Market Price Manipulation in a Sequential Trade Model*

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Abstract. The dynamic version of the Glosten and Milgrom (1985) model of asset pricing with asymmetric information is studied. It is shown that there is a unique equilibrium when the next-period value function of the informed trader, who knows the terminal value of the asset, is strictly convex and strictly monotone in terms of the market maker's prior belief. A characterization of the bid and ask prices and the informed trader's manipulative strategy in equilibrium is given. Finally, a computational method for simulating the equilibrium is presented.

Key Words: Market microstructure; Glosten-Milgrom; Dynamic trading; Price formation; Sequential trade; Asymmetric information; Bid-ask spreads.
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1 Introduction

This paper develops a model of dynamic informed trading from a canonical framework in the market microstructure literature and characterizes an equilibrium. In asymmetric information models of financial markets, trading behavior imperfectly reveals the private information held by traders. Informed traders who trade dynamically thus have an incentive not only to trade less aggressively but also to manipulate the market by trading in the wrong direction, undertaking short-term losses to confuse the market and then recouping the losses in the future. Dynamic trading and price manipulation by an informed trader have been a challenging issue in the literature of market microstructure.

There are two standard reference frameworks in the literature. The first is called the "continuous auction framework" first developed by Kyle (1985). The second is the "sequential trade framework" proposed by Glosten and Milgrom (1985). A large amount of research has been done involving the application of these two frameworks. Both frameworks are sufficiently simple and well behaved that they easily lend themselves to analysis of policy issues and empirical testing (see Madhavan (2000) and Biais et al. (2005) for extensive surveys of the literature).

One of the simplifying assumptions in Glosten and Milgrom (1985) is that traders can trade only once. In the original Glosten-Milgrom model manipulation does not occur because there is no chance to re-trade and as a result traders maximize their one-period payoff. In the Kyle model the informed trader's strategy is monotonic in the sense that she buys the asset when it is undervalued given her information and vice versa; dynamic price manipulation is ruled out by assumption.

This paper follows the strand of the sequential trade framework developed by Glosten and Milgrom. This paper considers markets where a risky asset is traded for finitely many periods between competitive market makers, two types of strategic informed traders and liquidity traders. In the beginning of the game, nature chooses the liquidation value of a risky asset to be high or low and tells the informed trader who trades dynamically. In each period there is a random determination of whether the informed trader or a liquidity trader trades. The market maker posts bid and ask prices for the next period, after which the trader buys or sells one unit. The termination value is revealed at the end of the game and the payoff for the informed trader is the sum of the termination value times net-holding of the asset and revenue from buying and selling the asset. Within the model described above we consider an equilibrium such that (a) the informed trader's strategy is optimal beginning at any history; (b) market makers make zero profit in each period under their common Bayesian belief conditional on the history and chosen trade; (c) liquidity traders trade for their exogenous liquidity needs.

There exist a large number of papers that theoretically and empirically examine price manipulation in different settings. For example, Jordan and Jordan (1996) examine Solomon Brothers' market corner of a Treasury note auction in May 1991. Felixson and Pelli (1999) examine closing price manipulation in the Finnish stock market. Mahoney (1999) examines stock price manipulation leading up to the Securities Exchange Act of 1934. Vitale (2000) examines manipulation in the foreign exchange market. Merrick et al. (2005) examine a case of manipulation involving a delivery squeeze on a bond futures contract traded in London. A recent paper by Aggarwal and Wu (2006) suggests that stock market manipulation may have important impacts on market efficiency. According to the empirical findings in Aggarwal and Wu (2006), while manipulative activities seem to have declined on the main exchanges, it is still a serious issue in both developed and emerging financial markets, especially in the over-the-counter markets.

The theoretical literature starts with manipulation by uninformed traders. Allen and Gale (1992) provide a model of strategic trading in which some equilibria involve manipulation. Furthermore, Allen and Gorton (1992) consider a model of pure trade-based uninformed manipulation in which an asymmetry in buys and sells by liquidity traders creates the possibility of manipulation. The first paper to consider manipulation by an informed trader within the discrete-time Glosten-Milgrom framework is Chakraborty and Yilmaz (2004). They show that when the market faces uncertainty about the existence of informed traders and there are a large number of trading periods long-lived informed traders will manipulate in every equilibrium. On the other hand, Back and Baruch (2004) study the equivalence of the Glosten-Milgrom model and the Kyle model in a continuous-time setting, and show that the equilibrium in the Glosten-Milgrom model is approximately the same as that in the Kyle model when the trade size is small and uninformed trades arrive frequently. They conclude that the continuous-time Kyle model is more tractable than the Glosten-Milgrom model, although most markets are organized as in the sequential trade models. More recently there has been an interest in the informed trader's dynamic strategy. Among others, Brunnermeier and Pedersen (2005) consider dynamic strategic behavior of large traders and show that "overshooting" occurs in equilibrium. Back and Baruch (2007) analyze different market systems by allowing the informed traders to trade continuously within the Glosten-Milgrom framework.

Despite the importance of dynamic trading strategies by informed traders in the literature, characteristics of price dynamics and information transmission have not yet been adequately studied because there is no closed-form solution for equilibrium in the dynamic Glosten-Milgrom framework and it is not yet known if equilibrium is unique either in the Kyle model (see Boulatov et al. (2005) for a further development) or the dynamic Glosten-Milgrom model in which strategic informed traders can trade repeatedly. In this paper we present a model of dynamic informed trading and show that there exists a unique equilibrium when the value functions of the informed traders are monotonic and strictly convex in terms of the market maker's prior belief. In addition, we characterize the equilibrium bid and ask prices and specify the necessary condition for manipulation to occur in equilibrium. Finally, we present a computational method to solve for the equilibrium and comparative statics from the simulations.

The paper uses the Markov property of equilibrium to prove the uniqueness of equilibrium assuming that the next-period value function is strictly convex and monotone in terms of market makers' prior belief. In each period time and the market maker's prior belief are the state variables. We truncate T-period serial problem into the problem of two-period decision making. Consider the last period. In order for manipulation to arise in equilibrium, there must be at least one more period to re-trade.

Therefore in the last period manipulation would not arise in equilibrium. So we know what is the relationship between the informed trader's payoff and the market maker's belief due to Bayes' rule. Given this relationship, the informed trader chooses the probability of each trade, buy or sell, in the second last period. Then we will obtain the relationship between the second last period payoff, which is the sum of second last and last period payoffs, and the market maker's prior belief in the second last period. This is repeated recursively to the first period. Since there is no closed-form solution to the informed trader's optimization problem, we are unable to obtain the closed-form value functions. However, we will show that if the value functions satisfy: strict concavity and monotonicity, then there is a unique solution to the informed trader's optimization problem. In other words, trade today will affect the prior belief tomorrow and we will prove the uniqueness of equilibrium by using backward induction.

It is worth mentioning that in Back and Baruch (2004) the equilibrium strategy of the informed trader is not accurately simulated. This is because their model is a continuous-time stationary case and their program tries to find the value functions as a fixed point. To do this they use an extrapolation method which requires calculating the slopes of the value functions. In the region of beliefs at which manipulation arises the slopes of the value functions are very small. In fact, Back and Baruch (2004) wrote that even though all the equilibrium conditions hold with a high order of accuracy, it appears from their plots that the strategies are not estimated very accurately when manipulation arises and this is probably an inevitable result of their estimation method, because the derivative of the value function is small where manipulation arises (see Back and Baruch (2004) p.464, last paragraph).

This difficulty prevents accurate analysis of the informed trader's manipulative strategy even if the value functions are known exactly. The method developed in this paper takes advantage of the deterministic construction of our model and allows us to accurately analyze the informed trader's manipulative strategy. Instead of using the extrapolation method we find bid and ask prices to make the informed trader indifferent between the two orders: buy and sell. In this way we can solve for the equilibrium directly. This method is one of the significant contributions that we make to the literature.

The paper is organized as follows. The second section presents the model. The third section proves the uniqueness of equilibrium under specific assumptions for the value functions of the informed traders. The fourth section characterizes an equilibrium and provides a numerical simulation of the model. The fifth section concludes.

2 The model

Trade occurs for finitely many periods, denoted by $t = 1, 2, \cdots, T$. Each interval of time accommodates one trade. There is a risky asset and a numeraire in terms of which the asset price is quoted. The terminal value of the risky asset, denoted by $\tilde{v}$, is a random variable which can take the value 0 or 1. The risk-free interest rate is assumed to be zero.

There are three classes of risk-neutral market participants: competitive market makers, an informed trader and a liquidity trader. Trade arises from both the informed trader, who knows the terminal value of the asset, and uninformed traders. The type of the trader arriving in period t is determined by a random variable $\tilde{\tau}_t$, which takes values from the set $\{i, l\}$. The letters i and l respectively denote the informed type and the liquidity type. The random variables $\{\tilde{\tau}_t : t = 1, ...T\}$ satisfy $\Pr(\tilde{\tau}_t = i) = \mu$.

There are two kinds of orders available to traders: sell or buy. Let $A = \{S, B\}$ where S denotes a sell order and B denotes a buy order. Let h_t denote the order that the market maker receives in period t, i.e. h_t is the realized order in period t. If the trader's type in period t is l, then the demand in that period is determined by the random variable $\tilde{Q}_t$ which takes values from A. The random variables $\{\tilde{Q}_t : t = 1, ..., T\}$ are i.i.d. and satisfy $\Pr(\tilde{Q}_t = B) = \gamma > 0$. For any given period t, the random variables $\tilde{\tau}_t$, $\tilde{Q}_t$, $\tilde{v}$ are mutually independent and $\tilde{\tau}_t$, $\tilde{Q}_t$ are i.i.d. across the periods $1, ..., T$.

The private information of the informed trader is determined by a random variable $\tilde{\theta} \in \Theta = \{H, L\}$. When $\theta = L$, the informed knows that the value of the asset is 0. We call this type of trader "low-type" and denote him by L. When $\theta = H$, the informed trader knows that the value of the asset is 1. We call this type of trader "high-type" and denote him by H. Only one type of trader is actually chosen by nature to trade for any given play of the game.

Knowledge of the game structure and of the parameters of the joint distribution of the state variables is common to all market participants. In each period market makers post bid and ask prices equal to the expected value of the asset conditional on the observed history of trades. The trader trades at those prices. Trading happens for finitely many periods after which all private information is revealed. The timing structure of the trading game is as follows:

1. In period 0 nature chooses the realization of the risky asset payoff. The informed trader observes θ.

2. In successive periods, indexed by $t = 1, ..., T$, having observed the realized trades in periods $1, ..., t-1$, the competitive market maker posts bid and ask prices. Nature chooses an informed trader of type θ with probability μ and a liquidity trader with probability $1 - \mu$. The trader learns market makers' price quotes.

3. If the trader is informed he takes the profit-maximizing quote. If the trader is a liquidity trader he trades according to his liquidity needs.

4. In the end of period T, payoffs realized.

Next we describe the details with regard to market makers' pricing strategy and the informed trader's trading strategy. When the trader chooses his order and the market maker posts the bid and ask prices in period t, each knows the entire trading history until and including period $t - 1$. A period t history $h^t := (h_1, \cdots, h_t)$ is a sequence of realized orders for periods until and including t. Let $\mathcal{H}^t := \underbrace{A \times \cdots \times A}_{t \ times}$. We assume that: $h^0 = \emptyset$.

In the model first we consider a Markov "subgame-perfect" equilibrium and then we will show that there is not other equilibrium which does not have the Markov property. In other words we focus on the equilibrium in which market makers' prior belief is the state variable in each period t. In this kind of equilibrium, we can present the next-period value of the game as a function of the prior belief in the next-period. Given that the informed trader's optimal strategy maximizes the current-value with respect to the zero-profit prices that market makers quote. In the current period t, there are three possible events; the informed trader is chosen to trade, a liquidity trader is chosen to trade and buy, and a liquidity trader is chosen to trade and sell.

For each type of informed trader a trading strategy specifies a probability distribution over trades in period t with respect to the bid and ask prices $p_t = (\alpha_t, \beta_t)$ posted in period t. The high-type informed trader buys the security with probability σ_H^t and sells with probability $1 - \sigma_H^t$, and the low-type buys and sells with probabilities σ_L^t and $1 - \sigma_L^t$ respectively. Since there are only trades, buy or sell, available to traders, choosing a probability of buy automatically determines a probability of sell. We call $\sigma^t = (\sigma_H^t, \sigma_L^t)$ a t-period strategy profile.

To determine bid and ask prices to be posted in period t the market maker updates his prior conditional on the arrival of an order of the relevant type. Suppose that the market makers' prior belief at period t is given by $b \in (0, 1)$. This belief, b, is resulted from some history h^{t-1}. Since we focus on the equilibrium with Markov property, we will not explicitly state h^t. The market maker's prior belief after history h^t on the event: $\tilde{\theta} = H$, which we denote by $\delta(h^t)$, is updated through Bayes' rule and thus formalized as:

$$\begin{aligned} \delta(h^{t-1}, B) \equiv & \quad \Pr(\tilde{\theta} = H | h^{t-1}, B) \\ = & \quad \frac{b \times \left[\mu\, \sigma_H^t + (1-\mu)\, \Pr(\tilde{Q}_t = B)\right]}{\mu\, [b \times \sigma_H^t + (1-b) \times \sigma_L^t] + (1-\mu)\, \Pr(\tilde{Q}_t = B)}, \end{aligned} \tag{1}$$

and

$$\delta(h^{t-1}, S) \equiv \frac{b \times \left[\mu\, (1 - \sigma_H^t) + (1-\mu)\, \Pr(\tilde{Q}_t = S)\right]}{\mu\, [b \times (1 - \sigma_H^t) + (1-b) \times (1 - \sigma_L^t)] + (1-\mu)\, \Pr(\tilde{Q}_t = S)}. \tag{2}$$

We assume that: $\delta(h^0) = \delta_0 = \Pr(\tilde{\theta} = H)$ for some $\delta_0 \in (0, 1)$. The market makers post bid and ask prices according to the zero-profit condition. Since the value of the asset is either 0 or 1, ask price in period t is equal to $\delta(h^{t-1}, B)$ and bid price in period t is equal to $\delta(h^{t-1}, S)$.

Now, we will define the informed trader's optimal strategy recursively in the sense that given the continuation value of the game in the next-period the informed trader's optimal strategy maximizes the current value of the game and this is true in all the periods. Suppose that the next-period $t+1$ value functions V_L^{t+1} and V_H^{t+1} are given as a function of the market makers' prior belief $\delta(h^t)$. Remember that as stated before we focus on an equilibrium with the Markov property in the sense that the market makers' prior belief $b = \delta(h^{t-1})$ is the state variable at period t. Thus time and the market makers' prior belief are the only state variables. Therefore the period-t value of the game for each type is expressed as: for $b = \delta(h^{t-1})$, and in response to prices $p_t = (\alpha_t, \beta_t)$

$$\begin{aligned} V_H^t(b) = \max_{\sigma_H \in [0,1]} & \big(\mu\sigma_H(1-\alpha_t + V_H^{t+1}(\alpha_t)) \\ & +\mu(1-\sigma_H)(\beta_t - 1 + V_H^{t+1}(\beta_t)) + (1-\mu) \times \left[\gamma V_H^{t+1}(\alpha_t) + (1-\gamma)V_H^{t+1}(\beta_t)\right]\big), \end{aligned} \tag{3}$$

and

$$\begin{aligned} V_L^t(b) \quad = \quad \max_{\sigma_L \in [0,1]} & \big(\mu\sigma_L(-\alpha_t + V_L^{t+1}(\alpha_t)) + \mu(1-\sigma_L)(\beta_t + V_L^{t+1}(\beta_t))\big) \\ & +(1-\mu) \times \big(\gamma V_L^{t+1}(\alpha_t) + (1-\gamma)V_L^{t+1}(\beta_t)\big)\big). \end{aligned} \tag{4}$$

Notice that after buy or sell order at period t, the market makers' posterior belief becomes α_t or β_t and thus the next-period value of the game for each type becomes a function of those variables.

Definition 1. *The high-type informed trader's strategy $\{\sigma_H^{t*} : t = 1, \cdots, T\}$ is optimal in response to prices $\{p_t : t = 1, \cdots, T\}$ if it prescribes a probability $\{\sigma_H^{t*} : t = 1, \cdots, T\}$ such that for every t, σ_H^{t*} solves (3) in response to p_t. The low-type informed trader's strategy $\{\sigma_L^{t*} : t = 1, \cdots, T\}$ is optimal in response to prices $\{p_t : t = 1, \cdots, T\}$ if it prescribes a probability $\{\sigma_L^{t*} : t = 1, \cdots, T\}$ such that for every t, σ_L^{t*} solves (4) in response to p_t.*

Next we define an equilibrium for our economy:

Definition 2. *An equilibrium consists of a pair of bid and ask prices $\{p_t^* = (\alpha_t^*, \beta_t^*)\}_{t \in \{1, \cdots, T\}}$, and an informed trader's strategies $\{\sigma^{t*}\}_{t=1,\cdots,T}$ such that for all $t \in \{1, \cdots, T\}$ and for every $b = \delta(h^{t-1})$ with $h^{t-1} \in \mathcal{H}^{t-1}$,*

(P1) *the pair of bid and ask prices p_t^* satisfies the zero-profit condition with respect to the market maker's posterior belief;*

(P2) *the informed trader's strategy profile $\{\sigma^{t*}\}_{t=1,\cdots,T}$ is optimal given the pair of bid and ask prices p_t^*;*

(B) *the pair of bid and ask prices $p_t^* = (\alpha_t^*, \beta_t^*)$ satisfies Bayes' rule.*

Now, we define a manipulative strategy. We say that a strategy is manipulative if it involves the informed trader undertaking a trade in any period that yields a strictly negative short-term profit. If this occurs in equilibrium it means that manipulation enables the informed trader to recoup the short-term losses.

Definition 3. *In response to a pair of bid and ask prices p_t for some $t \in \{1, \cdots, T\}$, a strategy profile $\{\sigma\}_{t=1,\cdots,T}$ is called manipulative for the high type in period t if $\sigma_H^t < 1$; or for the low type if $\sigma_L^t > 0$. Moreover we say that $\{\sigma\}_{t=1,\cdots,T}$ is manipulative for both types in period t if both conditions hold; or for only one type if only one of the two conditions holds.*

This is the same definition used by Chakraborty and Yilmaz (2004). Back and Baruch (2004) used the term "bluffing" instead. We call the situation where the informed trader chooses a totally mixed strategy "price manipulation." It's worth mentioning that in Huberman and Stanzl (2004) a price manipulation is defined as a round-trip trade. In this paper price manipulation occurs as a round-trip trade in equilibrium but not by definition. This is because if the informed trader trades against his short term profit incentive he incurs a loss which must be recouped, consequently price manipulation takes the form of a round-trip trade in equilibrium.

3 Preliminary Results

In this section we will prove the uniqueness of equilibrium if the next-period value function is strictly monotonic and convex in market makers' prior belief b. Take an arbitrary time period t. Since our plan is to use the Markov property of equilibrium, we will focus on the time period t given the property of the next-period value function in terms of market makers' prior in the next-period $t+1$. So in this section for the simplicity of notation we will eliminate t or $t+1$ from superscript or subscript in each variable unless specified. Fix a history h^t arbitrarily and let $b = \delta(h^t) \in (0,1)$ be market makers' prior belief after history h_t. Let $W_H = V_H^t$ and $W_L = V_L^t$ represent the current value of the game for both traders. Let $V_H = V_H^{t+1}$ and $V_L = V_L^{t+1}$ represent the continuation value of the remainder of the game for both traders. In other words, W_H and W_L are the value functions in the current period and V_H and V_L are the value functions in the next period. Suppose that the next-period value function V_H, and V_L satisfy the following two conditions:

(M) V_H is strictly decreasing in $[0,1]$ and V_L is strictly increasing in $[0,1]$;

(C) V_H and V_L are strictly convex in $[0,1]$.

Theorem 1. *The equilibrium exists if conditions* **(M)***, and* **(C)** *hold.*

Proof: For arbitrary belief b, define the best-response correspondence $BR^b = (BR_M^b, BR_H^b, BR_L^b)$ as follows:

$$
\begin{aligned}
BR_M^b(\sigma_L, \sigma_H) &= p = \{(\alpha,\beta) : (\alpha,\beta) \text{ satisfies Bayes' rule } (1 \,\&\, (2)\};\\
BR_H^b(p) &= \{\sigma_H : \sigma_H \text{ solves } (3)\};\\
BR_L^b(p) &= \{\sigma_L : \sigma_L \text{ solves } (4)\}.
\end{aligned}
$$

Then, the equilibrium $(p^*, \sigma_H^*, \sigma_L^*)$ is defined as a fixed-point of BR^b. When $t = T$, we must have: $\sigma_H^T = 1$ and $\sigma_L^T = 0$. Then fix $t \in \{1, \cdots, T-1\}$ arbitrarily. Notice that $[0,1]$ is a non-empty compact convex set. To apply Kakutani's fixed point theorem, we have to prove that BR^b is upper semi-continuous, convex and non-empty mapping.

Non-Emptiness: Fix p arbitrarily and then $V_H(b)$ becomes a continuous function of σ_H. For all $t < \infty$, V_H^{t+1} is well-defined and σ_H is defined in a compact set $[0,1]$. Therefore there is σ_H which solves (3). Thus $BR_H^b(p)$ is non-empty. Similarly we can say that $BR_L^b(p)$ is non-empty. Fix σ arbitrarily and then BR_M^b is non-empty and thus BR^b is non-empty. $\square$

Upper Semi-continuity: It is clear that BR_M^b is continuous in σ. It remains to show that BR_H^b and BR_L^b are upper semi-continuous. On the contrary, suppose that BR_H^b is not upper semi-continuous. Then there are two sequences $\{p^k\}_k$ converging to p and $\{\sigma_H^k\}_k$ converging to σ_H such that for every k $\sigma_H^k \in BR_H^b(p^k)$ but $\sigma_H \notin BR_H^b(p)$. Then there must be a different $\hat{\sigma}_H \in BR_H^b(p)$ such that: for some $\epsilon > 0$,

$$\begin{aligned}
&\mu(\hat{\sigma}_H(1-\alpha+V_H(\alpha))+(1-\hat{\sigma}_H)(\beta-1+V_H(\beta)))+(1-\mu)\times[\gamma V_H(\alpha)+(1-\gamma)V_H(\beta)]\\
&>\mu(\sigma_H(1-\alpha+V_H(\alpha))+(1-\sigma_H)(\beta-1+V_H(\beta)))+(1-\mu)\times[\gamma V_H(\alpha)+(1-\gamma)V_H(\beta)]+5\epsilon.
\end{aligned}$$

Since V_H is continuous and $p^k \to p$ and $\sigma_H^k \to \sigma_H$, for each k there exists a collection of strictly positive numbers $(\epsilon_1^k, \cdots, \epsilon_4^k)$ such that: for all $k' \geq k$,

$$|\alpha-\alpha^{k'}|<\epsilon_1^k; |\beta-\beta^{k'}|<\epsilon_2^k; |V_H(\alpha)-V_H(\alpha^{k'})|<\epsilon_3^k;\;\; |V_H(\beta)-V_H(\beta^{k'})|<\epsilon_4^k. \tag{5}$$

Take k_1 sufficiently large so that we have: $\epsilon > \max(\epsilon_1^{k_1}, \cdots, \epsilon_4^{k_1})$. Notice that: for all $k \geq k_1$,

$$\begin{aligned}
&\mu(\hat{\sigma}_H(1-\alpha+V_H(\alpha))+(1-\hat{\sigma}_H)(\beta-1+V_H(\beta)))+(1-\mu)\times[\gamma V_H(\alpha)+(1-\gamma)V_H(\beta)]\\
&-\mu(\hat{\sigma}_H(1-\alpha^k+V_H(\alpha^k))+(1-\hat{\sigma}_H)(\beta^k-1+V_H(\beta^k)))-(1-\mu)\times\left[\gamma V_H(\alpha^k)+(1-\gamma)V_H(\beta^k)\right]\\
&<\;\; \mu(\hat{\sigma}_H(\epsilon_1^{k_1}+\epsilon_3^{k_1})+(1-\hat{\sigma}_H)(\epsilon_2^{k_1}+\epsilon_4^{k_1}))+(1-\mu)\times(\gamma\epsilon_3^{k_1}+(1-\gamma)\epsilon_4^{k_1})\\
&<\;\; 2\mu\epsilon+(1-\mu)\epsilon<2\epsilon.
\end{aligned}\tag{6}$$

Therefore, for all $k \geq k_1$, we obtain:

$$\begin{aligned}
&\mu(\hat{\sigma}_H(1-\alpha^k+V_H(\alpha^k))+(1-\hat{\sigma}_H)(\beta^k-1+V_H(\beta^k)))+(1-\mu)\times\left[\gamma V_H(\alpha^k)+(1-\gamma)V_H(\beta^k)\right]\\
&>\mu(\sigma_H(1-\alpha+V_H(\alpha))+(1-\sigma_H)(\beta-1+V_H(\beta)))+(1-\mu)\times[\gamma V_H(\alpha)+(1-\gamma)V_H(\beta)]+3\epsilon\\
&>\mu(\sigma_H(1-\alpha^k+V_H(\alpha^k))+(1-\sigma_H)(\beta^k-1+V_H(\beta^k)))+(1-\mu)\times\left[\gamma V_H(\alpha^k)+(1-\gamma)V_H(\beta^k)\right]+\epsilon.
\end{aligned}$$

Then take k_2 sufficiently large so that we have:

$$\epsilon > \left[\mu(2-\alpha^{k_2}+V_H(\alpha^{k_2})-\beta^{k_2}-V_H(\beta^{k_2}))(\sigma_{HB}^{k_2}-\sigma_H)\right], \tag{7}$$

and then we have: for all $k \geq k_2$,

$$\begin{aligned}
&\mu(\hat{\sigma}_H(1-\alpha^k+V_H(\alpha^k))+(1-\hat{\sigma}_H)(\beta^k-1+V_H(\beta^k)))+(1-\mu)\times\left[\gamma V_H(\alpha^k)+(1-\gamma)V_H(\beta^k)\right]\\
&\mu(\sigma_H(1-\alpha^k+V_H(\alpha^k))+(1-\sigma_H)(\beta^k-1+V_H(\beta^k)))+(1-\mu)\times\left[\gamma V_H(\alpha^k)+(1-\gamma)V_H(\beta^k)\right]+\epsilon\\
&>\mu(\sigma_H^k(1-\alpha^k+V_H(\alpha^k))+(1-\sigma_H)^k(\beta^k-1+V_H(\beta^k)))+(1-\mu)\times\left[\gamma V_H(\alpha^k)+(1-\gamma)V_H(\beta^k)\right].
\end{aligned}$$

This contradicts with our assumption: $\sigma_H^k \in BR_H^b(p^k)$ for all k. We can prove the desired result for BR_L^b in a similar fashion. Finally we conclude that BR^b is upper semi-continuous. $\square$

Convexity: Given σ, $BR_M^b(\sigma_L, \sigma_H)$ consists of a unique point p which satisfies Bayes' rule. Therefore BR_M^b is convex. Now suppose that: $\sigma_H, \hat{\sigma}_H \in BR_H^b(p)$. Then for any $r \in (0,1)$ we have $r\sigma_H + (1-r)\hat{\sigma}_H \in BR_H^b(p)$. This implies that:

$$\mu(\hat{\sigma}_H(1-\alpha+V_H(\alpha)) + (1-\hat{\sigma}_H)(\beta-1+V_H(\beta))) + (1-\mu)\times[\gamma V_H(\alpha) + (1-\gamma)V_H(\beta)]$$
$$= \mu(\sigma_H(1-\alpha+V_H(\alpha)) + (1-\sigma_H)(\beta-1+V_H(\beta))) + (1-\mu)\times[\gamma V_H(\alpha) + (1-\gamma)V_H(\beta)].$$

Then we obtain:

$$(\hat{\sigma}_H - \sigma_H)(1-\alpha+V_H(\alpha)) + ((1-\hat{\sigma}_H)-(1-\sigma_H))(\beta-1+V_H(\beta)) = 0.$$

This implies that: $\hat{\sigma}_H = \sigma_H$ or $1-\alpha+V_H(\alpha) = \beta-1+V_H(\beta)$ because $(1-\hat{\sigma}_H) = 1-\hat{\sigma}_H$ and $(1-\sigma_H) = 1-\sigma_H$. If $\hat{\sigma}_H = \sigma_H$, then $r\sigma_H + (1-r)\hat{\sigma}_H \in BR_H^b(p)$ for any r because $r\sigma_H + (1-r)\hat{\sigma}_H = \sigma_H \in BR_H^b(p)$ by our assumption. If $1-\alpha+V_H(\alpha) = \beta-1+V_H(\beta)$, then for any $r \in (0,1)$ and $\bar{\sigma}_H = r\sigma_H + (1-r)\hat{\sigma}_H$ we have:

$$\bar{\sigma}_H(1-\alpha+V_H(\alpha)) + (1-\bar{\sigma}_H)(\beta-1+V_H(\beta)) = \sigma_H(1-\alpha+V_H(\alpha)) + (1-\sigma_H)(\beta-1+V_H(\beta)).$$

Then we conclude that: $\bar{\sigma}_H \in BR_H^b(p)$. We can prove the desired result for $BR_L^b(p)$ in a similar fashion. Finally we conclude that BR^b is convex. $\square$

Finally, by Kakutani's fixed point theorem, we conclude that: the equilibrium exists in period t. Since in period $t = T$ the equilibrium exists and V_L^T and V_H^T are both well-defined, by using mathematical induction we conclude that the equilibrium exists in all periods. $\blacksquare$

After Theorem 1 the equilibrium correspondence and the graph in each period t are well-defined and given by:

$$\begin{aligned}\mathcal{E}(b) &= \{\sigma : \sigma \text{ is an equilibrium strategy profile for belief } b\};\\ \mathcal{G}(b) &= \{(b,\sigma) : \sigma \in \mathcal{E}(b)\}.\end{aligned}$$

Proposition 1. *The equilibrium correspondence $\mathcal{E}$ is upper semi-continuous and the graph $\mathcal{G}$ is closed on* $[0,1]$.

Proof of Proposition 1: Now on the contrary, suppose that $\mathcal{E}$ is not upper semi-continuous. Then there is a sequence $\{b^k\}$ which converges to b and σ^k which converges to σ with $\sigma^k \in \mathcal{E}(b^k)$ for every k but $\sigma \notin \mathcal{E}(b)$. We respectively denote the sequences of prices which satisfy Bayes' rule with σ^k and b^k by p^k and σ and b by p. Then there must be $\hat{\sigma}$ with $\hat{\sigma} \in \mathcal{E}(b)$ and there exists $\epsilon > 0$ such that:

$$\mu(\hat{\sigma}_H(1-\alpha+V_H(\alpha)) + (1-\hat{\sigma}_H)(\beta-1+V_H(\beta))) + (1-\mu)\times[\gamma V_H(\alpha) + (1-\gamma)V_H(\beta)]$$
$$> \mu(\sigma_H(1-\alpha+V_H(\alpha)) + (1-\sigma_H)(\beta-1+V_H(\beta))) + (1-\mu)\times[\gamma V_H(\alpha) + (1-\gamma)V_H(\beta)] + 5\epsilon.$$

Since p is continuous in σ and b, we must have: $p^k \to p$. Take k_1 sufficiently large so that we have: $\epsilon > \max(\epsilon_1^{k_1}, \cdots, \epsilon_4^{k_1})$ ($\epsilon_1^{k_1}$, to $\epsilon_1^{k_4}$ are defined in (5)). Then by continuity of the value function (which condition (**C**) implies) and prices, a similar argument with (6) gives us:

$$\begin{aligned}
&\mu(\hat{\sigma}_H(1-\alpha^k+V_H(\alpha^k))+(1-\hat{\sigma}_H)(\beta^k-1+V_H(\beta^k)))+(1-\mu)\times\left[\gamma V_H(\alpha^k)+(1-\gamma)V_H(\beta^k)\right]\\
&>\mu(\sigma_H(1-\alpha+V_H(\alpha))+(1-\sigma_H)(\beta-1+V_H(\beta)))+(1-\mu)\times[\gamma V_H(\alpha)+(1-\gamma)V_H(\beta)]+3\epsilon\\
&>\mu(\sigma_H(1-\alpha^k+V_H(\alpha^k))+(1-\sigma_H)(\beta^k-1+V_H(\beta^k)))+(1-\mu)\times\left[\gamma V_H(\alpha^k)+(1-\gamma)V_H(\beta^k)\right]+\epsilon.
\end{aligned}$$

Since $\sigma^k \to \sigma$, we can take k_2 which satisfies (7). Then for all $k \geq k_2$, we obtain:

$$\begin{aligned}
&\mu(\hat{\sigma}_H(1-\alpha^k+V_H(\alpha^k))+(1-\hat{\sigma}_H)(\beta^k-1+V_H(\beta^k)))+(1-\mu)\times\left[\gamma V_H(\alpha^k)+(1-\gamma)V_H(\beta^k)\right]\\
&\mu(\sigma_H(1-\alpha^k+V_H(\alpha^k))+(1-\sigma_H)(\beta^k-1+V_H(\beta^k)))+(1-\mu)\times\left[\gamma V_H(\alpha^k)+(1-\gamma)V_H(\beta^k)\right]+\epsilon\\
&>\mu(\sigma_H^k(1-\alpha^k+V_H(\alpha^k))+(1-\sigma_H)^k(\beta^k-1+V_H(\beta^k)))+(1-\mu)\times\left[\gamma V_H(\alpha^k)+(1-\gamma)V_H(\beta^k)\right].
\end{aligned}$$

This contradicts with the assumption that $\sigma^k \in \mathcal{E}(b^k)$ for all k. The closedness of the graph $\mathcal{G}$ follows. ■

To simplify our proof we will make use of the symmetric setting of the model. Let: $\tilde{b} = 1 - b$ and $\tilde{\gamma} = 1 - \gamma$. Consider the same situation with our original economy except that now liquidity buys with probability $\tilde{\gamma}$ and market maker's belief is set as $\tilde{b}$. We call this economy "mirror economy." In what follows, ˜ stands for variables associated with the mirror economy.

Proposition 2. *Suppose that conditions* (**C**) *and* (**M**) *hold. Fix time t and prior* $b = \delta(h^{t-1})$.
1. Let $\sigma \in \mathcal{E}(b)$ *and* $\tilde{\sigma}_L = 1 - \sigma_H$, $\tilde{\sigma}_H = 1 - \sigma_L$. *Then we have:* $\tilde{\sigma} \in \tilde{\mathcal{E}}(\tilde{b})$.
2. Let (α, β) *denote the equilibrium prices associated with* σ *in the original economy and* $(\tilde{\alpha}, \tilde{\beta})$ *be the equilibrium prices associated with* $\tilde{\sigma}$ *in the mirror economy. Then, we have:* $\alpha = 1 - \tilde{\beta}$, $\beta = 1 - \tilde{\alpha}$.
3. For every t, we have: $V_L^t(b) = \tilde{V}_H^t(\tilde{b})$ *and* $V_H^t(b) = \tilde{V}_L^t(\tilde{b})$.

Proof:
By definition the period-t value of the game for each type in the mirror economy is expressed as: for $\tilde{b} = 1 - b = 1 - \delta(h^{t-1})$, and in response to prices $\tilde{p} = (\tilde{\alpha}, \tilde{\beta})$

$$\begin{aligned}
\tilde{V}_H^t(\tilde{b}) = \max_{\tilde{\sigma}_H\in[0,1]} &\Big(\mu\tilde{\sigma}_H(1-\tilde{\alpha}+\tilde{V}_H^{t+1}(\tilde{\alpha}))\\
&+\mu(1-\tilde{\sigma}_H)(\tilde{\beta}-1+\tilde{V}_H^{t+1}(\tilde{\beta}))+(1-\mu)\times\left[\tilde{\gamma}\tilde{V}_H^{t+1}(\tilde{\alpha})+(1-\tilde{\gamma})\tilde{V}_H^{t+1}(\tilde{\beta})\right]\Big),
\end{aligned}\tag{8}$$

and

$$\begin{aligned}
\tilde{V}_L^t(\tilde{b}) \quad = \quad \max_{\tilde{\sigma}_L\in[0,1]} &\Big(\mu\tilde{\sigma}_L(-\tilde{\alpha}+V_L^{t+1}(\tilde{\alpha}))+\mu(1-\tilde{\sigma}_L)(\tilde{\beta}+V_L^{t+1}(\tilde{\beta}))\Big)\\
&+(1-\mu)\times\Big(\tilde{\gamma}\tilde{V}_L^{t+1}(\tilde{\alpha})+(1-\tilde{\gamma})V_L^{t+1}(\tilde{\beta})\Big).
\end{aligned}\tag{9}$$

Also Bayes' rule dictates:

$$\tilde{\alpha} = \frac{\mu\tilde{\sigma}_H + (1-\mu)\tilde{\gamma}}{(1-\mu)\tilde{\gamma} + \mu\tilde{\sigma}_L(1-\tilde{b}) + \mu\tilde{\sigma}_H\tilde{b}} \cdot \tilde{b}; \tag{10}$$

and

$$\tilde{\beta} = \frac{\mu(1-\tilde{\sigma}_H) + (1-\mu)(1-\tilde{\gamma})}{(1-\mu)(1-\tilde{\gamma}) + \mu(1-\tilde{\sigma}_L)\cdot(1-\tilde{b}) + \mu(1-\tilde{\sigma}_H)\cdot\tilde{b}} \cdot \tilde{b}. \tag{11}$$

Having the description of the equilibrium in the mirror economy, now we consider the relationship of the two equilibria in the original economy and mirror economy recursively. When $t = T$, we have: $\tilde{\sigma}_L = (1-\sigma_H) = 1$ and $1 - \tilde{\sigma}_H = \sigma_L = 0$ because they do not manipulate in the last period, and so *1.* is proved. Then by Bayes' rule, (10) and (11), we have: $\alpha = 1 - \tilde{\beta}$, $\beta = 1 - \tilde{\alpha}$, which proves *2.*, and also since there is no more chance to trade, the equalities of those prices and the comparison of (3) and (9) or (4) and (8) give us: $V_L^T(b) = \tilde{V}_H^T(\tilde{b})$ and $V_H^T(b) = \tilde{V}_L^T(\tilde{b})$. This gives us *3.* and completes the proof for this case. □

When $t \neq T$, suppose that $\sigma \in \mathcal{E}(b)$ and (α, β) is the equilibrium prices associated with σ in the original economy. Moreover suppose that the next-period value functions satisfy the property that *3.* describes. Let: $\tilde{\sigma}_{LB} = (1-\sigma_H)$, $\tilde{\sigma}_{HS} = \sigma_L$. Then we have *2.* because:

$$\alpha = 1 - \tilde{\beta} \text{ and } \beta = 1 - \tilde{\alpha}. \tag{12}$$

By substituting *2.* into (9) and $\tilde{V}_L^{t+1}$, applying *3.* to $\tilde{V}_L^{t+1}$, we obtain:

$$\begin{aligned}(9) = \max_{\sigma_H \in \Delta(A)} &\big(\mu\sigma_H(1-\alpha_t + V_H^{t+1}(\alpha_t)) \\ &+\mu(1-\sigma_H)(\beta_t - 1 + V_H^{t+1}(\beta_t)) + (1-\mu)\times\left[\gamma V_H^{t+1}(\alpha_t) + (1-\gamma)V_H^{t+1}(\beta_t)\right]\big) = V_H^t(b),\end{aligned} \tag{13}$$

and similarly by substituting *2.* into (8) and $\tilde{V}_H^{t+1}$, applying *3.* to $\tilde{V}_H^{t+1}$, we obtain:

$$\begin{aligned}(8) = \max_{\sigma_L \in \Delta(A)} &\big(\mu\sigma_L(-\alpha_t + V_L^{t+1}(\alpha_t)) + \mu(1-\sigma_L)(\beta_t + V_L^{t+1}(\beta_t))\big) \\ &+(1-\mu)\times\big(\gamma V_L^{t+1}(\alpha_t) + (1-\gamma)V_L^{t+1}(\beta_t)\big)\big) = V_L^t(b).\end{aligned} \tag{14}$$

This shows that the current-period value functions also satisfy *3.* and it remains to show that *1.* is satisfies. If $\tilde{\sigma} \notin \tilde{\mathcal{E}}(\tilde{b})$, then there must be a different strategy profile $\overline{\sigma} \in \tilde{\mathcal{E}}(\tilde{b})$, which indicates that there is a different strategy profile $\underline{\sigma} \in \mathcal{E}(b)$. This is a contradiction to our assumption. □

Since the results hold for the last period T, by mathematical induction we conclude that the results hold for all the periods. ■

To show that an equilibrium exists uniquely we first show that under these conditions the bid-ask spread is always strictly positive. In this sense, there is no pure arbitrage opportunity for the informed traders. In what follows the proofs are kept in the Appendix unless otherwise specified.

Lemma 1. *Let: $\sigma \in \mathcal{E}(b)$ and $p = (\alpha, \beta)$ be an equilibrium price associated with σ and b. The followings hold:*

1. $\alpha > b > \beta$*;*

2. $\sigma_H > \sigma_L$*.*

Proof of Lemma 1 - 1: On the contrary suppose that for some b, $\alpha \leq \beta$. Notice that for $b \in (0, 1)$ α or β cannot be either 0 nor 1 by Bayes' rule. Then, by **(M)** we have:

$$\begin{aligned} 1 - \alpha + V_H(\alpha) &> \beta - 1 + V_H(\beta); & (15) \\ -\alpha + V_L(\alpha) &< \beta + V_L(\beta). & (16) \end{aligned}$$

Then, it must be the case that in equilibrium $\sigma_H = 1$ and $\sigma_L = 0$. Then, by Bayes' rule, we must have: $\alpha > b > \beta$, which contradicts with our assumption. □
Proof of Lemma 1 - 2: The result follows from **1** and Bayes' rule. □ ■

In equilibrium the high-type trader will not sell with probability one and the low-type trader will not buy with probability one. This means that an informed trader either trades on his information or assigns a positive probability to both buy and sell orders. In the latter case the informed trader is indifferent between buy and sell orders. This motivates the following lemma.

Lemma 2. *Let: $\sigma \in \mathcal{E}(b)$ and $p = (\alpha, \beta)$ be an equilibrium price associated with σ and b. Then, the following holds:*

$$W_H(b, \sigma) = \mu \left(1 - \alpha + V_H(\alpha)\right) + (1 - \mu) \left(\gamma V_H(\alpha) + (1 - \gamma) V_H(\beta)\right), \tag{17}$$

and

$$W_L(b, \sigma) = \mu \left(\beta + V_L(\beta)\right) + (1 - \mu) \left(\gamma V_L(\alpha) + (1 - \gamma) V_L(\beta)\right). \tag{18}$$

Proof of Lemma 2: Omitted. ■

Next, we consider the slopes of the value functions. By condition **(C)** we know that V_H' or V_L' exists except at most in a countable set. By Lemma 1 bid-ask spread $\alpha - \beta$ is strictly positive. If the low-type manipulates we have:

$$d_L \equiv \frac{V_L(\alpha) - V_L(\beta)}{\alpha - \beta} = \frac{\alpha + \beta}{\alpha - \beta} = 1 + \frac{2\beta}{\alpha - \beta} > 1. \tag{19}$$

Similarly if the high-type manipulates we have:

$$d_H \equiv \frac{V_H(\alpha) - V_H(\beta)}{\alpha - \beta} = \frac{\alpha + \beta - 2}{\alpha - \beta} = -1 - \frac{2 - 2\alpha}{\alpha - \beta} < -1. \tag{20}$$

This means that if the low-type manipulates, then the average slope between the ask and bid price in the value function is greater than 1. A similar argument also holds for the high-type. Thus we conclude the following.

Lemma 3. *Let: $\sigma \in \mathcal{E}(b)$ and $p = (\alpha, \beta)$ be an equilibrium price associated with σ and b.*

L. *If the low-type takes a manipulative strategy at b, then*

$$D^{+}V_L(\alpha) \equiv \lim_{h \to 0+} \frac{V_L(\alpha + h) - V_L(\alpha)}{h} \geq D^{-}V_L(\alpha) \equiv \lim_{h \to 0+} \frac{V_L(\alpha) - V_L(\alpha - h)}{h} > 1.$$

H. *If the high-type takes a manipulative strategy at b, then*

$$D^{-}V_H(\beta) \equiv \lim_{h \to 0+} \frac{V_H(\beta) - V_H(\beta - h)}{h} \leq D^{+}V_H(\beta) \equiv \lim_{h \to 0+} \frac{V_H(\beta + h) - V_H(\beta)}{h} < -1.$$

Proof of Lemma 3: By condition **(C)** $D^{-}V_H(\beta) \leq D^{+}V_H(\beta)$ holds. By Lemma 1 $\alpha - \beta$ is strictly positive and so $\alpha - \beta > h$ for sufficiently small and thus by condition **(C)** we have: $D^{+}V_H(\beta) < d_H < -1$. We can prove the desired result for **L.** in a similar fashion. ■

If both types take a manipulative strategy at b, then by the indifference conditions for both types the following is true:

$$[V_L(\alpha) - V_L(\beta)] - [V_H(\alpha) - V_H(\beta)] = 2. \tag{21}$$

Dividing both sides by the difference between the bid and ask prices (that is, bid-ask spread) we obtain:

$$d_L - d_H = \frac{2}{\alpha - \beta}. \tag{22}$$

Consider the ϵ-neighborhood of b. When b changes, α or β and thus $\alpha - \beta$ could change. If α increases, by condition **(C)**, d_L increases and $-d_H$ decreases. If both types manipulate in the ϵ-neighborhood then (22) must hold for different d_L, d_H, and $\alpha - \beta$. We now prove that there is only one pair of bid and ask prices α and β which satisfies (21). The next result holds irrespective of market makers' prior belief b.

Lemma 4. *Suppose that both types manipulate in an interval (b_0, b_1) with $b_0 < b_1$. There exists at most only one pair of equilibrium bid and ask prices in the interval (b_0, b_1).*

Proof of Lemma 4: First we define: for $\alpha \in [0, 1]$ and $\beta \in [0, 1]$,

$$\begin{aligned} H(\alpha, \beta) &= V_H(\alpha) - V_H(\beta) + 2 - \alpha - \beta; \\ L(\alpha, \beta) &= V_L(\alpha) - V_L(\beta) - \alpha - \beta. \end{aligned}$$

If both types manipulate in an (b_0, b_1), then it must be the case that there are sets of ask and bid prices associated with every $b \in (b_0, b_1)$ and for those prices $H(\alpha, \beta) = 0$ and $L(\alpha, \beta) = 0$ both hold.

Suppose that both types manipulate at the same time and there are two different equilibrium prices, say $p_1 = (\alpha_1, \beta_1)$ and $p_2 = (\alpha_2, \beta_2)$. Then we must have: $H(p_1) = H(p_2) = 0$ and $L(p_1) = L(p_2) = 0$.

Now we divide the interval between p_1 and p_2 into n sub-intervals $\{p^k\}_{k=1,\cdots,n+1}$; that is, $p_1 = p^1$, $p_2 = p^{n+1}$ and the length of each interval is given by $h_n := \frac{|p_1 - p_2|}{n}$.

Notice that:

$$\begin{array}{rcl} H(p_1) - H(p_2) & = & \lim_{n\to\infty} \sum_{k=1}^{n} \frac{H(p^{k+1})-H(p^k)}{p^{k+1}-p^k} \cdot (p_{k+1} - p_k); \\ L(p_1) - L(p_2) & = & \lim_{n\to\infty} \sum_{k=1}^{n} \frac{L(p^{k+1})-L(p^k)}{p^{k+1}-p^k} \cdot (p_{k+1} - p_k). \end{array} \tag{23}$$

By the definition of partial derivative, we have:

$$\begin{array}{rcl} \frac{H(p^{k+1})-H(p^k)}{p^{k+1}-p^k} & = & \frac{H(\alpha^{k+1},\beta^k)-H(\alpha^k,\beta^k)}{(\alpha^{k+1},\beta^k)-(\alpha^k,\beta^k)} + \frac{H(\alpha^k,\beta^{k+1})-H(\alpha^k,\beta^k)}{(\alpha^k,\beta^{k+1})-(\alpha^k,\beta^k)}; \\ \frac{L(p^{k+1})-L(p^k)}{p^{k+1}-p^k} & = & \frac{L(\alpha^{k+1},\beta^k)-L(\alpha^k,\beta^k)}{(\alpha^{k+1},\beta^k)-(\alpha^k,\beta^k)} + \frac{L(\alpha^k,\beta^{k+1})-L(\alpha^k,\beta^k)}{(\alpha^k,\beta^{k+1})-(\alpha^k,\beta^k)}. \end{array} \tag{24}$$

For each interval, we have:

$$\frac{H(p^{k+1}) - H(p^k)}{p^{k+1} - p^k} \leq D^{-}V_H(\alpha) - 1 - D^{+}V_H(\beta) - 1; \tag{25}$$

and

$$\frac{L(p^{k+1}) - L(p^k)}{p^{k+1} - p^k} \geq D^{-}V_L(\alpha) - 1 - D^{+}V_L(\beta) - 1. \tag{26}$$

Notice that: condition **(C)** and Lemma 1 indicates:

$$D^{+}V_L(\beta) - D^{+}V_H(\beta) < D^{-}V_L(\alpha) - D^{-}V_H(\alpha). \tag{27}$$

This indicates that for each interval we have: $\frac{H(p^{k+1})-H(p^k)}{p^{k+1}-p^k} < \frac{L(p^{k+1})-L(p^k)}{p^{k+1}-p^k}$. By (23) $H(p_1) - H(p_2) \neq L(p_1) - L(p_2)$, which makes it impossible for $H(p_1) = H(p_2) = 0$ and $L(p_1) = L(p_2) = 0$. If $H(p_1) = 0$ and $L(p_1) = 0$, then it must be: $H(p_2) \neq 0$ and $L(p_2) \neq 0$. Therefore we conclude that there is only one price for which both types manipulate. ■

Let: $\sigma \in \mathcal{E}(b)$ and $p = (\alpha, \beta)$ be an equilibrium price associated with σ and b. Depending on the equilibrium strategy, we can classify the equilibrium into four regimes:

Regime L: σ is manipulative for only the low-type;

Regime H: σ is manipulative for only the high-type;

Regime $\emptyset$: σ is not manipulative;

Regime LH: σ is manipulative for both types.

We denote the set of equilibrium strategies when Regime i arises at belief b by $R_i(b)$ for $i \in \{L, \cdots, LH\}$ and more formally it is defined as:

$$\begin{array}{rcl} R_L(b) & = & \{\sigma \in \mathcal{E}(b) : \sigma_H = 1 \;\;\&\;\; \sigma_L > 0\}; \\ R_H(b) & = & \{\sigma \in \mathcal{E}(b) : \sigma_H < 1 \;\;\&\;\; \sigma_L = 0\}; \\ R_\emptyset(b) & = & \{\sigma \in \mathcal{E}(b) : \sigma_H = 1 \;\;\&\;\; \sigma_L = 0\}; \\ R_{LH}(b) & = & \{\sigma \in \mathcal{E}(b) : \sigma_H < 1 \;\;\&\;\; \sigma_L > 0\}. \end{array} \tag{28}$$

We also define: $I_i = \{b \in (0,1) : \exists \sigma \in R_i(b)\}$. There is a possibility that several regimes co-exist in some beliefs. We will consider this co-existing possibility later. First we consider equilibrium within each regime. In Regime 3, there is only one equilibrium strategy in which the low-type sells and the high-type buys with probability 1. The next lemma shows that if Regime L or H arises in equilibrium, then there is only one equilibrium strategy in that regime.

Lemma 5. *If $\sigma, \hat{\sigma} \in R_L(b)$ or $\sigma, \hat{\sigma} \in R_H(b)$, then $\sigma = \hat{\sigma}$.*

Proof of Lemma 5: Since the argument is symmetric, we will prove the result for $R_H(b)$. Let $\sigma, \hat{\sigma} \in R_L(b)$ and suppose by way of contradiction $\sigma_H \neq \hat{\sigma}_H$ and without loss of generality we may assume: $\sigma_H < \hat{\sigma}_H$. Let (α, β) be an equilibrium price associated with σ and b, and $(\hat{\alpha}, \hat{\beta})$ be an equilibrium price associated with $\hat{\sigma}$ and b, respectively. Since $\sigma_L = \hat{\sigma}_L = 0$, we have by Bayes rule $\hat{\alpha} > \alpha$ and $\hat{\beta} < \beta$. By the indifference condition for the high-type we have:

$$
\begin{aligned}
1 - \alpha + V_H(\alpha) &= \beta - 1 + V_H(\beta); &(29)\\
1 - \hat{\alpha} + V_H(\hat{\alpha}) &= \hat{\beta} - 1 + V_H(\hat{\beta}). &(30)
\end{aligned}
$$

Subtracting (30) from (29) yields:

$$\hat{\alpha} - \alpha - V_H(\hat{\alpha}) + V_H(\alpha) = \beta - \hat{\beta} + V_H(\beta) - V_H(\hat{\beta}). \tag{31}$$

By Lemma 3 we know that: $V_H'(\beta) < -1$. By condition **(C)** and $\hat{\beta} < \beta$, the right hand side of (31) is strictly smaller than 0. However, since V_H is decreasing by condition **(M)**, the left hand side of (31) is strictly greater than 0, which makes (31) impossible to hold. ■

The remaining case is Regime LH. By Lemma 4 we know that there is at most only one pair of bid and ask prices within Regime LH. The next lemma establishes the unique relationship between prices and strategy. Since Lemma 4 gives us the uniqueness of prices if both manipulate, this gives us unique equilibrium strategy within Regime LH.

Lemma 6. *Let $\sigma, \hat{\sigma} \in \mathcal{E}(b)$ and we denote the equilibrium prices associated with σ and $\hat{\sigma}$ by (α, β) and $(\hat{\alpha}, \hat{\beta})$, respectively. If $\alpha = \hat{\alpha}$ and $\beta = \hat{\beta}$, then $\sigma = \hat{\sigma}$.*

Proof of Lemma 6: For the simplicity of notation, let $h = (1-\mu)\gamma + \mu\sigma_H$ and $l = (1-\mu)\gamma + \mu\sigma_L$. Suppose that in equilibrium, there are two different pairs of strategies, σ and $\hat{\sigma}$ in R_{LH}. Similarly with h and l, we define $\hat{h}$ and $\hat{l}$ associated with $\hat{\sigma}_{LB}$ and $\hat{\sigma}_H$. By the Bayes rule, we can write: $\alpha = \frac{hb}{hb+(1-b)l}$, and $\hat{\alpha} = \frac{\hat{h}b}{\hat{h}b+(1-b)\hat{l}}$. Since $\hat{\alpha} = \alpha$, we must have:

$$\hat{h}l = \hat{l}h. \tag{32}$$

Similarly, we have $\beta = \frac{(1-h)b}{(1-h)b+(1-b)(1-l)}$, and $\hat{\beta} = \frac{(1-\hat{h})b}{(1-\hat{h})b+(1-b)(1-\hat{l})}$. By equating them, we must have:

$$(1-\hat{h})(1-l) = (1-\hat{l})(1-h). \tag{33}$$

Combining the equations (32) and (33) gives $\hat{h} - h = \hat{l} - l \equiv \Delta$. Then, by substituting it into (32) we obtain:

$$(h + \Delta)l = (l + \Delta)h. \tag{34}$$

Therefore, we must have $h = l$ and $\hat{h} = \hat{l}$, which indicates: $\sigma = \hat{\sigma}$. ■

Lemma 7. *If $\sigma, \hat{\sigma} \in R_{LH}(b)$, then $\sigma = \hat{\sigma}$.*

Proof of Lemma 7: Proved by Lemma 4 and Lemma 6. ■

Proposition 3. *Equilibrium exists uniquely within each regime.*

Proof of Proposition 3:
Let: $\sigma \in \mathcal{E}(b)$. We will show that there is only one $\sigma \in R_i$ for every $i \in \{1, \cdots, 4\}$. In Regime L and H, by Lemma 5 the result follows. In regime $\emptyset$, it is obvious. In regime LH, by Lemma 4 and Lemma 7 the result follows. ■

Now we consider the possibility of co-existence of different regimes for one prior belief. We will prove that two different regimes do not co-exist by eliminating the possibility of each combination of regimes. Next four lemmas establish this result.

Lemma 8. *Regime L or H does not co-exist with Regime $\emptyset$.*

Proof of Lemma 8: Since the argument is symmetric, we will prove the result for $R_H(b)$. Let $\sigma \in R_H(b)$ and suppose by way of contradiction $\hat{\sigma} \in R_{\emptyset}(b)$. Let (α, β) be an equilibrium price associated with σ, and $(\hat{\alpha}, \hat{\beta})$ be an equilibrium price associated with $\hat{\sigma}$, respectively. Since $(1 - \sigma_L) = \hat{\sigma}_{LS} = 1$, we have by Bayes' rule $\hat{\alpha} > \alpha$ and $\hat{\beta} < \beta$. We have:

$$1 - \alpha + V_H(\alpha) = \beta - 1 + V_H(\beta); 1 - \hat{\alpha} + V_H(\hat{\alpha}) \geq \hat{\beta} - 1 + V_H(\hat{\beta}). \tag{35}$$

Subtracting (35) from (35) yields:

$$\hat{\alpha} - \alpha - V_H(\hat{\alpha}) + V_H(\alpha) \leq \beta - \hat{\beta} + V_H(\beta) - V_H(\hat{\beta}). \tag{36}$$

By Lemma 3 we know that: $V_H'(\beta) < -1$. By condition **(C)** and $\hat{\beta} < \beta$, the right hand side of (30) is strictly smaller than 0. However, since V_H is decreasing by condition **(M)**, the left hand side of (30) is strictly greater than 0, which makes (30) impossible to hold. ■

After Lemma 8 and Proposition 5 it remains to show that Regime L does not co-exist with Regime H. The next proposition explores this remaining case and shows the uniqueness of equilibrium under the conditions **(M)** and **(C)**.

Lemma 9. *Regime L does not co-exist with Regime H.*

Proof of Lemma 9:

By way of contradiction suppose that at prior b Regime L and Regime H co-exist.

We denote one pair of prices associated with Regime L by (α_1, β_1), and the other associated with Regime H by (α_2, β_2). Then by the indifference condition we must have:

$$\begin{aligned} -\alpha_1 + V_L(\alpha_1) &= \beta_1 + V_L(\beta_1); && (37) \\ 1 - \alpha_1 + V_H(\alpha_1) &\geq \beta_1 - 1 + V_H(\beta_1); && (38) \end{aligned}$$

and also

$$\begin{aligned} -\alpha_2 + V_L(\alpha_2) &\leq \beta_2 + V_L(\beta_2); && (39) \\ 1 - \alpha_2 + V_H(\alpha_2) &= \beta_2 - 1 + V_H(\beta_2). && (40) \end{aligned}$$

Consider Bayes' rule:

$$\alpha_1 = \frac{\bar{h}b}{\bar{h}b + l(b)(1-b)} \ \ \& \ \ \alpha_2 = \frac{h(b)b}{h(b)b + \bar{l}(1-b)};$$

$$\beta_1 = \frac{(1-\bar{h})b}{(1-\bar{h})b + (1-l(b))(1-b)} \ \ \& \ \ \beta_2 = \frac{(1-h(b))b}{(1-h(b))b + (1-\bar{l})(1-b)}.$$

Case 1: $\alpha_1 > \alpha_2$: Let $\bar{h} = (1-\mu)\gamma + \mu$ and $\bar{l} = (1-\mu)\gamma$ so that we have: $h(b) \leq \bar{h}$ and $l(b) \geq \bar{l}$.

Since $\alpha_2 > \beta_2$, we must have:

$$\alpha_2 = \frac{b}{b + \frac{\bar{l}}{h(b)}(1-b)} > \beta_2 = \frac{b}{b + \frac{(1-\bar{l})}{(1-h(b))}(1-b)}.$$

Therefore we must have:

$$\frac{\bar{l}}{h(b)} < \frac{(1-\bar{l})}{(1-h(b))}. \tag{41}$$

Now by Bayes' Rule, we must have:

$$\alpha_1 = \frac{\bar{h}b}{\bar{h}b + l(b)(1-b)} > \alpha_2 = \frac{h(b)b}{h(b)b + \bar{l}(1-b)}.$$

$\Longleftrightarrow$

$$\bar{h}\bar{l} > h(b)l(b). \tag{42}$$

$\Longleftrightarrow$

$$\bar{h}\bar{l} - h(b)\bar{l} > h(b)l(b) - h(b)\bar{l}. \tag{43}$$

$\Longleftrightarrow$

$$(\bar{h} - h(b))\bar{l} > h(b)(l(b) - \bar{l}). \tag{44}$$

$\iff$

$$(\bar{h}-1+1-h(b))\bar{l} > h(b)(l(b)-1+1-\bar{l}). \tag{45}$$

$\iff$

$$(1-\frac{1-\bar{h}}{1-h(b)})\bar{l}(1-h(b)) > h(b)(1-\bar{l})(1-\frac{1-l(b)}{1-\bar{l}}). \tag{46}$$

$\iff$

$$(1-\frac{1-\bar{h}}{1-h(b)})\frac{\bar{l}}{1-\bar{l}} > \frac{h(b)}{1-h(b)}(1-\frac{1-l(b)}{1-\bar{l}}). \tag{47}$$

By (41) we must have: $\frac{\bar{l}}{1-\bar{l}} < \frac{h(b)}{1-h(b)}$. In addition $\frac{1-\bar{h}}{1-h(b)} < 1$ and $\frac{1-l(b)}{1-\bar{l}} < 1$. Therefore in order for (47) to hold, we must have:

$$1-\frac{1-\bar{h}}{1-h(b)} > 1-\frac{1-l(b)}{1-\bar{l}}. \tag{48}$$

Therefore we must have: $\frac{1-\bar{h}}{1-h(b)} < \frac{1-l(b)}{1-\bar{l}}$, which indicates $\beta_1 < \beta_2$ by Bayes' rule. Now subtracting (40) from (38) yields:

$$-\alpha_1+\alpha_2+V_H(\alpha_1)-V_H(\alpha_2) \geq \beta_1-\beta_2+V_H(\beta_1)-V_H(\beta_2). \tag{49}$$

Since $\beta_1 < \beta_2$, by condition **(C)** we have: $\frac{V_H(\beta_1)-V_H(\beta_2)}{\beta_1-\beta_2} < -1$, which indicates the RHS of (49) > 0. However $\alpha_1 \geq \alpha_2$ indicates the LHS of (49) ≤ 0. Therefore (49) is impossible. □

Case 2: $\alpha_2 \geq \alpha_1$: Suppose that we have: $\alpha_2 \geq \alpha_1$. Then by Proposition 2 in the mirror economy, we must have: $1-\tilde{\beta}_2 \geq 1-\tilde{\beta}_1$. Therefore we have: $\tilde{\beta}_2 \leq \tilde{\beta}_1$. Then by taking the contrapositive of the proved statement in the first part of this proof, which is: $\alpha_1 > \alpha_2 \to \beta_1 < \beta_2$, we obtain: $\beta_2 \leq \beta_1 \to \alpha_1 \leq \alpha_2$. By applying this result to the mirror economy, we obtain: $\tilde{\beta}_2 \leq \tilde{\beta}_1 \to \tilde{\alpha}_1 \leq \tilde{\alpha}_2$, which indicates: $1-\beta_1 \leq 1-\beta_2$. Therefore we conclude: $\beta_2 \leq \beta_1$.

Now subtracting (37) from (39) yields:

$$\alpha_1-\alpha_2+V_L(\alpha_2)-V_L(\alpha_1) \leq \beta_2-\beta_1+V_L(\beta_2)-V_L(\beta_1). \tag{50}$$

Since $\beta_1 \geq \beta_2$, by condition **(M)** we must have: the RHS of (50) ≤ 0. However $\alpha_1 < \alpha_2$ indicates by condition **(C)** we have: $\frac{V_L(\alpha_2)-V_L(\alpha_1)}{\alpha_2-\alpha_1} > 1$ and thus the LHS of (50) > 0. Therefore (50) is impossible. □ ■

Lemma 10. *Regime LH does not co-exist with Regime $\emptyset$.*

Proof of Lemma 10:

Let $\sigma \in R_{LH}(b)$ and suppose by way of contradiction $\hat{\sigma} \in R_{\emptyset}(b)$. Let (α,β) be an equilibrium price associated with σ, and $(\hat{\alpha},\hat{\beta})$ be an equilibrium price associated with $\hat{\sigma}$, respectively. By Bayes' rule $\hat{\alpha} > \alpha$ and $\hat{\beta} < \beta$. We have:

$$1-\alpha+V_H(\alpha) = \beta-1+V_H(\beta); \tag{51}$$
$$1-\hat{\alpha}+V_H(\hat{\alpha}) \geq \hat{\beta}-1+V_H(\hat{\beta}). \tag{52}$$

Subtracting (52) from (51) yields:

$$\hat{\alpha} - \alpha - V_H(\hat{\alpha}) + V_H(\alpha) \leq \beta - \hat{\beta} + V_H(\beta) - V_H(\hat{\beta}). \tag{53}$$

By Lemma 3 we know that: $V_H'(\beta) < -1$. By condition **(C)** and $\hat{\beta} < \beta$, the right hand side of (52) is strictly smaller than 0. However, since V_H is decreasing by condition **(M)**, the left hand side of (52) is strictly greater than 0, which makes (52) impossible to hold. ■

Lemma 11. *Regime LH does not co-exist with Regime L or H.*

Proof of Lemma 11:
Let $\sigma \in R_{LH}(b)$ and suppose by way of contradiction $\hat{\sigma} \in R_H(b)$. Let (α, β) be an equilibrium price associated with σ, and $(\hat{\alpha}, \hat{\beta})$ be an equilibrium price associated with $\hat{\sigma}$, respectively. We have:

$$\begin{aligned} 1 - \alpha + V_H(\alpha) &= \beta - 1 + V_H(\beta); && (54)\\ 1 - \hat{\alpha} + V_H(\hat{\alpha}) &= \hat{\beta} - 1 + V_H(\hat{\beta}); && (55) \end{aligned}$$

and

$$\begin{aligned} -\alpha + V_L(\alpha) &= \beta + V_L(\beta); && (56)\\ -\hat{\alpha} + V_L(\hat{\alpha}) &\leq \hat{\beta} + V_L(\hat{\beta}). && (57) \end{aligned}$$

Subtracting (55) from (54) yields:

$$\hat{\alpha} - \alpha - V_H(\hat{\alpha}) + V_H(\alpha) = \beta - \hat{\beta} + V_H(\beta) - V_H(\hat{\beta}), \tag{58}$$

and similarly
Case 1: $\alpha > \hat{\alpha}$: Since $\hat{\alpha} > \hat{\beta}$, we must have:

$$\hat{\alpha} = \frac{b}{b + \frac{\bar{l}}{h_H(b)}(1-b)} > \hat{\beta} = \frac{b}{b + \frac{(1-\bar{l})}{(1-h_H(b))}(1-b)}.$$

Therefore we must have:

$$\frac{\bar{l}}{h_H(b)} < \frac{(1-\bar{l})}{(1-h_H(b))}. \tag{59}$$

Now by Bayes' Rule, we must have:

$$\alpha = \frac{h(b)b}{h(b)b + l(b)(1-b)} > \hat{\alpha} = \frac{h_H(b)b}{h_H(b)b + \bar{l}(1-b)}.$$

$\iff$

$$\bar{l}h(b) > h_H(b)l(b). \tag{60}$$

$\iff$

$$\bar{l}h(b) - \bar{l}h_H(b) > h_H(b)l(b) - \bar{l}h_H(b). \tag{61}$$

$\Longleftrightarrow$

$$(h(b) - h_H(b))\bar{l} > h_H(b)(l(b) - \bar{l}). \tag{62}$$

$\Longleftrightarrow$

$$(h(b) - 1 + 1 - h_H(b))\bar{l} > h_H(b)(l(b) - 1 + 1 - \bar{l}). \tag{63}$$

$\Longleftrightarrow$

$$(1 - \frac{1 - h(b)}{1 - h_H(b)})\bar{l}(1 - h_H(b)) > h_H(b)(1 - \bar{l})(1 - \frac{1 - l(b)}{1 - \bar{l}}). \tag{64}$$

$\Longleftrightarrow$

$$(1 - \frac{1 - h(b)}{1 - h_H(b)})\frac{\bar{l}}{1 - \bar{l}} > \frac{h_H(b)}{1 - h_H(b)}(1 - \frac{1 - l(b)}{1 - \bar{l}}). \tag{65}$$

By (59) we must have: $\frac{\bar{l}}{1-\bar{l}} < \frac{h_H(b)}{1-h_H(b)}$. In addition $\frac{1-l(b)}{1-\bar{l}} < 1$, which indicates $1 - \frac{1-h(b)}{1-h_H(b)} > 0$ by (65). Therefore in order for (65) to hold, we must have:

$$1 - \frac{1 - h(b)}{1 - h_H(b)} > 1 - \frac{1 - l(b)}{1 - \bar{l}}. \tag{66}$$

Therefore we must have: $\frac{1-h(b)}{1-h_H(b)} < \frac{1-l(b)}{1-\bar{l}}$, which indicates $\beta < \hat{\beta}$ by Bayes' rule. By condition **(M)** the LHS of (58) is strictly smaller than 0. If $\beta < \hat{\beta}$ then by Lemma 3 we must have: $\frac{V_H(\beta)-V_H(\hat{\beta})}{\beta-\hat{\beta}} < -1$ which indicates that the RHS of (58) is strictly greater than 0. This is a contradiction. □

Case 2: $\alpha \leq \hat{\alpha}$: By the same logic in the proof for case 2 of Lemma 9, we must have: $\hat{\beta} \leq \beta$.

Now subtracting (57) from (56) yields:

$$\hat{\alpha} - \alpha - V_L(\hat{\alpha}) + V_L(\alpha) \geq \beta - \hat{\beta} + V_L(\beta) - V_L(\hat{\beta}). \tag{67}$$

Since $\hat{\beta} \leq \beta$, by condition **(M)** we must have: the RHS of (67) ≥ 0. If $\alpha = \hat{\alpha}$, then we must have: $\beta = \hat{\beta}$, which is impossible because as the proof in 7, if both prices are the same, then $\sigma = \hat{\sigma}$. This is a contradiction with our assumption. Now suppose $\alpha < \hat{\alpha}$. However by condition **(C)** we have: $\frac{V_L(\hat{\alpha})-V_L(\alpha)}{\hat{\alpha}-\alpha} > 1$ and thus the LHS of (67) < 0. Therefore (67) is impossible. □ ■

We have proved that two different regimes do not co-exist. This result indicates that when one regime shifts to a different regime as prior belief increases/decreases, there is no jump in the equilibrium strategy. The following Lemma and Proposition formally state this intuition.

Lemma 12. *Consider the boundary of each regime, which is a prior $b \in (0, 1)$ such that for any sufficiently small $\epsilon > 0$, $b - \epsilon \in I_i$ and $b + \epsilon \in I_j$ for $j \neq i$ and $i, j \in \{L, H, \emptyset\}$. Let $\sigma = \mathcal{E}(b)$ and then we must have: $\sigma_H = 1$ and $\sigma_L = 0$.*

Proof of Lemma 12:

By Proposition 6 and Proposition 4, the equilibrium strategy is expressed by a continuous function of prior belief denoted by $\mathcal{E}(b)$. The result is clear when I_L shifts to/from $I_\emptyset$ and I_H shifts to/from $I_\emptyset$

because of the definition of Regime $\emptyset$ and the continuity of the equilibrium strategy. Consider I_L shifts to/from I_H. Since the argument is symmetric, we only consider I_L shifts to/from I_H. Take a sequence $\{\epsilon_k\}_k$ with $\epsilon_k > 0$ which converges to 0. For every $b^k = b - \epsilon_k$, we have $\sigma^k = R_L(b^k)$ and every $\hat{b}^k = b + \epsilon_k$, we have$\hat{\sigma}^k = R_H(\hat{b}^k)$. Take a limit of k to infinity. Then we have $\sigma^k \to \sigma$ and $\hat{\sigma}^k \to \sigma$ by the continuity of the equilibrium strategy. The only possibility is: $\sigma_H = 1$ and $\sigma_L = 0$ by the continuity of the equilibrium strategy. ■

Proposition 4. *The equilibrium strategy $\sigma = \mathcal{E}(b)$ is continuous in b on $(0, 1)$.*

Proof of Proposition 4:
By Proposition 6, $\mathcal{E}$ is a function of prior belief b. By Theorem 1 the equilibrium correspondence $\mathcal{E}$ is upper semi-continuous. Therefore we conclude that it is continuous. ■

Since there is no jump in the equilibrium strategy, by Bayes rule the following result about bid and ask prices is immediate.

Lemma 13. *Bid and ask prices are continuous in b.*

Proof of Lemma 13:
By Lemma 4 and Bayes' rule, the result follows. ■

In the end we will prove that Regime LH does not arise. Let $\sigma^* \in \mathcal{E}(b)$ and (α, β) denote the equilibrium ask and bid prices. We denote payoff from buy by π^B_θ and payoff from sell by π^S_θ for $\theta \in \{L, H\}$. Payoff from buy is a function of ask price and payoff from sell is a function of bid price. When manipulation arises, those two payoff functions intersect. If both types manipulate at the same time, then the two pairs of the two payoff functions intersect for the same bid and ask prices. We will prove that this will not happen. To do so, we define the following functions to consider the single-crossing property of the functions for each type:

$$\begin{aligned} A(b, \sigma_L, \sigma_H) &= \tfrac{[(1-\mu)\gamma + \mu\sigma_H]b}{(1-\mu)\gamma + \mu b\sigma_H + \mu(1-b)\sigma_L}; \\ B(b, \sigma_L, \sigma_H) &= \tfrac{[(1-\mu)(1-\gamma) + \mu(1-\sigma_H)]b}{(1-\mu)(1-\gamma) + \mu b(1-\sigma_H) + \mu(1-b)(1-\sigma_L)}. \end{aligned} \tag{68}$$

By the definition of the functions, obviously we have:

$$A(b, \sigma^*_L, \sigma^*_H) = \alpha \;\; \& \;\; B(b, \sigma^*_L, \sigma^*_H) = \beta. \tag{69}$$

Lemma 14. *Suppose that $b \in I_H$. Then the following holds:*

$$\pi^B_H(A(b, 0, 1)) \;\; < \;\; \pi^S_H(B(b, 0, 1)). \tag{70}$$

Proof of Lemma 14:
Since $b \in I_H$, there must be σ^*_H which satisfies:

$$\pi^B_H(A(b, 0, \sigma^*_H)) \;\; = \;\; \pi^S_H(B(b, 0, \sigma^*_H)). \tag{71}$$

Notice that as σ_H increases, $A(b,0,\sigma_H)$ increases and $B(b,0,\sigma_H)$ decreases. Take $\sigma'_H > \sigma^*_H$. Then since V_H is strictly decreasing, we have: $\pi^B_H(A(b,0,\sigma'_H)) < \pi^B_H(A(b,0,\sigma^*_H))$. Moreover since when the high-type manipulates, the slope of the value function satisfies: $V'_H(\beta) < -1$ by Lemma 3, the strict convexity of the value function implies: $\pi^S_H(B(b,0,\sigma'_H)) > \pi^S_H(B(b,0,\sigma^*_H))$. Since $\sigma^*_H \in (0,1)$, by (71) we must have: (70). ■

Lemma 15. *If* $\pi^B_H(A(b,0,1)) < \pi^S_H(B(b,0,1))$ *and* $\pi^B_L(A(b,0,1)) < \pi^S_L(B(b,0,1))$*, then* $b \in I_H$.

Proof of Lemma 15:
Suppose that $\pi^B_H(A(b,0,1)) < \pi^S_H(B(b,0,1))$ and $\pi^B_L(A(b,0,1)) < \pi^S_L(B(b,0,1))$. Then it is clear that: $b \notin I_\emptyset$. Notice that when $\sigma_H = 0$ we have:

$$\pi^B_H(A(b,0,0)) = 1 - b + V_H(b) > \pi^S_H(B(b,0,0)) = b - 1 + V_H(b). \tag{72}$$

Since π^B_θ's and π^S_θ's are continuous in σ_H, the intermediate value theorem is applicable. Thus there must be $\sigma^*_H \in (0,1)$ which satisfies (71). By the same argument with the above, π^B_H is monotonically decreasing in terms of σ_H and π^B_H is monotonically increasing and thus there is only one σ^*_H.

On the contrary suppose that $b \notin I_H$. Then, we must have:

$$\pi^B_L(A(b,0,\sigma^*_H)) \geq \pi^S_L(B(b,0,\sigma^*_H)). \tag{73}$$

We consider two cases where the inequality (73) is strict or the equality holds.
(Case 1) The expression (73) holds with strict inequality. Since $\pi^B_L(A(b,0,1)) < \pi^S_L(B(b,0,1))$, by the intermediate value theorem there must be $\bar{\sigma}_H \in (\sigma^*_H, 1)$ such that:

$$\pi^B_L(A(b,0,\bar{\sigma}_H)) = \pi^S_L(B(b,0,\bar{\sigma}_H)). \tag{74}$$

As $A(b,0,\bar{\sigma}_H) > B(b,0,\bar{\sigma}_H)$ for $\bar{\sigma}_H \neq 0$ by Bayes rule, (74) gives:

$$\frac{V_L(A(b,0,\bar{\sigma}_H)) - V_L(B(b,0,\bar{\sigma}_H))}{A(b,0,\bar{\sigma}_H) - B(b,0,\bar{\sigma}_H)} > 1.$$

Therefore as σ_H decreases, the RHS of (74) decreases and the LHS of (74) increases. Therefore (73) is impossible to hold since $\bar{\sigma}_H > \sigma^*_H$. □
(Case 2) The expression (73) holds with equality. By (71) and (73) we have:

$$\begin{aligned} \pi^B_L(A(b,0,\sigma^*_H)) &= \pi^S_L(B(b,0,\sigma^*_H)); \\ \pi^B_H(A(b,0,\sigma^*_H)) &= \pi^S_H(B(b,0,\sigma^*_H)). \end{aligned} \tag{75}$$

Since $b \notin I_H$ by our assumption, by Theorem 1 we must have: $b \in I_r$ for $r \in \{\emptyset, L, LH\}$. By Lemma 9, Lemma 11, and Lemma 8 we know that there is no other pair of bid and ask prices to satisfy the indifference conditions for other regimes at the same time with (75). This is a contradiction. ■

Lemma 16. *Suppose that $b \in I_L$. Then the following holds:*

$$\pi_L^B(A(b,0,1)) \quad > \quad \pi_L^S(B(b,0,1)). \tag{76}$$

Proof of Lemma 16:
Proved symmetrically with Lemma 14. ■

Lemma 17. *Suppose that $b \in I_{LH}$. Then the following holds:*

$$\pi_H^B(A(b,0,1)) \quad < \quad \pi_H^S(B(b,0,1)); \tag{77}$$
$$\pi_L^B(A(b,0,1)) \quad > \quad \pi_L^S(B(b,0,1)). \tag{78}$$

Proof of Lemma 17:
Proved symmetrically with Lemma 14. ■

Lemma 18. *Both types do not manipulate at the same time in an interval (b_0, b_1) with $b_0 \neq b_1$.*

Proof of Lemma 18:
By Lemma 7 R_{LH} is a function of prior b. Let $\sigma = R_{LH}(b)$ and we define: $h(b) = (1-\mu)\gamma + \mu\sigma_H$, and $l(b) = (1-\mu)\gamma + \mu\sigma_L$. In this proof we have to consider an interval but not a point unlike Lemma 7. Thus we need to define h or l as a function of prior b and in this way we consider the interval of priors in Regime LH.

On the contrary, suppose that they manipulate at the same time in an interval (b_0, b_1) with $b_0 < b_1$. By Lemma 4 we know that there exists only one pair of equilibrium bid and ask prices for which both types manipulate. Therefore ask and bid prices associated with every $b \in (b_0, b_1)$ must be same. Consider Bayes' rule for the interval. Since it gives us constant bid and ask prices for the interval, it is continuously differentiable. Therefore, if they manipulate in the interval, the first derivative of the bid and ask prices must be equal to $zero$. By taking the derivatives for $b \in (b_0, b_1)$ and setting the numerators at $zero$, we obtain:

$$l(b) \cdot h(b) = (1-b)b[h(b) \cdot l'(b) - h'(b) \cdot l(b)], \tag{79}$$

and

$$[1-h(b)][1-l(b)] = (1-b)b[h'(b) \cdot (1-l(b)) - (1-h(b)) \cdot l'(b)]. \tag{80}$$

By (79) we obtain:

$$l'(b) = \frac{l(b)}{(1-b)b} + \frac{h'(b)\, l(b)}{h(b)}. \tag{81}$$

By substituting (81) into (80) and organizing the terms we obtain:

$$[1-h(b)][1-l(b)] = (1-b)bh'(b)[(1-l(b)) - \frac{(1-h(b))\, l(b)}{h(b)}] - (1-h(b))l(b). \tag{82}$$

Therefore we obtain:

$$1 - h(b) = (1-b)bh'(b) \times \frac{h(b) - l(b)}{h(b)} \tag{83}$$

By Lemma 1 we must have $h(b) - l(b) \neq 0$. Thus we obtain:

$$h'(b) = \frac{h(b)[1-h(b)]}{b(1-b)(h(b)-l(b))} > 0. \tag{84}$$

Then by (81) we obtain:

$$l'(b) = \frac{l(b)[1-l(b)]}{b(1-b)(h(b)-l(b))} > 0. \tag{85}$$

Consider the case where Regime L shifts to Regime LH. In the end of Regime L, we must have: $\sigma_H = 1$. It is impossible to increase σ_H out of 1 but by (84) $h'(b) > 0$. Similarly it is impossible to have a shift from Regime $\emptyset$ to Regime LH.

The last case that we have to prove is that Regime H does not shift to Regime LH as prior increases. On the contrary suppose that it does. Then there has to be a prior b_H such that $b_H \in I_H$ but $b_H + \epsilon \notin I_H$ for sufficiently small $\epsilon > 0$. Notice that when $b = 1$, the low-type does not manipulate. So $1 \notin I_{LH}$. So there has to be a prior $b_{LH}(\leq 1)$ in which $b_{LH} - \epsilon \in I_{LH}$ for sufficiently small $\epsilon > 0$ but $b_{LH} \notin I_{LH}$.

In the end of Regime H, we must have: $\sigma_L = 0$ and $\sigma_H < 1$ because by (84) and (85) if Regime LH arises in prior $b + \epsilon$, we must have: $h(b+\epsilon) > h(b)$. Notice that: $l(b+\epsilon) > l(b) = \gamma(1-\mu)$.

Let $\sigma^* \in \mathcal{E}(b_H)$ and $\sigma^{**} \in \mathcal{E}(b_{LH})$. Notice: $b_{LH} \notin I_H$ because $b_{LH} - \epsilon \in I_{LH}$ means that the low-type manipulates at $b_{LH} - \epsilon$ and in the interval I_H the low-type does not manipulate, which contradicts with $l'(b) > 0$ in Regime LH by the continuity of equilibrium strategies. For the same reason we have: $b_{LH} \notin I_\emptyset$. The remaining possibility is: $b_{LH} \in I_L$. In other words, we may have: $\sigma_H^{**} = 1$ and $\sigma_L^{**} > 0$. So it remains to show that this last case does not happen.

The idea here is that we look at the two boundary points b_H and b_{LH}. If Regime LH arises in an interval (b_0, b_1), the indifference conditions for both types hold for the only one pair of prices irrespective of priors in that interval.

We denote the equilibrium prices at Regime LH by (α, β). The indifference condition for the low-type dictates:

$$-\alpha + V_L(\alpha) \quad = \quad \beta + V_L(\beta). \tag{86}$$

Next consider the low-type's payoffs at b_H. Lemma 14 implies:

$$-A(b_H,0,1) + V_L(A(b_H,0,1)) \quad < \quad B(b_H,0,1) + V_L(B(b_H,0,1))).$$

Consider b_{LH}. Lemma 17 implies:

$$\pi_L^B(A(b_{LH},0,1)) \quad < \quad \pi_L^S(B(b_{LH},0,1)).$$

Therefore by the intermediate value theorem there must be $\bar{b} \in (b_H, b_{LH})$ such that:

$$\pi_L^B(A(\bar{b},0,1)) \quad = \quad \pi_L^S(B(\bar{b},0,1)).$$

Then we obtain:

$$\alpha - A(\bar{b},0,1) + V_L(A(\bar{b},0,1)) - V_L(\alpha) = B(\bar{b},0,1) - \beta + V_L(B(\bar{b},0,1)) - V_L(\beta). \tag{87}$$

By Bayes' rule we must have: $\alpha < A(\bar{b},0,1)$ and $\beta > B(\bar{b},0,1)$. Since the low-type manipulates at α and V_L is strictly convex, we must have: $\frac{V_L(A(\bar{b},0,1))-V_L(\alpha)}{A(\bar{b},0,1)-\alpha} > 1$. Therefore the RHS of (87) is strictly positive. On the other hand since $\beta > B(\bar{b},0,1)$ and V_L is monotonically increasing, the LHS of (87) is strictly negative. This is a contradiction. ■

Proposition 5. *Both types do not manipulate at the same time. In other words, Regime LH does not arise.*

Proof of Proposition 5:
Suppose not. Then, there is a $b \in \Delta(\Theta)$ and $\sigma \in \mathcal{E}(b)$ such that: $\sigma_H < 1$ and $\sigma_L > 0$. Since σ is continuous, there is ϵ such that for every b' within the ϵ-neighborhood of b and $\sigma' \in \mathcal{E}(b')$ $\sigma'_H < 1$ and $\sigma'_L > 0$ holds. This gives us the interval at which both types manipulate at the same time, which contradicts with Proposition 18. ■

Proposition 6. *The equilibrium exists uniquely when the value functions in every period satisfy the conditions* **(M)** *and* **(C)**.

Proof of Proposition 6:
Proposition 5 indicates: $I_{LH} = \emptyset$. Lemma 8 indicates that $I_L \cap I_\emptyset = \emptyset$ and $I_H \cap I_\emptyset = \emptyset$. Lemma 9 indicates that $I_L \cap I_H = \emptyset$. In the end Proposition 3 completes our proof. ■

To end this section we remark that in the last period neither type manipulates because there is no opportunity to re-trade. As a result, we can calculate the value functions in the last period and show that they satisfy **(C)** and **(M)**. Therefore, in the second last period there is a unique equilibrium.

Remark 1. *Suppose that $T = 3$. Equilibrium exists uniquely (without* **(C)** *and* **(M)***).*

4 Characterization and Simulation

In the previous section we proved that if the value function is monotone and strictly convex in the market-maker's belief, then the equilibrium exists uniquely. In this section we will characterize the equilibrium prices under the two conditions **(C)** and **(M)**. Moreover, we will demonstrate some simulation results of the model.

We redefine ask and bid functions as $\alpha : \Delta(\Theta) \to [0,1]$ and $\beta : \Delta(\Theta) \to [0,1]$. Moreover we extend the definitions for the functions $h(b)$ and $l(b)$ to all regimes, that is, for $\sigma \in \mathcal{E}(b)$, $h(b) = (1-\mu)\gamma + \mu\sigma_H$, $l(b) = (1-\mu)\gamma + \mu\sigma_L$. In addition, define: $P(b) = h(b) \times b + l(b) \times (1-b)$. In words P is the market maker's expectation that buy order comes.

Lemma 19. *Ask and bid prices, $\beta(b)$ and $\alpha(b)$, satisfy the following:*
1. for every prior $b \in (0,1)$, $1 > \alpha(b) > 0$ and $1 > \beta(b) > 0$;
2. for every prior $b \in (0,1)$, bid-ask spread is non-zero: that is, $\alpha(b) - \beta(b) \neq 0$.

Proof of Lemma 19:
1. For every $b \in (0,1)$, $1 > P(b) > 0$ by Lemma 1. In addition $h(b)b \neq 0$ and $(1-h(b))b \neq 0$. By Bayes' rule we conclude: $1 > \alpha(b) > 0$ and $1 > \beta(b) > 0$. $\square$
2. The proof is done by Lemma 1. $\square$ $\blacksquare$

Lemma 20. *In each regime, bid and ask prices are piecewise differentiable in every period. Moreover the value functions are piecewise differentiable in every period.*

Proof of Lemma 20:
Notice that: $\sigma_H \in R_2(b)$ or $\sigma_L \in R_1(b)$ solves the following equation:

$$\begin{aligned}
&1 - \tfrac{b\times[\mu\,\sigma_H+(1-\mu)\,\gamma]}{\mu\,[b\times\sigma_H]+(1-\mu)\,\gamma} + V_H(\tfrac{b\times[\mu\,\sigma_H+(1-\mu)\,\gamma]}{\mu\,b\times\sigma_H+(1-\mu)\,\gamma}) \\
&\quad = \tfrac{b\times[(1-\mu)\,(1-\gamma)+\mu(1-\sigma_H)]}{\mu\,[b\times(1-\sigma_H)+(1-b)]+(1-\mu)\,(1-\gamma)} - 1 + V_H(\tfrac{b\times[\mu\,(1-\sigma_H)+(1-\mu)\,(1-\gamma)]}{\mu\,[b\times(1-\sigma_H)+(1-b)]+(1-\mu)\,(1-\gamma)}); \\
&-\tfrac{b\times[\mu+(1-\mu)\,\gamma]}{\mu\,[b+(1-b)\times\sigma_L]+(1-\mu)\,\gamma} + V_L(\tfrac{b\times[\mu+(1-\mu)\,\gamma]}{\mu\,[b+(1-b)\times\sigma_L]+(1-\mu)\,\gamma}) \\
&\quad = \tfrac{b\times+(1-\mu)\,(1-\gamma)}{\mu\,(1-b)\times(1-\sigma_L)+(1-\mu)\,(1-\gamma)} + V_L(\tfrac{b\times(1-\mu)\,(1-\gamma)}{\mu\,(1-b)\times(1-\sigma_L)+(1-\mu)\,(1-\gamma)}).
\end{aligned} \tag{88}$$

Notice that: V^T's are continuously differentiable on [0, 1]. Suppose that the next-period value functions are piecewise differentiable. By the Implicit Function Theorem, σ_H or σ_L are piecewise continuously differentiable in terms of b. Bid and ask prices are a continuously differentiable function in terms of b or σ_H or σ_L. Therefore we conclude that bid and ask prices are piecewise differentiable. By Lemma 2 the current value function is also piecewise differentiable. By mathematical induction the result holds for every period. $\blacksquare$

Suppose that the low-type manipulates. Let: $F_l(b) = l(b) - l'(b)b(1-b)$. Then, we have:

$$\alpha'(b) = \frac{\bar{h} \cdot F_l(b)}{P(b)} \text{ and } \beta'(b) = \frac{(1-\bar{h}) \cdot (1 - F_l(b))}{1 - P(b)}.$$

Therefore, we have:

$$\alpha'(b)b = \alpha(b) \cdot \frac{F_l(b)}{P(b)} \text{ and } \beta'(b)b = \beta(b) \cdot \frac{1 - F_l(b)}{1 - P(b)}, \tag{89}$$

and

$$\alpha''(b) = h(b) \times \frac{F_l'(b)P(b) - F_l(b) \times 2P'(b)}{(h(b)b + l(b)(1-b))^3}; \tag{90}$$

$$\beta''(b) = [1-h(b)] \times \frac{-F_l'(b)(1-P(b)) + (1-F_l(b)) \times 2P'(b)}{([1-h(b)]b + [1-l(b)](1-b))^3}. \tag{91}$$

Proposition 7. *Suppose that the next-period value functions satisfy* **(C)** *and* **(M)**. *Ask and bid prices, $\alpha(b)$ and $\beta(b)$ are strictly increasing in b.*

Proof of Proposition 7:
By Proposition 5 we know that Regime 4 does not arise. When nobody manipulates, by the Bayes rule we can show that bid and ask prices decrease as market makers' belief b decreases. It remains to show that the result holds in Regime L and H. Since the argument is symmetric, suppose that the low-type manipulates at b. Then, as the low-type's indifference condition for σ we have:

$$-\alpha(b) + V_L(\alpha(b)) = \beta + V_L(\beta(b)). \tag{92}$$

Taking the first derivative we obtain:

$$\alpha'(b)(-1 + V_L'(\alpha(b))) = \beta'(b)(1 + V_L'(\beta(b))). \tag{93}$$

By Lemma 3 $-1 + V_L'(\alpha(b)) > 0$. By condition **(M)** $(1 + V_L'(\beta(b))) > 0$. Therefore (93) indicates that: $\alpha'(b) > 0$ if and only if $\beta'(b) > 0$. By (89) if $\alpha'(b) \leq 0$ we must have: $1 - F_l(b) \leq 0$, which is impossible. ■

Let $\sigma_H : [0,1] \to [0,1]$ be the high-type's equilibrium strategy of buying as a function of b and similarly let $\sigma_L : [0,1] \to [0,1]$ be the low-type's equilibrium strategy of buying as a function of b. By taking the first and second derivatives in the bid and ask prices with $\sigma_H(b) = 1$ and $\sigma_L(b) = 0$, we obtain the following result.

Proposition 8. *Suppose that nobody manipulates in equilibrium. In equilibrium, the ask price is strictly concave and the bid price is strictly convex.*

Proof of Proposition 8: Omitted. ■

By Lemma 4, we know that if Regime LH arises in equilibrium, then there is a unique pair of bid and ask prices. The following Lemma shows that in Regime L and H, the slope from the origin in equilibrium ask price is decreasing and the slope from the origin in equilibrium bid price is increasing.

Lemma 21. *In Regime L and H, $\frac{\alpha(b)}{b}$ is decreasing and $\frac{\beta(b)}{b}$ is increasing in terms of b.*

The previous lemma does not say that the ask price is strictly concave and the bid price is strictly convex, although convex or concave functions satisfy the properties in Lemma 21. To see how bid and ask prices behave in equilibrium we have written a computer program. In our program we use a calibrating method called "linear interpolation." Since in the last period of the game neither type manipulates, we can calculate the value functions in the last period as well as the bid and ask prices. We then split the interval $[0,1]$ into n segments and linearly interpolate the value function for each type in each interval. The first case we consider is manipulation by a high-type. We do this by seeing whether or not a pair of ask and bid prices exist that make the high-type indifferent between buy and sell orders in each interval of the market makers belief. Similarly, we consider a second case where the low-type manipulates and a third case where both types manipulate. Using the bid and ask prices we obtain from this procedure, we calculate the current period value functions and repeat the procedure in the following periods. To simplify the following calculation let $f_H(b) = (1-\mu)\gamma + \mu\sigma_H(b)$, and $f_L(b) = (1-\mu)\gamma + \mu\sigma_L(b)$.

Consider the case where only the low-type manipulates. Expressing $f_L(b)$ as a function of the ask price $\alpha(b)$ gives:

$$f_L(b) = \frac{\alpha(b) \times (1-b) - b(1-\alpha(b))f_H(b)}{\alpha(b) \times (1-b)}. \tag{94}$$

Since the high-type does not manipulate, $f_H(b) = (1-\mu)\gamma + \mu$, which we denote by H. Then,:

$$\beta(b) = \frac{\alpha(b) \times b \times (1-H)}{\alpha(b) - b \times H}. \tag{95}$$

We construct a new function $\tilde{V}_L$ by a linear interpolation of V_L. Define for each $\alpha_L \in [b_k, b_{k+1}]$,

$$\tilde{V}_L(\alpha_L) = (\alpha_L - b_k)\frac{V_L(b_{k+1}) - V_L(b_k)}{(b_{k+1} - b_k)} + V_L(b_k), \tag{96}$$

and for each $\beta_L \in [b_j, b_{j+1}]$,

$$\tilde{V}_L(\beta_L) = (\beta_L - b_j)\frac{V_L(b_{j+1}) - V_L(b_j)}{(b_{j+1} - b_j)} + V_L(b_j). \tag{97}$$

Let $m_k^L = \frac{V_L(b_{k+1}) - V_L(b_k)}{b_{k+1} - b_k}$ and $m_j^L = \frac{V_L(b_{j+1}) - V_L(b_j)}{(b_{j+1} - b_j)}$. By substituting the bid price (95) into the indifference condition for the low-type and rearranging yields:

$$\begin{aligned}&\alpha_L^2(m_k^L - 1) + \alpha_L\left(bH(1-m_k^L) + b(H-1)(1+m_j^L) - [V_L(b_j) - V_L(b_k) + b_k m_k^L - b_j m_j^L]\right)\\ &+[V_L(b_j) - V_L(b_k) + b_k m_k^L - b_j m_j^L]bH = 0.\end{aligned} \tag{98}$$

Thus we obtain the following lemma.

Lemma 22. *If the low-type manipulates in equilibrium, then the equilibrium price α_L solves (98). Moreover, the equilibrium price α_L satisfies:*

$$\tilde{L} \equiv \frac{\alpha_L(1-b) - b(1-\alpha_L)H}{\alpha_L(1-b)} \leq (1-\mu)(1-\gamma) + \mu. \tag{99}$$

Inequality (99) states that the low-type's strategy of selling cannot exceed 1. We may calculate the low-type's strategy of selling from the ask price and H. If it exceeds 1, then this strategy is not feasible. The details for Regime 2 and 4 can be found in the Appendix.

In our computer simulation we look for equilibrium pairs of bid and ask prices that satisfy the conditions for Regime 1 through 4. If there are two regimes, we ask the computer to report them. Then, we repeat this procedure recursively. The following figures describe the simulation results. The x-axis shows market makers' prior belief $b \in [0, 1]$ and we have used a grid size of 100 in $[0, 1]$. The y-axes in Figures 1 to 4 show the value functions and figures 5 to 8 show the equilibrium prices.

Notice that we can use the Markov property of the model and this backward method because we know that there exists a unique equilibrium if the next-period value function satisfies **(C)** and **(M)** by Proposition 6. In each figure, each curve shows the equilibrium variable for each period. Since the equilibrium is unique, given the values at period $t + 1$, we can compute the values at period t. Those curves in the figures show them.

In the figures of the bid and ask prices, there is a region of beliefs in which bid or ask prices are different between periods. Given these beliefs, manipulation arises. Since the informed trader's strategy is different between the current period and the next period and so forth, bid or ask prices are also different. The results of the simulation also show that the high-type manipulates in a region of beliefs close to 0 and the low-type manipulates in a region of beliefs close to 1. This result is somewhat counter-intuitive, because for example if the high-type manipulates in a region of beliefs close to 0, bid price will be very low and he can only obtain a little amount of money. However, motivation for manipulation is to affect the future payoffs through market maker's belief updating. Therefore, they would manipulate when the bid and ask spread is small and the slope of the next-period value function is steep.

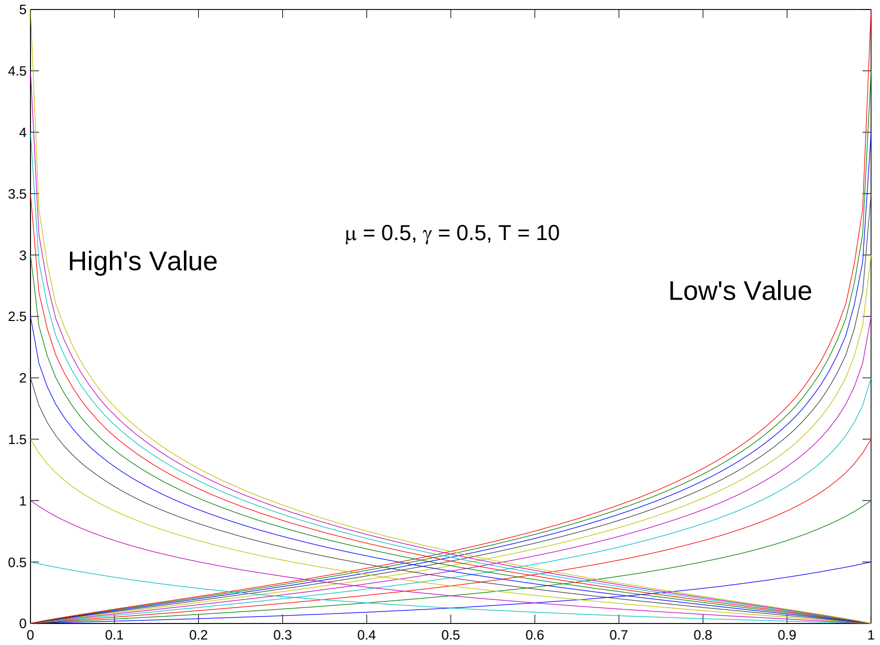

Figure 1: $\mu_0 = 0.5, \gamma = 0.5, T = 10$

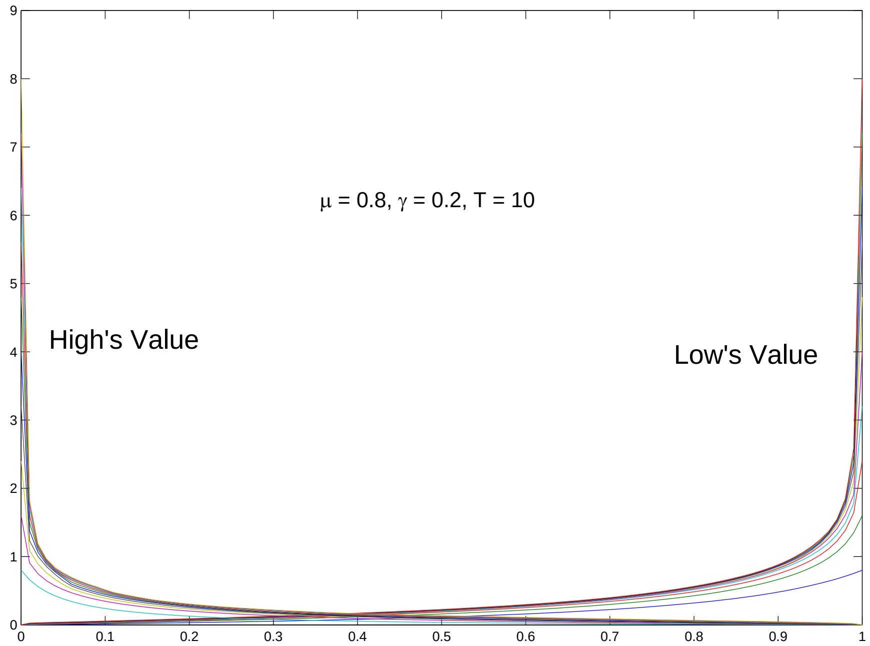

Figure 2: $\mu_0 = 0.8, \gamma = 0.2, T = 10$

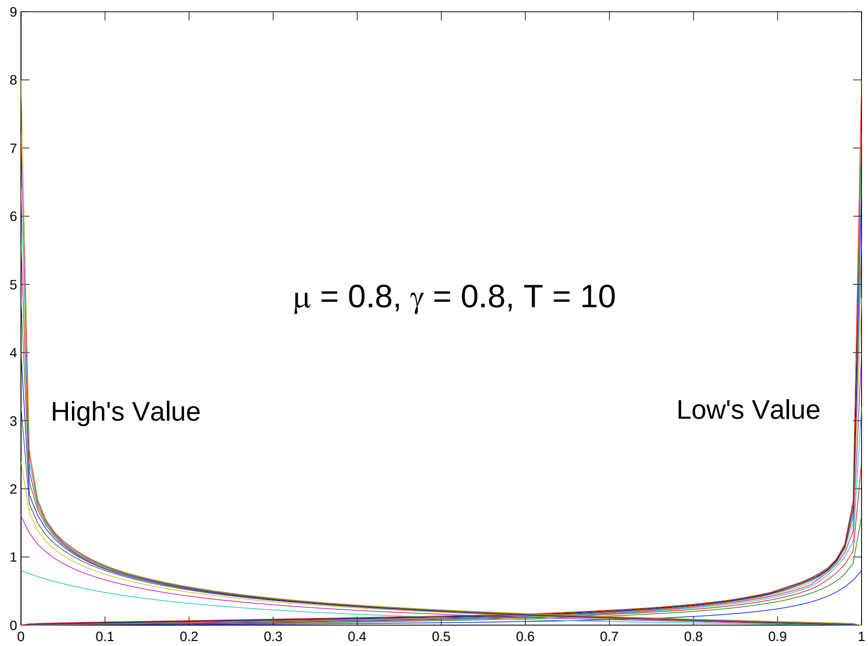

Figure 3: $\mu_0 = 0.8, \gamma = 0.8, T = 10$

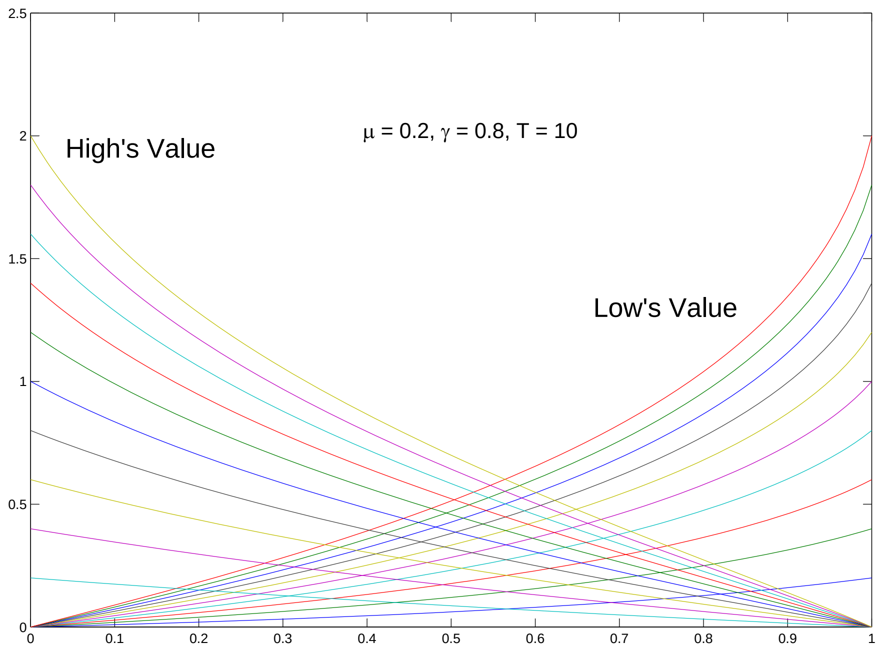

Figure 4: $\mu_0 = 0.2, \gamma = 0.8, T = 10$

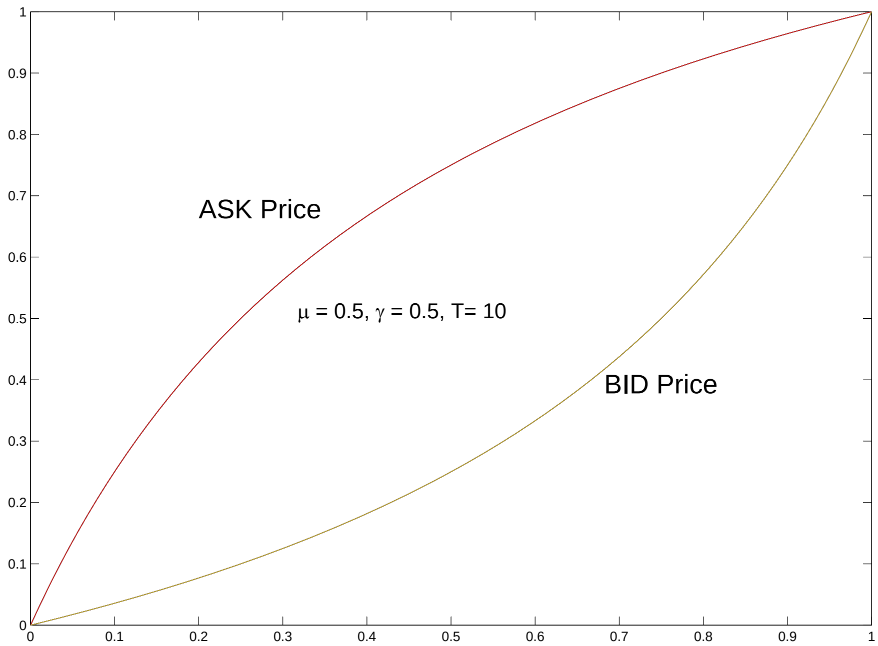

Figure 5: $\mu_0 = 0.5, \gamma = 0.5, T = 10$

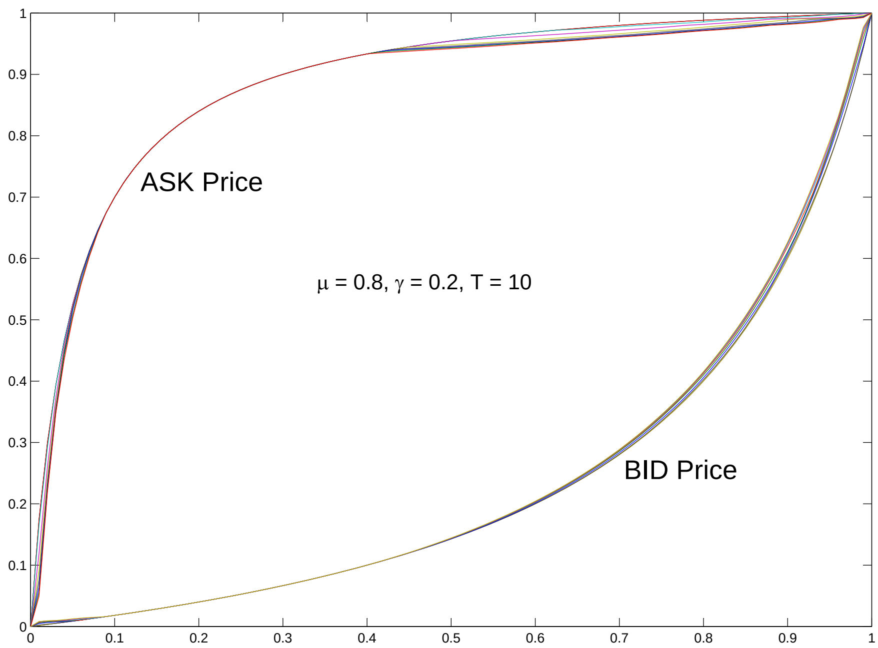

Figure 6: $\mu_0 = 0.8, \gamma = 0.2, T = 10$

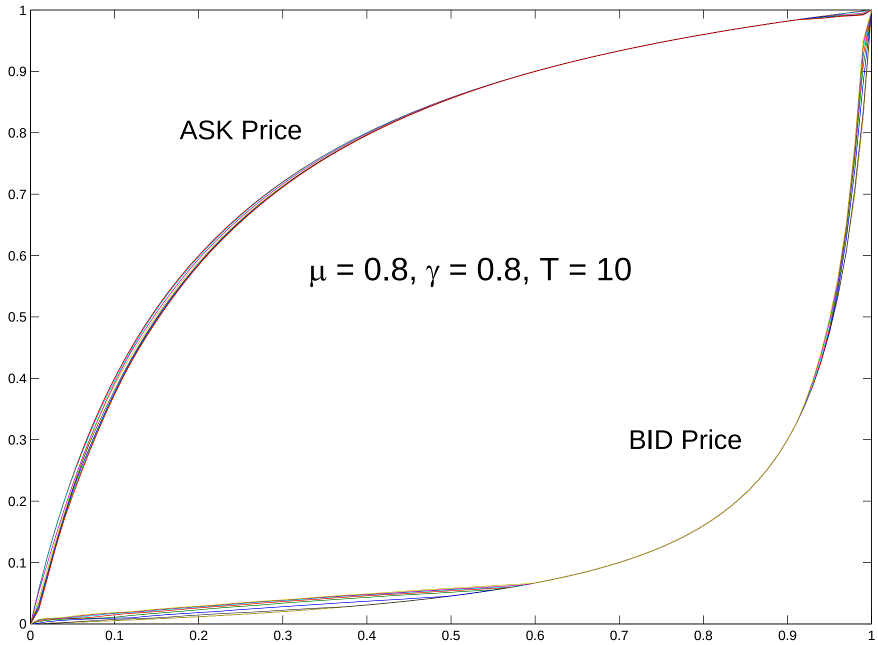

Figure 7: $\mu_0 = 0.8, \gamma = 0.8, T = 10$

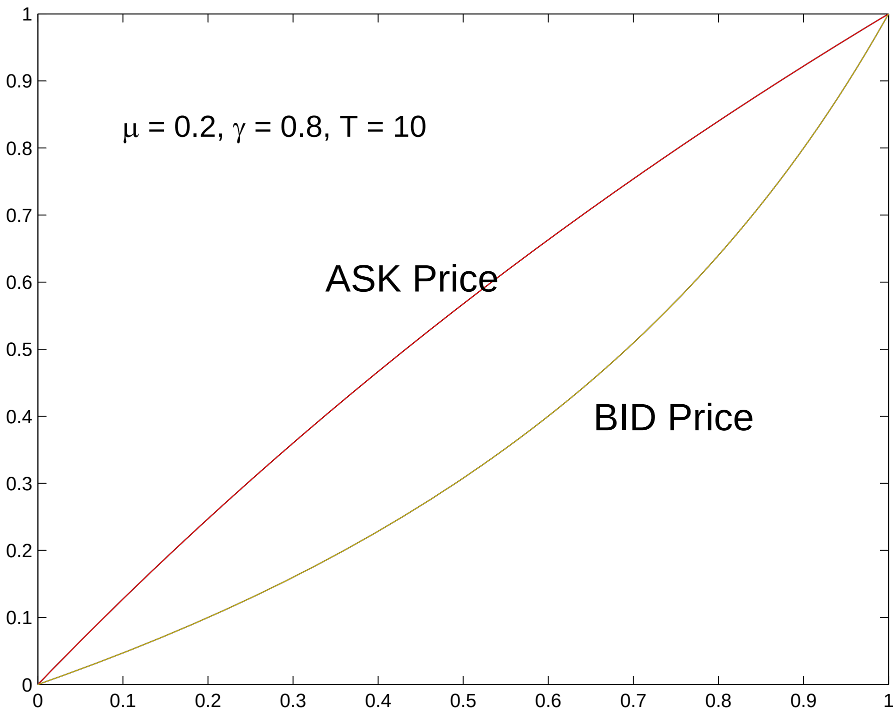

Figure 8: $\mu_0 = 0.2, \gamma = 0.8, T = 10$

Suppose that the information structure is as follows: $T = 4, \mu = 0.8, \gamma = 0.2$. Suppose that the ask price $\alpha \in [b_k, b_{k+1}]$ and the bid price $\beta \in [b_j, b_{j+1}]$. Then, by linear interpolation, we can approximate the sum of the payoffs from taking each action for each type as follows:

$$highbuy(b) = 1 - \alpha + (\alpha - b_k)\frac{V_H(b_{k+1}) - V_H(b_k)}{b_{k+1} - b_k} + V_H(b_k); \quad (100)$$

$$highsell(b) = \beta - 1 + (\beta - b_j)\frac{V_H(b_{j+1}) - V_H(b_j)}{b_{j+1} - b_j} + V_H(b_j); \quad (101)$$

$$lowbuy(b) = \alpha + (\alpha - b_k)\frac{V_L(b_{k+1}) - V_L(b_k)}{b_{k+1} - b_k} + V_L(b_k); \quad (102)$$

$$lowsell(b) = \beta + (\beta - b_j)\frac{V_L(b_{j+1}) - V_L(b_j)}{b_{j+1} - b_j} + V_L(b_j). \quad (103)$$

The table below shows some of the numerical results for $b = 0.01$, 0.02, and 0.03. The high-type manipulates in each case.

b	0.01	0.02	0.03
$\alpha(b)$	0.1654	0.2919	0.3931
$\beta(b)$	0.0023	0.0041	0.0052
σ_{HB}	0.9309	0.9599	0.9970
$highbuy(b) = highsell(b)$	1.1058	0.8708	0.7227
$lowbuy(b)$	-0.1143	-0.1983	-0.2590
$lowsell(b)$	0.0031	0.0055	0.0071

Consider the indifference condition for the high-type. Rearanging yields:

$$2 - \alpha(b) - \beta(b) = V_H(\beta(b)) - V_H(\alpha(b)). \quad (104)$$

By Lemma 3 we know that if manipulation arises in some region of a belief, then the value function V_H needs to be steep enough in that region. In other words, manipulation changes the market maker's posterior belief from $\alpha(b)$ to $\beta(b)$. Consequently, if manipulation arises, then there is a difference between $V_H(\alpha(b))$ and $V_H(\beta(b))$. However, if this difference is too large, then (104) does not hold. This effect can be seen in the results of our numerical simulation which show that manipulation arises in the region of beliefs where the value function is steep and the bid and ask spread is small.

From our numerical simulation we make the following conjectures. First, there is an interval of beliefs in which each type manipulates. Our first conjecture is that both types do not manipulate at the same time. Our second conjecture is that as the time period increases, manipulation occurs at wider ranges of the market maker's belief. In the previous example manipulation arises between $b = 0.1$ to 0.4 for $t = 5$, $b = 0.1$ to 0.5 for $t = 6$ and 7[1]. Our last, and most important conjecture, is that the value functions are strictly convex, the ask price is strictly concave and the bid price is strictly convex. If the

[1]We ran many other examples and this conjecture was observed in all cases.

third conjecture is proved, it would complete the proof of uniqueness result in a general case. To do so, proving the first conjecture would simplify the proof for a general case, because in this way we can focus on one type.

5 Remarks

Notice that in the last period the informed trader trades on their information because there is no chance to re-trade. This means that in the last period of the game the informed trader's unique equilibrium strategy is to trade honestly. By taking the first and second derivative of the last-period value function we can see that the value of the game in the last period is strictly monotonic and convex in the market maker's belief. The idea behind proving the uniqueness of equilibrium in the general case is to prove the uniqueness of the equilibrium strategy, supposing the existence of unique next-period value functions V_L and V_H, which are monotonic and convex in terms of the market maker's belief.

From the results of our numerical simulation we make several conjectures. First, in equilibrium both types do not manipulate at the same time. Second, the bid and ask price is strictly monotonic and the bid price is strictly convex. Third, the value functions are monotonic and strictly convex. These conjectures are important steps towards a proof of the uniqueness of equilibrium in the general case. One specific difficulty in proving uniqueness for the general case can be seen by considering the indifference condition for the low-type. If the low-type manipulates the following must hold:

$$-\alpha(b) + V_L(\alpha(b)) = \beta(b) + V_L(\beta(b)). \tag{105}$$

If the equilibrium is unique, then the equilibrium strategy should be continuous. Therefore, if the low-type manipulates at belief b then he should manipulate in an ϵ-neighborhood of b. This means that in an ϵ-neighborhood of b, a similar indifference condition must hold. The question comes down to the properties of V_L, α and β that make the two indifference conditions hold in an ϵ-neighborhood of b. Ultimately, the question becomes how the low-type changes his strategy in an ϵ-neighborhood of b. Similar questions arise if we consider the indifference condition for the high type. Regime 4 is the most difficult case because we have to consider how both types change their strategies in an ϵ-neighborhood of b. This is why proving our first conjecture that both types do not manipulate at the same time is a key step in proving uniqueness in the general case.

The last difficulty is to prove that the value functions are strictly convex in the market maker's prior belief. If V_L is monotone and strictly convex we can write the current-period value function as:

$$W_L(b) = \mu\left(\beta(b) + V_L(\beta(b))\right) + (1-\mu)\left(\gamma V_L(\alpha(b)) + (1-\gamma)V_L(\beta(b))\right). \tag{106}$$

Notice that if β and V_L are strictly convex and monotone, then $V_L(\beta(b))$ is strictly convex. The problem is the ask price. If α is strictly concave, then we cannot determine if $V_L(\alpha(b))$ is strictly concave. This prevents us from determining if W_L is strictly convex.

Proving the uniqueness of equilibrium in the general case is a challenging endeavor and this paper opens up the path to it. In this paper we have presented a model of dynamic informed trading, in which there exists a unique equilibrium under two conditions for the value functions. We have presented a computational method to solve for an equilibrium and have made several conjectures for proving the uniqueness of equilibrium in the general case.

Appendix: Proof of Theorem 1

In order to prove the existence of equilibrium, we consider the equilibrium strategies (σ_L^*, σ_H^*) to be a fixed point of the collection of their best response correspondences $BR = \{BR^t\}_{t=1\cdots,T}$ with $BR^t : [\Delta(A)]^2 \rightrightarrows [\Delta(A)]^2$ such that for each t, $(\sigma_L^*, \sigma_H^*) = BR^t(\sigma_L^*, \sigma_H^*)$. Let $U_n^t : \Delta(A) \times [0,1]^2 \to I\!R$ denote the payoff function for the type $n \in N$ trader in period t. More formally, for $n \in \{H, L\}$,

$$U_n^t(\sigma_n, p_t) = \sum_{t'=t}^{T} \left[\sigma_{nB}(\theta - \alpha_{t'}) - \sigma_{nS}(\theta - \beta_{t'})\right]. \tag{107}$$

Then, we define the informed trader's best response correspondence: for every $t \in \{1, \cdots, T\}$ and given p_t,

$$BR^t(\sigma_L, \sigma_H) = \left\{(\sigma_L, \sigma_H) \in [\Delta(A)]^2 | \sigma_n \in \arg\max_{\sigma \in \Delta(A)} U_n^t(\sigma, p_t) \;\; \forall n \in N\right\}. \tag{108}$$

Therefore, when $b(h_t) = b_t$, $\alpha_t^*(b(h_t)) = \alpha_t$ and $\beta_t^*(b(h_t)) = \beta_t$, continuation value of the game for the high-type in period t is:

$$V_H^t(b_t) = \max_{\sigma_H \in \Delta(A)} [\sigma_{HB}(1 - \alpha_t + V_H^{t+1}(b(h_t, B))) + \sigma_{HS}(\beta_t - 1 + V_H^{t+1}(b(h_t, S)))], \tag{109}$$

and one for the low type is:

$$V_L^t(b_t) = \max_{\sigma_L \in \Delta(A)} [-\sigma_{LB}\alpha_t + V_L^{t+1}(b(h_t, B)) + \sigma_{LS}(\beta_t + V_L^{t+1}(b(h_t, S)))]. \tag{110}$$

Thus, an equilibrium defined in Definition 2 is a fixed point of the best response correspondence BR, and α_t and β_t are respectively updated by Bayes rule.

Lemma 23. *The payoff function U_n^t is continuous. In addition, for every t, BR^t is a upper semi-continuous correspondence.*

Proof: Since the argument is symmetric, we only consider the high-type's payoff function and the value function. Note that U_H^t is continuous in his strategy and also the market maker's quotes (β_t, α_t). Then, U_H^t is a continuous numerical function.

We respectively denote the sequences of prices associated with σ^k and $\hat{\sigma}^k$ by p^k and $\hat{p}^k$ and also σ and $\hat{\sigma}$ by p and $\hat{p}$. Then, since the prices are continuous in strategies, we have $p^k \to p$ and $\hat{p}^k \to \hat{p}$.

Now on the contrary, suppose that there exists a sequence as above but $\hat{\sigma} \notin BR(\sigma_H, \sigma_L)$. Without loss of generality, we suppose that there exists a $\epsilon > 0$ and $\bar{\sigma}_H \in \Delta(E)$ such that:

$$U_H^t(\bar{\sigma}_H, p) > U_H^t(\hat{\sigma}_H, p) + 3\epsilon. \tag{111}$$

For k large enough, by continuity of the payoff function and prices, we have:

$$\begin{aligned} U_H^t(\bar{\sigma}_H, p^k) &> U_H^t(\bar{\sigma}_H, p) - \epsilon > U_H^t(\hat{\sigma}_H, p) + 2\epsilon \\ &> U_H^t(\hat{\sigma}_H^k, p) + \epsilon > U_H^t(\hat{\sigma}_H^k, p^k). \end{aligned} \tag{112}$$

This contradicts with the fact that $(\hat{\sigma}_H^k, \hat{\sigma}_L^k) \in BR^t(\sigma_H^k, \sigma_L^k)$ for all k. ■

Lemma 24. *The set* $[\Delta(A)]^2$ *is non-empty, compact and convex.*

Proof: The set of strategies $\Delta(A)$ is non-empty, compact and convex. The set $[\Delta(A)]^2$ is a Cartesian product of those sets and thus the result follows. ■

Lemma 25. *The informed trader's best response correspondence* BR^t *is non-empty and convex-valued for every* $t \in \{1, \cdots, T\}$.

Proof: We will prove this by mathematical induction. Since the argument is symmetric, we only consider the high type. Consider the last period $t = T$. Then, the high type and low type trade on their information. In this sense, BR^T is non-empty and convex-valued. Next we suppose that in period $t+1$, BR^{t+1} is non-empty and convex-valued. Then, we will prove that in period t, BR^t is also non-empty and convex-valued.

By the assumption for the inductive hypothesis, we know that V_H^{t+1} is well-defined. Now, fix a history h^{t-1} arbitrarily. Then, given V_H^{t+1}, the right hand side of the expression in (109) is linear in the strategies σ_H. Therefore the expression in (109) has a maximum so that the set BR^t is non-empty.

Second, we will prove that it is also convex-valued. Take two different strategies $(\bar{\sigma}_H, \bar{\sigma}_L) \in BR^t(\bar{\sigma}_H, \bar{\sigma}_L)$ and $(\bar{\bar{\sigma}}_H, \bar{\sigma}_L) \in BR^t(\bar{\sigma}_H, \bar{\sigma}_L)$. We denote the prices associated with the strategies $(\bar{\sigma}_H, \bar{\sigma}_L)$ by $\bar{p}_t$. Then, the following must hold: $U_H^t(\bar{\sigma}_H, \bar{p}_t) = U_H^t(\bar{\bar{\sigma}}_H, \bar{p}_t)$.

Let $\hat{\sigma}_H^t = \gamma\bar{\sigma}_H + (1-\gamma)\bar{\bar{\sigma}}_H$ for some $\gamma \in (0, 1)$. By using linearity of the payoff function, we conclude: $(\hat{\sigma}_H, \bar{\sigma}_L) \in BR^t(\bar{\sigma}_H, \bar{\sigma}_L)$. Therefore, BR^t is convex-valued. ■

Proof of Theorem 1: By Lemma 23 to Lemma 25, we can apply the Kakutani's fixed point theorem to the best response correspondence BR^t on $[\Delta(A)]^2$ for all $t \in \{1, \cdots, T\}$. ■

Appendix: Proofs

Let: $P(b) = f_H(b) \times b + f_L(b) \times (1-b)$.

Lemma 26. *Fix a history h^t arbitrarily and suppose that $b = b(h^t)$. Suppose that V_H is monotonically decreasing and strictly convex in the market maker's prior b and that V_L is monotonically increasing and strictly convex in b. Suppose that only one type manipulates at the belief b. Then, P is increasing at the belief b.*

Proof of Lemma 26:
Case 1: When the high-type manipulates at the belief b.
By the indifference condition, we know that if the high-type manipulates, then the slope between bid and ask prices should be:

$$d_H(b) = \frac{\alpha(b) + \beta(b) - 2}{\alpha(b) - \beta(b)}. \tag{113}$$

Thus, we obtain:

$$d'_H(b) = \frac{\alpha'(b)(1 - \beta(b)) - \beta'(b)(1 - \alpha(b))}{[\alpha(b) - \beta(b)]^2} \times 2. \tag{114}$$

Suppose that the high-type manipulates at the belief b. Then by the continuity of the equilibrium strategy, we must have $d'_H(b) > 0$ because the equilibrium strategy is unique. By the Bayes rule, we have:

$$\log(1 - \alpha(b)) = \log(1 - b) + \log(1 - \mu)\gamma - \log P(b), \tag{115}$$

and

$$\log(1 - \beta(b)) = \log(1 - b) + \log((1 - \mu)(1 - \gamma) + \mu) - \log(1 - P(b)). \tag{116}$$

By taking the first derivative of the above equations, we obtain:

$$\frac{-\alpha'(b)}{1 - \alpha(b)} = \frac{-1}{1 - b} - \frac{P'(b)}{P(b)}, \tag{117}$$

and

$$\frac{-\beta'(b)}{1 - \beta(b)} = \frac{-1}{1 - b} + \frac{P'(b)}{1 - P(b)}. \tag{118}$$

Notice that:

$$\alpha'(b)(1 - \beta(b)) - \beta'(b)(1 - \alpha(b)) \quad = \quad (1 - \alpha(b))(1 - \beta(b)) \times P'(b) \times \left(\frac{1}{P(b)} + \frac{1}{1 - P(b)} \right).$$

Therefore, we have:

$$d'_H(b) = \frac{(1 - \alpha(b))(1 - \beta(b)) \times P'(b) \times \left(\frac{1}{P(b)} + \frac{1}{1 - P(b)} \right)}{[\alpha(b) - \beta(b)]^2} \times 2. \tag{119}$$

and thus we conclude $P'(b) > 0$. □

Case 2: When the low-type manipulates at the belief b.
By the indifference condition, we know that if the low-type manipulates, then the slope between bid and ask prices should be:

$$d_L(b) = \frac{\alpha(b) + \beta(b)}{\alpha(b) - \beta(b)}. \tag{120}$$

Thus, we obtain:

$$d'_L(b) = \frac{\alpha(b)\beta'(b) - \alpha'(b)\beta(b)}{[\alpha(b) - \beta(b)]^2} \times 2. \tag{121}$$

Suppose that the low-type manipulates at the belief b. Then by the continuity of the equilibrium strategy, we must have $d'_L(b) > 0$ because the equilibrium strategy is unique. By the Bayes rule, we have:

$$\log \alpha(b) = \log b + \log((1-\mu)\gamma + \mu) - \log P(b), \tag{122}$$

and

$$\log \beta(b) = \log b + \log(1-\mu)(1-\gamma) - \log(1 - P(b)). \tag{123}$$

By taking the first derivative of the above equations, we obtain:

$$\frac{\alpha'(b)}{\alpha(b)} = \frac{1}{b} - \frac{P'(b)}{P(b)}, \tag{124}$$

and

$$\frac{\beta'(b)}{\beta(b)} = \frac{1}{b} + \frac{P'(b)}{1 - P(b)}. \tag{125}$$

Notice that:

$$\alpha(b)\beta'(b) - \alpha'(b)\beta(b) \quad = \quad \alpha(b)\beta(b) \times P'(b) \times \left(\frac{1}{1 - P(b)} + \frac{1}{P(b)}\right). \tag{126}$$

Thus, we can write:

$$d'_L(b) = \frac{\alpha(b)\beta(b) \times P'(b) \times \left(\frac{1}{1-P(b)} + \frac{1}{P(b)}\right)}{[\alpha(b) - \beta(b)]^2} \times 2. \tag{127}$$

Thus, we must have $P'(b) > 0$. ■

Proof of Lemma 21:

Regime 1: By the Bayes rule, we have:

$$b \times ((1-\mu)\gamma + \mu) = P(b) \times \alpha(b), \tag{128}$$

and

$$b(1-\mu)(1-\gamma) = (1 - P(b)) \times \beta(b). \tag{129}$$

By Lemma 26, for $k < 1$, we have:

$$\frac{P(b)}{P(kb)} = \frac{\alpha(kb)}{k\alpha(b)} > 1, \tag{130}$$

and

$$\frac{1 - P(b)}{1 - P(kb)} = \frac{\beta(kb)}{k\beta(b)} < 1. \tag{131}$$

Dividing both sides by b and arranging terms yields the desired results. □

Regime 2: The proof will be done in a similar fashion with the previous lemma. □ ■

Appendix: Numerical Simulation

Let $m_k^H = \frac{V_H(b_{k+1})-V_H(b_k)}{b_{k+1}-b_k}$ and $m_j^H = \frac{V_H(b_{j+1})-V_H(b_j)}{(b_{j+1}-b_j)}$. Moreover, let $(1-\mu)(1-\gamma)+\mu = L$.

Lemma 27. *When the high-type manipulates, then equilibrium ask price α_H solves:*

$$\begin{aligned}
&(m_k^H-1)\alpha_H^2 + \alpha_H\left(-b_k m_k^H + 1 + V_H(b_k) + (b+L(1-b))(1-m_k^H)\right)\\
&\quad +\alpha_H\left[L(1-b) + [b_j - 1 + L(1-b)]\, m_j^H - V_H(b_j)\right]\\
&\qquad -(-b_k m_k^H + 1 + V_H(b_k))(b+L(1-b))\\
&\qquad\quad +(-L(1-b) + [b(1-b_j) - b_j L(1-b)] m_j^H + V_H(b_j)\,(b+L(1-b)) = 0. \qquad (132)
\end{aligned}$$

Moreover, equilibrium ask price α_H must satisfy:

$$\tilde{H} \equiv \frac{\alpha(1-L)(1-b)}{b(1-\alpha)} \leq (1-\mu)\gamma + \mu. \qquad (133)$$

Proof of Lemma 27: To avoid lengthy calculations, we only explain the key steps. Then, the equation in question is:

$$1-\alpha+(\alpha-b_k)m_k^H + V_H(b_k) = \frac{(b-\alpha)+\alpha L(1-b)}{(b-\alpha)+L(1-b)} - 1 + \left(\frac{(b-\alpha)+\alpha L(1-b)}{(b-\alpha)+L(1-b)} - b_j\right)m_j^H + V_H(b_j).$$

It is very similarly done with Lemma 22. Finally, we obtain the desired result. ■

Let $m_k^D = \frac{D(b_{k+1})-D(b_k)}{b_{k+1}-b_k}$ and $m_j^D = \frac{D(b_{j+1})-D(b_j)}{b_{j+1}-b_j}$.

Lemma 28. *If both types manipulate, then the equilibrium ask price α_M and strategy σ_{HB}^* satisfies:*

$$\begin{aligned}
&m_k^D\alpha_M^2 + \alpha_M\left[-b_k m_k^D + D(b_k) - bH^* m_k^D + (b(H^*-1)+b_j)m_j^D - (D(b_j)+2)\right]\\
&+ \quad bH^*(D(b_j) + 2 - b_j m_j^D + b_k m_k^D - D(b_k)) = 0, \qquad (134)
\end{aligned}$$

and

$$H^* = (1-\mu)\gamma + \mu\sigma_{HB}^*. \qquad (135)$$

Proof of Lemma 28: For the simplicity of notation, let us define: $D = V_L - V_H$. If both types manipulate, this means that there is a pair of ask and bid prices which satisfies: $D(\alpha) = D(\beta) + 2$. By using the linear interpolating method, we check if there is a pair of α, b and H which satisfies the indifference conditions for both types:

$$(\alpha - b_k)m_k^D + D(b_k) = \left[\frac{\alpha b(H-1)}{(bH-\alpha)} - b_j\right] m_j^D + D(b_j) + 2. \qquad (136)$$

Then, we obtain the desired result. ■

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