V}_1 \in (V) \wedge \mathcal{V}_2 \notin (V)$ or $\mathcal{V}_1 \notin (V) \wedge \mathcal{V}_2 \in (V)$. ■

As is well-known in the literature, β defines a topology on $\mathbb{UF}$ called the Stone topology $\mathcal{O}_S$ which is compact, totally disconnected and hence Hausdorff (see for example Sikorski, 1969). Since the restriction $\alpha \mid_{\mathbb{UF}}$ coincides with β , we clearly have

$$\mathcal{O} \mid_{\mathbb{UF}} = \mathcal{O}_S \ ,$$

where $\mathcal{O} \mid_{\mathbb{UF}}$ denotes the restriction of $\mathcal{O}$ to $\mathbb{UF}$. The next proposition shows that $\mathbb{UF}$ is dense in $\mathbb{F}$ with respect to $\mathcal{O}$.

Proposition 76 *$\mathbb{UF}$ is dense in $\mathbb{F}$.*

Proof. Let $\mathcal{V} \in \mathbb{F}$ and $\mathcal{V} \in (V)$, i.e., $V \in \mathcal{V}$. Then there is an ultrafilter $\overline{\mathcal{V}}$ containing $\mathcal{V}$ so that $V \in \mathcal{V} \subseteq \overline{\mathcal{V}}$. Hence $\overline{\mathcal{V}} \in (V)$. ■

In what follows we endow $\mathcal{P}$ with the topology of pointwise convergence as in Armstrong (1980). We then endow $\mathbb{CR}$ with the topology of pointwise convergence and assume that all relevant subsets such as $\mathbb{CR}^{T,C}$ and $\mathbb{CR}^{QT,SP}$ inherit this topology and become topological subspaces of $\mathbb{CR}$. Thus a net F_i converges to F in $\mathbb{CR}$ if and only if $\forall \widehat{R}\ F_i\left(\widehat{R}\right) \to F\left(\widehat{R}\right)$ if and only if $\forall \widehat{R}\ \forall x, y\ F_i\left(\widehat{R}\right)(x,y) \to F\left(\widehat{R}\right)(x,y)$. We will show in the next theorem that $\rho : \mathbb{CR}^{QT} \to \mathbb{F}$ is continuous.

As noted in Armstrong (1980, p.67), $\mathcal{P}$ is a totally disconnected compact Hausdorff space with respect to the topology of pointwise convergence.

Lemma 77 *Suppose $R_i \to R$ is a convergent net in $\mathcal{P}$. Then $R(x,y) = 1^+$ if and only if $\exists i_0\, \forall i \geq i_0\, R_i(x,y) = 1^+$.*

Proof. Note that $R_i \to R$ if and only if $\forall x, y\, R_i(x,y) \to R(x,y)$ if and only if $\forall x, y\, \exists i_0\, \forall i \geq i_0\, R_i(x,y) = R(x,y)$ and our assertion is immediate. ■

Lemma 78 *$\mathcal{P}^{QT}$ is closed in $\mathcal{P}$ and hence compact Hausdorff.*

Proof. Let R_i be a net in $\mathcal{P}^{QT}$ which converges to $R \in \mathcal{P}$. We must show that $R \in \mathcal{P}^{QT}$. To this end suppose $R(x,y) = 1^+ \wedge R(y,z) = 1^+$. Then by Lemma 77, $\exists i_0\, \forall i \geq i_0\, R_i(x,y) = 1^+ \wedge R_i(y,z) = 1^+$. Since R_i are quasi-transitive, $\forall i \geq i_0\, R_i(x,z) = 1^+$. Again by Lemma 77 , we obtain $R(x,z) = 1^+$ and thus $R \in \mathcal{P}^{QT}$. ■

Corollary 79 *$\mathbb{CR}^{QT} = \left(\mathcal{P}^{QT}\right)^{\Gamma}$ is compact Hausdorff.*

Theorem 80 *Suppose $|\mathbb{A}| \geq 3$. Then $\rho : \mathbb{CR}^{QT} \to \mathbb{F}$ is continuous.*

Proof. Let $F_i \to F$ be a convergent net in $\mathbb{CR}^{QT}$. We must show that $\mathcal{V}_{F_i} \!\to \mathcal{V}_F$ in $\mathbb{F}$. Recall that $F_i \to F$ if and only if $\forall \widehat{R}\, \forall x, y\; F_i\left(\widehat{R}\right)(x,y) \to F\left(\widehat{R}\right)(x,y)$. Let (V) be a basic neighborhood of $\mathcal{V}_F$ in $\mathbb{F}$, where $V \in \mathcal{T}$. In view of Lemma 25 we then have

$$\begin{aligned} \mathcal{V}_F &\in (V) \text{ if and only if } V \in \mathcal{V}_F \\ &\quad \text{if and only if} \\ \exists \widehat{R}\; \widehat{R}(V)(x,y) &= 1^+ \text{ , } \widehat{R}(V^c)(y,x) = 1^+ \text{ , and } F\left(\widehat{R}\right)(x,y) = 1^+ \text{ .} \end{aligned}$$

We claim that $\exists i_0$ such that $\forall i \geq i_0\, \mathcal{V}_{F_i} \in (V)$. Note that $\exists i_0$ such that $\forall i \geq i_0\, F_i\left(\widehat{R}\right)(x,y) = 1^+$. This implies that $\forall i \geq i_0\, V \in \mathcal{V}_{F_i}$ and hence $\forall i \geq i_0\, \mathcal{V}_{F_i} \in (V)$ as desired. ■

Corollary 81 *$\mathbb{F}$ is compact.*

Proof. Recall that $\mathbb{F}$ is the image of $\mathbb{CR}^{QT}$ under a continuous map ρ . ■

We can also prove that $\sigma^{a\prime} : \mathbb{UF} \to \mathbb{CR}^{QT,SP}$ is continuous.

Theorem 82 *Suppose $|\mathbb{A}| \geq 3$. Then $\sigma^{a\prime} : \mathbb{UF} \to \mathbb{CR}^{QT,SP}$ is continuous.*

Proof. Assuming $\mathcal{V}_i \to \mathcal{V}$, we must show that

$$\forall \widehat{R}\, \forall x,y\; \sigma^{a\prime}(\mathcal{V}_i)\left(\widehat{R}\right)(x,y) {\to} \sigma^{a\prime}(\mathcal{V})\left(\widehat{R}\right)(x,y) \; .$$

First, suppose $\sigma^{a\prime}(\mathcal{V})\left(\widehat{R}\right)(x,y) = 1$. This holds if and only if

$$\left\{t : \widehat{R}(t)(x,y) = 1^+\right\} \in \mathcal{V} \Leftrightarrow \mathcal{V} \in \left[\left\{t : \widehat{R}(t)(x,y) = 1^+\right\}\right] \; .$$

Then $\exists i_0$ such that $\forall i \geq i_0\, \mathcal{V}_i \in \left(\left\{t : \widehat{R}(t)(x,y) = 1^+\right\}\right)$, which holds if and only if $\left\{t : \widehat{R}(t)(x,y) = 1^+\right\} \in \mathcal{V}_i$ if and only if $\sigma^{a\prime}(\mathcal{V}_i)\left(\widehat{R}\right)(x,y) = 1$. This proves our assertion provided $\sigma^{a\prime}(\mathcal{V})\left(\widehat{R}\right)(x,y) = 1$. Next, suppose $\sigma^{a\prime}(\mathcal{V})\left(\widehat{R}\right)(x,y) \neq 1$, which holds if and only if $\left\{t : \widehat{R}(t)(x,y) = 1^+\right\} \notin \mathcal{V}$ if and only if $\left\{t : \widehat{R}(t)(x,y) \neq 1^+\right\} \in \mathcal{V}$ if and only if $\mathcal{V} \in \left[\left\{t : \widehat{R}(t)(x,y) \neq 1^+\right\}\right]$ since $\mathcal{V}$ is an ultrafilter. Then as in the earlier argument we can deduce that $\exists i_0$ such that $\forall i \geq i_0\; \sigma^{a\prime}(\mathcal{V}_i)\left(\widehat{R}\right)(x,y) \neq 1$. ■

Combining Theorem 80 and 82 we obtain the following corollary:

Corollary 83 *Suppose* $|\mathbb{A}| \geq 3$ *. Then* $\sigma^{a\prime} : \mathbb{UF} \to \mathbb{CR}^{NQT,SP,A,RM}$ *is a homeomorphism. In particular* $\mathbb{CR}^{NQT,SP,A,RM}$ *is compact and totally disconnected.*

We can also prove that $\sigma' : \mathbb{UF} \to \mathbb{CR}^{QT,SP}$ is continuous provided $|\mathbb{A}| \geq 3$. The proof is almost verbatim to that of Theorem 82. The only change to be made is to replace $\left\{t : \widehat{R}(t)(x,y) = 1^+\right\}$ by $\left\{t : \widehat{R}(t)(x,y) = 1\right\}$ in the proof of Theorem 82. We thus obtain the following corollaries:

Corollary 84 *Suppose* $|\mathbb{A}| \geq 3$ *. Then* $\sigma' : \mathbb{UF} \to \mathbb{CR}^{T,C,RM}$ *is a homeomorphism. In particular* $\mathbb{CR}^{T,C,RM}$ *is compact and totally disconnected.*

Corollary 85 *Suppose* $|\mathbb{A}| \geq 3$ *. Then* $\widetilde{\Phi} : \mathbb{CR}^{T,C,RM} \to \mathbb{CR}^{NQT,SP,A,RM}$ *is a homeomorphism.*

In the sequel we will show that when there are infinitely many alternatives, $\mathbb{CR}^{QT,SP}$ becomes nowhere dense in $\mathbb{CR}^{QT}$, and $\mathbb{CR}^{T,C}$ becomes nowhere dense in $\mathbb{CR}^{QT,SP}$. The former assertion in particular has an important economic implication: In the presence of a large number of social alternatives the set of Arrovian collective choice rules that fall into Arrow's impossibility theorem is in some sense "negligible" in the totality of quasi-transitive Arrovian collective choice rules.

Lemma 86 *Suppose* $|\mathbb{A}| \geq 3$ *. Then* $\mathbb{CR}^{QT,SP}$ *is closed in* $\mathbb{CR}^{QT}$ *.*

Proof. Generally, a basic open neighborhood U_F of $F \in \mathbb{CR}^{QT}$ looks like

$$U_F = \left\{ \begin{array}{c} G \in \mathbb{CR}^{QT} : G\left(\widehat{R}_i\right)(x_{ij}, y_{ij}) = F\left(\widehat{R}_i\right)(x_{ij}, y_{ij}); \\ i = 1, \cdots, n; ij = i1, \cdots, ij_i \end{array} \right\}.$$

Let $F \in \mathbb{CR}^{QT} \setminus \mathbb{CR}^{QT,SP}$. Then there exist $\widehat{R}_0$ and distinct x_0 and y_0 such that $\widehat{R}_0(t)(x_0, y_0) = 1^+ \vee \widehat{R}_0(t)(y_0, x_0) = 1^+$ for almost all $t \in T$ and $F\left(\widehat{R}_0\right)(x_0, y_0) \neq 1^+ \wedge F\left(\widehat{R}_0\right)(y_0, x_0) \neq 1^+$. Define

$$U_F = \left\{ \begin{array}{c} G \in \mathbb{CR}^{QT} : G\left(\widehat{R}_0\right)(x_0, y_0) = F\left(\widehat{R}_0\right)(x_0, y_0) \\ \wedge G\left(\widehat{R}_0\right)(y_0, x_0) = F\left(\widehat{R}_0\right)(y_0, x_0) \end{array} \right\}.$$

Note that U_F is an open neighborhood of F such that $U_F \cap \mathbb{CR}^{QT,SP} = \varnothing$.

■

Remark 87 *Note that since $\mathbb{F}$ is only T_0 , even though $\mathbb{UF}$ is compact it may not be closed in $\mathbb{F}$. Thus we cannot conclude the above assertion from Corollary 55.*

Corollary 88 *Suppose $|\mathbb{A}| \geq 3$. Then $\mathbb{CR}^{QT,SP}$ is compact Hausdorff.*

Theorem 89 *Suppose $|\mathbb{A}| = \infty$. Then $\mathbb{CR}^{QT,SP}$ is nowhere dense in $\mathbb{CR}^{QT}$.*

Proof. Since $\mathbb{CR}^{QT,SP}$ is closed in $\mathbb{CR}^{QT}$ by Corollary 88, it suffices to show that $\mathbb{CR}^{QT,SP}$ has empty interior. Let $F \in \mathbb{CR}^{QT,SP}$ and let U_F be an open neighborhood of F . We may assume that U_F is a basic open neighborhood of F so that

$$U_F = \left\{ \begin{array}{c} G \in \mathbb{CR}^{QT} : G\left(\widehat{R}_i\right)(x_{ij}, y_{ij}) = F\left(\widehat{R}_i\right)(x_{ij}, y_{ij}) ; \\ i = 1, \cdots, n; ij = i1, \cdots, ij_i \end{array} \right\} .$$

Then since $|\mathbb{A}| = \infty$, we can safely choose $x_0 \neq y_0$ from the complement of the list $\{(x_{ij}, y_{ij}) : i = 1, \cdots, n; ij = i1, \cdots, ij_i\}$. Choose $\widehat{R}_0 \in \Gamma$ such that $\widehat{R}_0(t)(x_0, y_0) = 1^+ \vee \widehat{R}_0(t)(y_0, x_0) = 1^+$ for almost all $t \in T$ and choose the symmetric factor $\sim_0$ of any $\succsim \in \mathcal{P}^{T,C} \subseteq \mathcal{P}^{QT}$. Now define $\widetilde{F} \in \mathbb{CR}^{QT}$ by

$$\widetilde{F}\left(\widehat{R}\right) = \begin{cases} F\left(\widehat{R}\right) & \widehat{R} \neq \widehat{R}_0 \\ \sim_0 & \widehat{R} = \widehat{R}_0 \end{cases} .$$

Note that $\widetilde{F} \notin \mathbb{CR}^{QT,SP}$ and $\widetilde{F} \in U_F$. ■

Lemma 90 *$\mathbb{CR}^{T,C}$ is closed in $\mathbb{CR}^{QT,SP}$.*

Proof. First, we show that $\mathbb{CR}^{T,C}$ is closed in $\mathbb{CR}$. Then since $\mathbb{CR}^{T,C} \subseteq \mathbb{CR}^{QT,SP}$, $\mathbb{CR}^{T,C}$ will be closed in $\mathbb{CR}^{QT,SP}$. Let $F_i \in \mathbb{CR}^{T,C}$ be a net converging to $F \in \mathbb{CR}$. We must show that $F \in \mathbb{CR}^{T,C}$. To this end it suffices to show that if $R_i \in \mathcal{P}^{T,C}$ is a net converging to $R \in \mathcal{P}$, then $R \in \mathcal{P}^{T,C}$, i.e., $\mathcal{P}^{T,C}$ is closed in $\mathcal{P}$. Suppose $R(x, y) = 1 \wedge R(y, z) = 1$. Then $\exists i_0 \, \forall i \geq i_0 \, R_i(x, y) = 1 \wedge R_i(y, z) = 1$ and hence $\exists i_0 \, \forall i \geq i_0 \, R_i(x, z) = 1$. Thus $R(x, z) = 1$, i.e., R is transitive. We must show that R is complete as well. Given x and y , suppose $R(x, y) = 0$. Then $\exists i_0 \, \forall i \geq i_0 \, R_i(x, y) = 0$.

Since R_i is complete, $\exists i_0 \, \forall i \geq i_0 \, R_i(y,x) = 1$, and hence $R(y,x) = 1$, i.e., R is complete. ■

Lemma 91 *Suppose $|\mathbb{A}| = \infty$. Then $\mathcal{P}^{T,C}$ is nowhere dense in $\mathcal{P}^{QT}$.*

Proof. As noted in the previous proof, $\mathcal{P}^{T,C}$ is closed in $\mathcal{P}$ and since $\mathcal{P}^{T,C} \subseteq \mathcal{P}^{QT}$, $\mathcal{P}^{T,C}$ is closed in $\mathcal{P}^{QT}$. We only need to show that $\mathcal{P}^{T,C}$ has empty interior in $\mathcal{P}^{QT}$. To this end, let $R \in \mathcal{P}^{T,C}$ and U_R be an open neighborhood of R . We may assume that U_R is a basic open set, i.e.,

$$U_R = \left\{ R' \in \mathcal{P}^{QT} : R'(x_i, y_i) = R(x_i, y_i), i = 1, \cdots, n \right\} .$$

Let $D = \{(x_i, y_i), i = 1, \cdots, n\}$ and find a finite subset $I \subseteq \mathbb{A}$ such that $D \subseteq I \times I$. Now define $R' \in \mathcal{P}$ by

$$R'(x,y) = \begin{cases} R(x,y) & (x,y) \in I \times I \\ 0 & (x,y) \notin I \times I \end{cases} .$$

Note that since $|\mathbb{A}| = \infty$, there exists a (x,y) such that $R'(x,y) = R'(y,x) = 0$. Thus R' is not complete. Furthermore, since R' is transitive on $I \times I$, it is transitive on $\mathbb{A} \times \mathbb{A}$ as well. In particular, R' is quasi-transitive. ■

The following theorem indicates that in the presence of many social alternatives the set of the so-called Arrovian social welfare functions, i.e., $\mathbb{CR}^{T,C}$ is in some sense "negligible" in the totality of quasi-transitive collective choice rules that give rise to ultrafilters of decisive coalitions.

Theorem 92 *Suppose* $|\mathbb{A}| = \infty$. *Then* $\mathbb{CR}^{T,C}$ *is nowhere dense in* $\mathbb{CR}^{QT,SP}$.

Proof. In view of Lemma 90, we only need to show that $\mathbb{CR}^{T,C}$ has empty interior. Define $\widehat{R}_0 \in \Gamma$ by $\widehat{R}_0 (T) (x, y) = \widehat{R}_0 (T) (y, x) = 1$. Let $F \in \mathbb{CR}^{T,C}$ and let U_F be an open neighborhood of F . We may assume that U_F is a basic open neighborhood of F so that

$$U_F = \left(\prod_{\widehat{R} \in \Gamma} W_{\widehat{R}} \right) \cap \mathbb{CR}^{QT,SP} ,$$

where $W_{\widehat{R}}$ is an open subset of $\mathcal{P}^{QT}$ containing $F\left(\widehat{R}\right)$, and $W_{\widehat{R}} = \mathcal{P}^{QT}$ for all but finitely many $\widehat{R}$. By Lemma 91 , we can choose $R' \in \left(\mathcal{P}^{QT} \setminus \mathcal{P}^{T,C}\right) \cap W_{\widehat{R}_0}$. Now define a modification $\widetilde{F}$ of F by

$$\widetilde{F}\left(\widehat{R}\right) = \begin{cases} F\left(\widehat{R}\right) & \widehat{R} \neq \widehat{R}_0 \\ R' & \widehat{R} = \widehat{R}_0 \end{cases} .$$

Note that $\widetilde{F} \in \left(\mathbb{CR}^{QT} \setminus \mathbb{CR}^{T,C}\right) \cap \left(\prod_{\widehat{R} \in \Gamma} W_{\widehat{R}} \right)$ and by our construction, $\widetilde{F}$ satisfies **SP** as well and hence belongs to $\left(\mathbb{CR}^{QT,SP} \setminus \mathbb{CR}^{T,C}\right) \cap U_F$. ■

7 Large Society and Dictatorial Coalitions

We give yet another natural generalization of Arrow's dictators:

Definition 93 *Given* $F \in \mathbb{CR}$ *and* $V \in \mathcal{V}_F$, *if* V *is minimal with respect to* *set inclusion modulo $\mathcal{N}$, V is said to be a* ***dictatorial coalition.***

Proposition 94 (Uniqueness of a Dictatorial Coalition) *Suppose $|\mathbb{A}| \geq 3$ and $F \in \mathbb{CR}^{QT}$. Then a dictatorial coalition is unique modulo $\mathcal{N}$.*

Proof. Suppose there are two dictatorial coalitions $V_1, V_2 \in \mathcal{V}_F$. Since $\mathcal{V}_F$ is a filter under the present assumptions, we obtain $V_1 \cap V_2 \in \mathcal{V}_F$. Now since $V_1 \cap V_2 \subseteq V_i$, we deduce by minimality that $V_1 = V_2$ modulo $\mathcal{N}$, i.e., $V_1 \triangle V_2 \in \mathcal{N}$, where $\triangle$ denotes symmetric difference. ■

Definition 95 *Suppose $V, V' \in \mathcal{T}$. If $V' \subseteq V$ implies either $V' \in \mathcal{N}$ or $V \setminus V' \in \mathcal{N}$, then V is said to be an* ***atom.***

Proposition 96 (Atomicity of a Dictatorial Coalition) *Suppose $|\mathbb{A}| \geq 3$ and $F \in \mathbb{CR}^{QT,SP}$. Then a dictatorial coalition is an atom.*

Proof. Suppose a dictatorial coalition V is not an atom. Then there exists $V' \subseteq V, V' \in \mathcal{T}$ such that $V' \notin \mathcal{N}$ and $V \setminus V' \notin \mathcal{N}$. If $V' \in \mathcal{V}_F$, it contradicts the minimality of V . On the other hand if $V' \notin \mathcal{V}_F$, we obtain by Corollary 51 that $V \setminus V' \in \mathcal{V}_F$, which again contradicts the minimality of V . ■

Proposition 97 *Suppose $|\mathbb{A}| \geq 3$ and $F \in \mathbb{CR}^{QT}$. Then if $V \in \mathcal{V}_F$ is an atom, it is dictatorial.*

Proof. By way of contradiction, assume that $V \in \mathcal{V}_F$ is not dictatorial. Then there must be a $V' \in \mathcal{V}_F$ such that $V^c \cap V' \in \mathcal{N}$ and $V \setminus V \cap V' \notin \mathcal{N}$.

Since V is assumed to be an atom, we obtain $V \cap V' \in \mathcal{N}$, which leads to a contradiction in light of Proposition 21. ■

The following theorem is yet another version of Arrow's impossibility theorem.

Theorem 98 *Suppose* $|\mathbb{A}| \geq 3$ *and and* $F \in \mathbb{CR}^{QT,SP}$ *. Let* $(T, \mathcal{T}, \mathcal{N})$ *be a finite disjoint union of atoms. Then there exists a unique dictatorial coalition* $V \in \mathcal{T}$ *.*

Proof. By assumption we can write $T = T_1 \cup T_2 \cup \cdots \cup T_N$, where $T_1, T_2 \cdots$ are mutually disjoint atoms. Then by Corollary 51 either T_1 or T_1^c is decisive. If T_1 is decisive our assertion holds by Proposition 97. Now suppose T_1^c is decisive. Repeating the preceding argument we eventually capture a dictatorial T_i . The uniqueness follows from Proposition 94. ■

Remark 99 *Note that when* $T = \{1, 2, \cdots N\}$ *with* $\mathcal{T} = 2^T$ *and* $\mathcal{N} = \{\varnothing\}$ *, the statement of the above theorem means precisely the same thing as that of Arrow's impossibility theorem since dictatorial coalitions in this case are singletons.*

Consider $(T, \mathcal{T}, \mathcal{N})$ where $T = \{1, 2, 3, \cdots\}$, $\mathcal{T} = 2^T$, and $\mathcal{N} = \{\varnothing\}$. Note that $(T, \mathcal{T}, \mathcal{N})$ is a disjoint union of atoms, i.e., singletons. We demonstrate in this case that a collective choice rule $F \in \mathbb{CR}^{QT,SP}$ may or may not admit a dictator, where $|\mathbb{A}| \geq 3$. Denote the initial segment $\{1, 2, 3, \cdots n\}$ by I_n . Given F , there are two possibilities:

Case 100 *$I_n \in \mathcal{V}_F$ for some n .*

Case 101 *$I_n \notin \mathcal{V}_F$ for any n .*

If Case 1 holds, I_n contains a dictator and hence F admits a dictator. On the other hand, if Case 2 holds and if there is a dictator $n_0 \in \{1, 2, 3, \cdots\}$, which is necessarily unique in virtue of Proposition 94 , we reach a contradiction that is

$$\begin{aligned} n_0 &\in \bigcap_{n=1}^{\infty} I_n^c \\ &= \varnothing \ . \end{aligned}$$

Therefore, F does not admit a dictator. For example, let $\mathcal{V}$ be the ultrafilter on $\mathcal{T}$ consisting of all sets with finite complement in $T = \{1, 2, 3, \cdots\}$ and let $F = \sigma'(\mathcal{V})$. Since $\mathcal{V}_F = \mathcal{V}$ falls into Case 2, F admits no dictators.

We define large society to be one in which all members are negligible. Hence large society is the opposite extreme of finite society considered by Arrow.

Definition 102 ***Large society*** *is an atomless triple $(T, \mathcal{T}, \mathcal{N})$.*

In view of Proposition 96, there cannot be any dictatorial coalition in large society. We formally state this fact in the following proposition:

Proposition 103 *Suppose $|\mathbb{A}| \geq 3$ and $F \in \mathbb{CR}^{QT,SP}$. In large society $(T, \mathcal{T}, \mathcal{N})$, there cannot exist a dictatorial coalition.*

Remark 104 *When large society* $(T, \mathcal{T}, \mathcal{N})$ *arises from an atomless measure space, where* $\mathcal{T}$ *is a* σ *-algebra and* $\mathcal{N}$ *the collection of null sets with respect to some measure* λ *on* $\mathcal{T}$ *, Kirman and Sondermann (1972, p.271) showed that there exists an arbitrary small decisive coalition. To be precise, they showed that for any* $\epsilon > 0$ *, there exists a decisive coalition* $V \in \mathcal{T}$ *such that* $\lambda(V) < \epsilon$ *.*

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