# Gambling Reputation: Repeated Bargaining with Outside Options*

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**Abstract**

We study repeated bargaining between a privately informed long-run player and a sequence of short-run players in discrete time. In each period, a disagreement invokes outside options ruled by an imperfect yet unbiased third party. The payoff-relevant outside options also partially reveal private information held by the long-run player. The information externality of outside options significantly affects the players' bargaining postures in our repeated setting. Under natural assumptions, we characterize the interaction of outside options and reputation effects as a modified gambler's ruin process, and explicitly construct the generically unique Markov perfect equilibrium. The gambler's ruin technique enables a sharp characterization of the equilibrium payoff and behavior. Our novel analytical tools can be applied to other repeated interactions with informative outside options.

**Keywords**: bargaining, outside option, third party, reputation, gambler's ruin
**JEL codes**: C61, C78

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# 1 Introduction

Many real world negotiations take place in the shadow of a third party, such as an expert, an arbitrator or even a court. Consider, for example, a firm in disputes with its employees or customers regarding wage increase or damage compensation.[1] These disputes often involve repeated interaction between a single *privately informed* long-run player and a sequence of short-run players. The recent high-profile litigations surrounding Merck, a pharmaceutical firm, offer an interesting case in point. Merck refused to settle and contested every case in court. After losing the first case with a compensation verdict of $253 million in 2005, it continued to fight in court over the following two years. After winning most of them, the firm ended up settling further 27,000 cases out of court for $4.85 billion in total, an amount far smaller than experts predicted at the beginning.[2]

In these examples, the bargainers obtain random outside options ruled by a third party when they fail to reach an agreement. Moreover, the outside options represent not merely a disagreement point in a repeated setup; they partially reveal the informed party's private information. Our goal is to explore how this linkage between *outside options* and *incomplete information* determines bargaining strategies and outcomes, and to provide a useful analytical tool to study a variety of other related applications.

The bilateral bargaining literature has long recognized the fundamental roles of outside options and incomplete information in determination of bargaining strategies and outcomes. Various details of outside options have been introduced in recent literature to study interesting applications, from settlement bargaining (e.g. Spier (1992)) to final-offer arbitration and mediation (e.g. Yildiz (2007) and Goltsman, Hörner, Pavlov and Squintani (2009)). Bargaining theory with incomplete information explores uncertainties regarding the players' *payoffs* (Fudenberg, Levine and Tirole (1985), Gul, Sonnenschein and Wilson (1986) and Cho (1990), among many others) as well as their *behaviors* (Myerson (1991), Kambe (1999), Abreu and Gul (2000) and Abreu and Pearce (2007)).

The interaction between outside options and incomplete information has received little attention to date. In a notable exception, Compte and Jehiel (2002) show that introducing outside options to the Myerson-Abreu-Gul setup of single-sale bargaining with irrational behavioral types may cancel out the delay and inefficiency that such informational asym-

[1] The sheer existence of collective governance arrangements such as courts is a demonstration of the prominence of these applications.

[2] Source: New York Times, http://www.nytimes.com/2007/11/09/business/09merck.html

metry otherwise generates. In a recent paper, Atakan and Ekmekci (2009) consider search market as a way of endogenizing outside options.

In this paper, we highlight another interplay between outside options and incomplete information in a *repeated* bargaining model, partly motivated by the aforementioned real world applications: outside options provide informative signals as well as determinant of the players' immediate disagreement payoffs. In our model, a long-run player (e.g. firm) faces disputes with a sequence of short-run players (e.g. customers/employees). The long-run player has private information regarding his responsibility towards a transfer (e.g. damage compensation/wage increase) to each short-run player. He is either "*good*" (responsible) or "*bad*" (not responsible). In each period, a new short-run player enters the game and the two parties bargain. If they reach an agreement, a transfer is made from the long-run player to the short-run player who subsequently leaves the game. If they disagree, an imperfect yet unbiased third party is called upon to judge the long-run player's responsibility, at a cost to each party.

The uninformed players in our model therefore learn from two sources of information: any decisions made by an imperfect third party, or "hard" information, and the actions of the informed player, or "soft" information. The presence of third party signals alters the players' intertemporal incentives in a significant way, as our analysis will demonstrate.

In order to derive sharp insights into our objectives, we assume that the good, non-responsible type adopts an *insistent strategy*; that is, he accepts a demand if and only if it is less than or equal to a low constant cutoff. We develop a novel algorithm to construct a Markov perfect equilibrium, and show that this is the generically unique equilibrium satisfying a natural monotonicity requirement.

Remarkably, the equilibrium bargaining strategies and reputation dynamics resemble the gambler's ruin process. There exist two critical threshold levels of reputation, $0 < p^* < p^{**} < 1$, the players' posterior belief about the long-run player being good. When reputation falls into the interval $(p^*, p^{**})$, the long-run player will turn down any equilibrium demand made by the short-run players, receiving outside options ruled by the third party. The belief updating is therefore driven by third party signals alone, and the dynamics parallel those of a gambler who faces a sequence of bets with money (reputation) and two boundaries (i.e. one below $p^*$ and the other above $p^{**}$).

When reputation is above the upper threshold $p^{**}$, the long-run player accepts the short-run players' low equilibrium demand for sure. There is no learning, and the full benefits of reputation are reaped. When reputation is below the lower threshold $p^*$, the bad type builds reputation by randomizing between accepting and rejecting the demand and, hence, outside options are invoked only occasionally. Here, the negative reputational effect of a bad third party signal is reduced, and may even be overturned, by the long-run player's sheer act of rejection. It is worth noting that the bad type reveals himself only when he *voluntarily* gives up reputation building; a third party signal, due to its imperfectness, can never lead to full revelation.

These reputation dynamics provide one possible explanation of the bargaining postures adopted by Merck. The huge compensation verdict in the first case may not have damaged the firm's reputation so severely in view of its decision to fight, and winning many other cases would have eventually taken its reputation to a level high enough to induce favorable settlements.[3]

We apply the gambler's ruin technique to further derive the properties of bargaining strategies and reputation dynamics, as the discount factor $\delta$ or third party precision $q$ goes to 1. First, as the long-run player becomes extremely patient, the lower threshold $p^*$ converges to 0 while the upper threshold $p^{**}$ remains unchanged. Thus, the players adopt incompatible bargaining postures over a wider range of beliefs. Note that, in the equilibrium, the bad long-run player reveals himself only below $p^*$, but surprisingly, despite this lower threshold approaching 0, the probability of reputation building, i.e. the posterior reaching $p^{**}$, converges to a level strictly between 0 and 1. Second, as third party signals become perfectly informative, $p^*$ tends to 0 while $p^{**}$ increases, and the probability of reputation building converges to 0. The two limiting cases also differ crucially in another respect: the number of signals invoked is larger when the discount factor goes to 1 than when the precision of signals improves. Furthermore, letting $\tau(\delta)$ be the stopping time at which reputation hits either boundary, we derive from the gambler's ruin property that $\lim_{\delta \to 1} \mathbf{E}[\delta^{\tau(\delta)}] = 1$. This allows us to explicitly characterize the limit equilibrium payoffs.

We also investigate strategic behavior of the short-run players. In models with reputation dynamics (e.g. Benabou and Laroque (1992), Mailath and Samuelson (2001), Bar-Isaac (2003) and Mathis, McAndrews and Rochet (2009)) the uninformed players face competitive environments and, consequently, respond continuously to their beliefs.

[3]One possible way to test our predictions would be to analyze the performance of Merck's stock prices during the process of the firm's "gambling reputation."

The short-run players in our repeated bargaining model however behave in a more sophisticated pattern, and this feature is crucial for our equilibrium dynamics. Remarkably, the short-run players essentially make one of just two demands in equilibrium even though they are allowed to choose any distribution over the real line, a prediction supported by observations.[4] This property gives rise to distinct arguments for equilibrium characterization that involve a discontinuous value function.

The presence of informative outside options is a salient feature of many repeated interactions beyond the bargaining setup that we consider. It is not difficult to see that our analysis and tools can be readily adapted to capture other seemingly different situations. We elaborate on two examples below:

- Repeated Sales. A seller serves a sequence of identical buyers. The seller privately knows his unit production cost. Each buyer only consumes one unit of the product and his valuation is commonly known to be higher than the high cost. Each buyer makes an offer. The high cost seller insists on a high price while the low cost seller behaves strategically. A disagreement invokes a random and fair but imperfect third party arbitration which results in an informative signal about the seller's private cost.[5] Applying our analysis to this model, we will obtain gambler's ruin dynamics: transactions are conducted with direct involvement of third parties when the belief on high cost seller lies between two thresholds, while the low cost seller bets his reputation until it reaches one of the boundaries.

- Entry Deterrence. An incumbent faces a sequence of potential entrants over spatially separated markets. The incumbent has private information about technology or consumer brand loyalty, and this stochastically affects the parties' profits. Each entrant decides whether to enter and the incumbent decides whether to start a price war. The incumbent who possesses a superior technology or high consumer loyalty insists on fighting, while the incumbent with a regular technology or low consumer

[4]For instance, in her study of repeated shareholder litigations involving long-run underwriters, Alexander (1991) finds that, beyond very few exceptions, the estimated strength of the case does not matter for the settlement amount.

[5]Gambetta (1993) and Dixit (2009) report an intriguing example of the Sicilian Mafia's role as an arbitrator. Reputation in a repeated bargaining model without informative outside options has been previously considered by Schmidt (1993). The equilibrium dynamics in his setup drastically differ from ours.

loyalty behaves strategically. We can interpret entry as "disagreement" and the profits after entry as "informative outside options." Applying our analysis to this model, we will again derive gambler's ruin dynamics: entry is deterred only when the incumbent's reputation is high, and the incumbent will fight for sure when the reputation is between two thresholds, betting on the random payoffs to improve his reputation. It is very important to note the difference between this model and the standard chain-store model *á la* Kreps and Wilson (1982) and Milgrom and Roberts (1982). In the above model, the incumbent is not building a reputation for being tough *per se*. Such an incumbent will not scare the entrant away; rather, the incumbent is building a reputation of having a superior technology or high consumer loyalty, convincing the potential entrants that entry will not be profitable.

The rest of the paper is organized as follows. The next section describes a model of repeated bargaining with a third party. Section 3 then presents our main characterization results. In Section 4, we further discuss the properties of our equilibrium by introducing the gambler's ruin technique. Finally, we offer some concluding remarks in Section 5. All proofs are relegated to Appendix, and the formal statements and proofs of some results appear in the Supplementary Material for space reasons.

# 2 The Model

## 2.1 Description

We consider a discrete time model. Periods are indexed by $t = 1, 2, \ldots$. A single long-run player 1 faces an infinite sequence of short-run players 2 with a new player 2 entering in every period. Each player 2 brings a claim to player 1.

Player 1 privately knows his type $\theta \in \{G, B\}$, where $G$ stands for good and $B$ for bad. Type $B$ is responsible for each claim, while type $G$ is not. As discussed in Remark 1 below, this assumption is imposed primarily for simplicity. Nonetheless, it is valid in applications, such as the aforementioned Merck example, where the long-run player faces repeated disputes all related to some foregone act.

The stake involved in each period, denoted by $H > 0$, is fixed and commonly known. Each player 1-player 2 pair attempts to settle their dispute via voluntary bargaining, but should they fail to reach an agreement, an external third party is called upon to determine whether player 1 is responsible (or, equivalently, type $B$) or not responsible (or type $G$). Both players are committed to obey the third party's decision: if the decision is $B$, player 1 should pay $H$ to player 2 and, otherwise, player 1 should pay nothing.

The third party is imperfectly informative but unbiased: independently of the true type of player 1 and history of the game, he makes an error with probability $1-q$, where $q \in \left(\frac{1}{2}, 1\right)$ is common knowledge. Specifically, when player 1's type is $B$ (or $G$), the third party decision is $G$ (or $B$) with probability $1-q$. Third party decisions are therefore conditional i.i.d. signals with precision $q$. Note that the signals are correlated over time although their precision is constant. The third party could be drawn from a pool each period.[6]

The timing of the stage game in period $t$ is as follows. Player 2 makes a take-it-or-leave-it demand $s_t \in \mathbb{R}$, which player 1 can either accept or reject. If $s_t$ is accepted, then player 1 pays $s_t$ to player 2; if the demand is rejected, a third party is called upon to make a decision. At the end of a period, player 2 leaves the game forever.

Note that if player 1 is type $B$ his expected transfer to player 2 under third party resolution is equal to $qH$; if he is type $G$ the corresponding amount is $(1-q)H$. To focus on interesting cases, it is assumed throughout that $c_1 + c_2 < qH - (1-q)H = (2q-1)H$.[7] Also, we assume that $(1-q)H - c_2 > 0$, that is, each short-run player's expected payoff from third party resolution is always positive.

A third party signal is publicly observable and so are the details of an agreement. However, the value of a rejected demand is unobservable to later short-run players.[8] The prior belief that player 1 is $G$ is $p_1 \in (0, 1)$. Short-run players update their beliefs from this prior and the public history that they observe. Let $p_t \in [0, 1]$ denote player 2's belief that player 1 is good at the beginning of period $t$. This will be referred to as player 1's "reputation."

*Remark 1.* We have assumed that type $B$ ($G$) is always responsible (not responsible). The analysis remains the same by instead assuming the following structure. For each

[6]We further discuss the unbiasedness assumption in Section 5.

[7]If the third party is too costly, the players would never seek an external decision.

[8]The assumption of unobservable rejected demands allows us to avoid signaling issues. It is also consistent with reality. See, for instance, the shareholders-auditor bargaining documented by Alexander (1991) and Palmrose (1991), where the details of disagreement are private information while the terms of agreement are publicly available. As discussed in Section 5, our results are robust to the possibility of endogenous confidential agreement or open disagreement.

claim, type $G$ is responsible with probability $q' < \frac{1}{2}$ and type $B$ is responsible with a probability $1 - q'$. Player 1 knows his type, but not his responsibility due to some randomness. The third party makes a decision on player 1's responsibility with precision $q''$. It is readily verified that this is isomorphic to the model above with third party precision $q = q'q'' + (1 - q')(1 - q'')$.

*Remark 2.* Our model assumes that the stake of each claim is fixed and the same amount of transfer from player 1 to player 2 results from the third party decision. However, we can alternatively interpret the model as representing the following scenario. If the third party is called upon, he decides whether or not player 1 is responsible and, in the event of the decision being $B$, also identifies a random transfer whose expectation is commonly known to the players as $H$.

## 2.2 Insistent behavior of the good type

The motivation of our paper is to model and study the dynamics of bargaining postures. In particular, we seek to address whether and how the bad long-run player, the responsible type, could profitably build a reputation for not being responsible by strategically adjusting his bargaining postures. To analyze reputation effects in bargaining, Myerson (1991) and Abreu and Gul (2000) propose an incisive approach that introduces the possibility of a player who adopts a simple bargaining rule that never yields to a demand above some cutoff. We follow this approach in our repeated setup by modeling type $G$'s bargaining posture as an "insistent strategy."

In our model, under a third party decision, good player 1 would incur an expected payment equal to $\overline{C} \equiv (1-q)H + c_1$ while each player 2 can guarantee himself an expected payment $\underline{C} \equiv (1 - q)H - c_2$. Therefore, we assume that type $G$ accepts a demand if and only if it does not exceed a cutoff $C \in \left(\underline{C}, \overline{C}\right]$. As usual, acceptance of any demand strictly greater than $C$ reveals that player 1 is type $B$, both on and off the equilibrium path. Given the insistent strategy assumption, in what follows whenever we refer to player 1 we shall mean type $B$ unless otherwise stated.

*Remark 3.* We assume the cutoff $C$ in the above range in order to to focus on interesting cases, for the same reason that the cutoff in $r$-insistent behavior of Myerson (1991) is taken to be strictly higher than the player's outside value. It is indeed true *a priori* that the cutoff can be anything in the real line. However, modeling insistent bargaining postures with cutoffs outside the considered range will not deliver any insights. In particular, if $C \in [0, \overline{C}]$, the equilibrium dynamics are trivial as we show in the Supplementary Material.

*Remark 4.* As will become clear in the analysis below, the insistent strategy assumption with $C \in \left(\underline{C}, \overline{C}\right]$ can in fact be rationalized in our repeated setup. That is, the assumed strategy of the good type can arise as an equilibrium behavior.

## 2.3 Equilibrium notion

We focus on perfect Bayesian equilibria in Markov strategies in which, at the beginning of each period before a demand is made, any relevant past history is summarized by the short-run players' belief that it induces.[9] A Markov strategy of player 2 in any given period, $d$, maps his belief to a probability distribution over all possible demands. That is,

$$d : [0, 1] \to \triangle(\mathbb{R}).$$

For (bad) player 1, given the current belief and demand made by player 2, a Markov strategy, $r$, specifies a probability of rejection. That is,

$$r : [0, 1] \times \mathbb{R} \to [0, 1].$$

If $(r, d)$ is a Makrov strategy profile, we write player 1's *discounted average expected payment* at belief $p$ as $S(p)$ with discount factor $\delta \in (0, 1)$. That is, it is the discounted average expected *sum* of transfers to player 2 and, if any, third party costs incurred by player 1. Note that we have already surpressed the dependence of $S$ on the strategy profile and the discount factor. Player 2 maximizes his expected stage game payoff while player 1 *minimizes* his discounted average expected payment.

A strategy profile $(r, d)$, together with a system of beliefs, forms a Markov perfect equilibrium if the usual conditions are satisfied. We will invoke a natural restriction of beliefs: when the type is revealed, the game proceeds as if it has complete information. Note also that the Markov property implies that player 1, when his type is known by player 2, will accept a demand equal to the best that he could expect from a third party signal, that is, $qH + c_1$.

[9] Allowing for non-Markov behavior will generate a large number of equilibrium possibilities in our repeated setup, thereby making it difficult to draw a clear conclusion. We further discuss non-Markov equilibria in Section 5 below.

# 3 Equilibrium Characterization

## 3.1 Some preliminary results

We first present some useful properties of a Markov equilibrium. Our first Lemma is an important implication of the insistent strategy assumption on the good type.

**Lemma 1** ***(Player 1 plays a cutoff strategy)*** *Fix any $\delta$ and Markov equilibrium. Also, fix any belief $p$, and consider a demand $s > C$ (where $C$ is the good type's cutoff). The following is true on or off the equilibrium path:*

- *If player 1 accepts $s$ with a positive probability, then he must accept any $s' < s$ with probability 1.*
- *If player 1 rejects $s$ with a positive probability, then it must reject any $s' > s$ with probability 1.*

In equilibrium, the short-run player can make a demand which will be rejected for sure; let us refer to such a demand as a *losing* demand; a demand that is made and accepted with a positive probability in equilibrium will be referred to as a *serious* demand.[10] Next, we obtain an important property regarding player 2's demand in any equilibrium. If player 1 is patient enough, there are only two serious demands despite the fact that *a priori* anything in the real line is possible. Any other demands must be either off the equilibrium path, or offered and rejected for sure in equilibrium.

**Proposition 1** ***(Characterization of serious demands)*** *Fix any $\delta > \frac{c_1+c_2}{(2q-1)H+c_1+c_2}$. In any Markov equilibrium, before player 1 reveals his type, a serious demand is either $C$ or $qH - c_2$.*

Let us argue informally that the only acceptable equilibrium demand other than $C$ is $qH - c_2$.[11] Suppose to the contrary that a higher demand $s > qH - c_2$ is acceptable. Then, the acceptance must occur for sure. This is because, otherwise, player 2 could profitably

[10] The terminology is borrowed from Hörner and Vieille (2009).

[11] Player 2 clearly has no incentive to demand anything between $C$ and $qH - c_2$ since only the bad type would accept such a demand and the expected payoff under third party resolution conditional on player 1 being bad is $qH - c_2$.

deviate by demanding slightly less and the deviation would be met with sure acceptance (by Lemma 1). But then, the bad type has an incentive to reject $s$ and show that he is good, a contradiction.

## 3.2 Main characterization

### 3.2.1 Intuition

Despite the technical complexity, our equilibrium characterization is driven by a simple economic intuition. We first spell out this intuition to facilitate the reading of the paper.

If player 1's reputation is sufficiently high, player 2's expected payoff from resolving the dispute via the third party is low and, moreover, he has to pay a cost to obtain a signal. Thus, he would make a low demand $C$ that will be accepted by both types of player 1. Let us denote this belief threshold as $p^{**}$. If player 1's reputation is sufficiently low, player 2 expects to win a large amount under third party resolution and, therefore, he would make a large demand not acceptable to the good type. How should the bad type respond?

If the bad type accepts it, he reveals his type and consequently his future payments will be high. He cannot therefore accept it with probability 1; otherwise, the equilibrium posterior following rejection must be 1, and the bad type would mimic the good type by rejecting the demand. The bad type should also be reluctant to reject the large demand for sure. A third party signal is imperfect but nonetheless informative. Thus, it will hurt his reputation on average. Moreover, from a very low reputation, he needs many pieces of good luck (good third party signals) in order to reach a reputation high enough that player 2 begins to make low demands. This suggests that, when his reputation is very low, the bad type should play a mixed strategy: he rejects the high demand with an interior probability. Moreover, given Proposition 1 above, the demand must be $qH - c_2$.

However, when reputation is close to $p^{**}$, the bad type may still wish to fully mimic the good type, reject the high demand with probability 1 and count on the chance that the signal favors him. If he is lucky, his reputation will jump above $p^{**}$. Therefore, when reputation is below $p^{**}$, there must exist some threshold at which player 1's incentives change. We shall denote this reputation level as $p^*$. Remarkably, we show that player 1's randomization at any $p \in (0, p^*)$ is such that the posterior immediately after rejection, but before a third party signal, must be *constant* and indeed *equal to* $p^*$.

#### 3.2.2 Formal statement

**Theorem 1** *There exists $\bar{\delta} \in (0,1)$, independent of $C$, such that, for any $\delta > \bar{\delta}$, we have the following:*

1. *there exist two reputation thresholds, $p^*$ and $p^{**}$, $0 < p^* < p^{**} < 1$, such that the following is a Markov equilibrium outcome:*

    - *If $p \in (0, p^*)$, player 2 demands $qH - c_2$ for sure; player 1 rejects it with probability $r(p) = \frac{p}{p^*}\frac{1-p^*}{1-p} \leq 1$.*
    - *If $p \in [p^*, p^{**})$, player 2 makes a losing demand.*
    - *If $p = p^{**}$, player 2 demands $C$ with probability $x$ and makes a losing demand with probability $1 - x$ for some $x \in [0, 1)$; player 1 accepts $C$ for sure.*
    - *If $p \in (p^{**}, 1]$, player 2 demands $C$ for sure; player 1 accepts it for sure.*

2. *The above is the generically unique Markov equilibrium outcome with the property that $S(p)$ is non-increasing in reputation $p$.*[12]

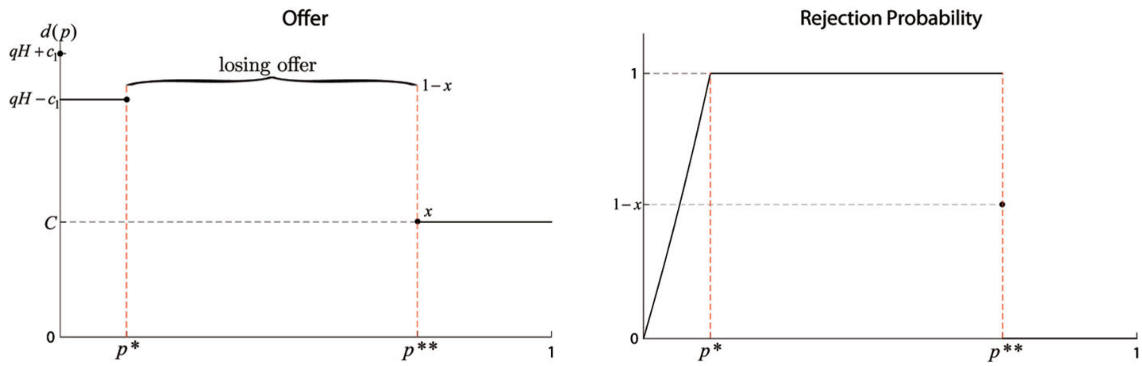
The monotonicity of $S(p)$ simply states that reputation is valuable. Benabou and Laroque (1992) and Mathis, McAndrews and Rochet (2009) also derive sharp equilibrium characterizations with the assumption of monotone value function in their repeated setups. In the single sale bargaining setup, a similar property is invoked by Fudenberg, Levine and Tirole (1987). It is difficult to prove that this intuitive property must hold in all equilibria. This is also the case in our setup. We are not able to show uniqueness without the aid of this monotonicity property.

Figure 1 below illustrates this equilibrium. The left panel describes player 2's demand at different reputation levels; the right panel illustrates player 1's corresponding rejection probability.

Figure 2 below illustrates player 1's expected equilibrium payment at different reputation levels. It is a decreasing step function with a finite number of jumps. When reputation is low ($p \in (0, p^*)$), $S(p) = qH + \delta c_1 - (1-\delta)c_2 \equiv \overline{S}$; when it is high ($p = (p^{**}, 1]$),

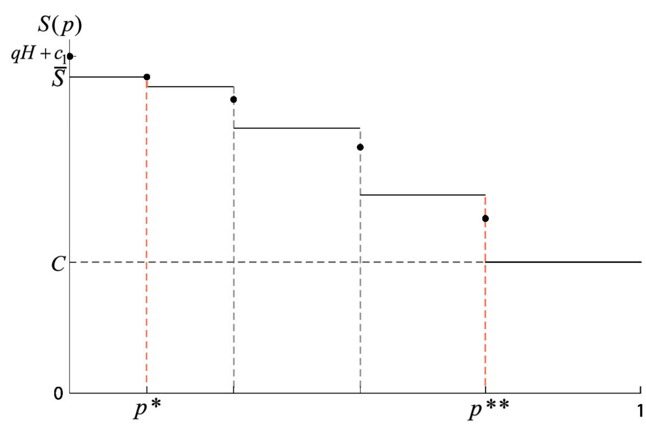
[12] Our uniqueness characterization is generic in the sense that, for a given $C$, it holds for all but at most a countable set of values of $\delta$ which are roots of a series of polynomials. In the Supplementary Material, we clarify the non-generic case and also characterize the equilibria therein.

Figure 1: Equilibrium strategies




$S(p) = C$. The reputation gain of player 1 is $S(0) - S(p_1)$, the difference between the expected payments when his type is known and when there is a prior belief $p_1$ that he could be good. We immediately derive that this amounts just to the total costs of a single third party signal for low initial reputation: if $p_1 \in (0, p^*)$, $S(0) - S(p_1) = (1-\delta)(c_1 + c_2)$.

Figure 2: Equilibrium payments



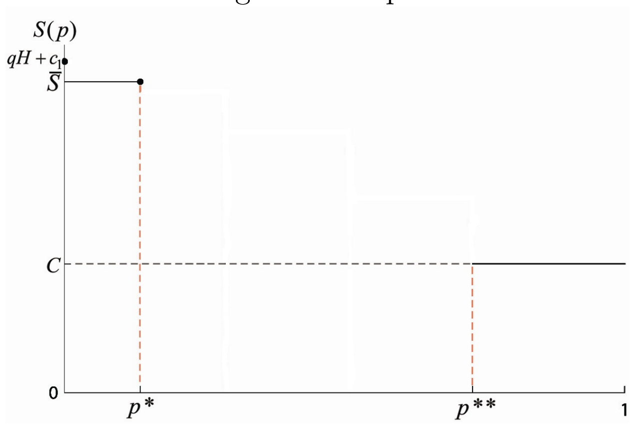
### 3.3 The algorithm of equilibrium construction

We now sketch the technique behind the equilibrium construction which we believe to be interesting and useful in itself. The rigorous arguments, as well as a proof for the uniqueness result, appear in Appendix.

**Step 1:** From the intuition above, we can deduce player 1's payments for the proposed equilibrium at $p \in (0, p^*)$ and $p \in (p^{**}, 1]$.[13] But, note that we do *not* yet know the values of $p^*$ and $p^{**}$. All we have is the payment levels in the two reputation regions. The intriguing aspect of our construction is that we identify $p^*$ and $p^{**}$ in the process of installing correct incentives through continuation payments.

If $p \in (0, p^*)$, player 2 demands $qH - c_2$ and player 1 is indifferent between rejecting and accepting it and, hence, his expected payment is given by accepting and revealing his type. If his type is revealed, the demand will be $qH + c_1$ in every period thereafter (and he is going to accept it). Thus, for $p \in (0, p^*)$, $S(p) = qH + \delta c_1 - (1-\delta)c_2 = \overline{S}$. Let us summarize these payments in the following illustration.

Figure 3: Step 1




**Step 2:** We have to construct continuation payments to support $\overline{S}$ as equilibrium payment and make player 1 indifferent at $p \in (0, p^*)$ and reject at $p^*$ with probability 1. What is the continuation payment from rejecting at $p^*$? In the current period, player 1 expects to spend $qH + c_1$ from the third party. As of the next period, the continuation payment depends on the signal. A bad signal (which happens with probability $q$) takes

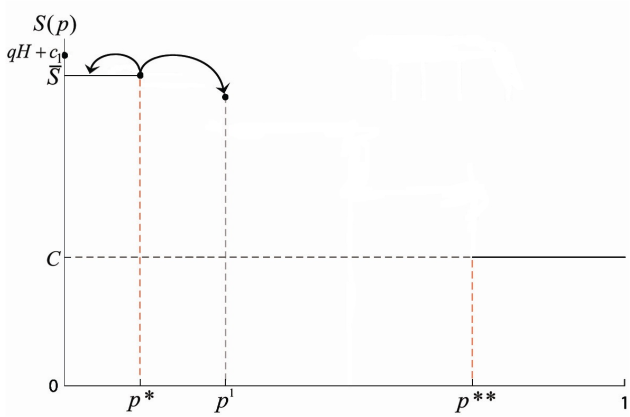
[13] The detailed proof for this statement is rather involved. See Proposition 5 in Appendix.

the belief below $p^*$ but then the continuation payment is given by $\overline{S}$. After a good signal the belief improves to, say, $p^1 > p^*$ with continuation payment $S(p^1)$.[14] Thus we obtain the following Bellman equation

$$S(p^*) = (1-\delta)(qH + c_1) + \delta q\overline{S} + \delta(1-q)S(p^1). \tag{1}$$

Since $S(p^*) = \overline{S}$, equation (1) pins down $S(p^1)$. It is easy to check that $S(p^1) \in \big(C, \overline{S}\big)$ when $\delta$ is sufficiently large.[15] The next figure illustrates these arguments.

Figure 4: Step 2




**Step 3:** Having obtained the value of $S(p^1)$, we now turn to the continuation payment that supports $S(p^1)$ in equilibrium. At $p^1$, the proposed equilibrium requires player 1 to reject player 2's demand for sure. The current period's expected payment is $qH + c_1$. At the next period, if the third party signal is good, the belief improves to some $p^2 > p^1$ with continuation payment $S(p^2)$; otherwise, the belief goes back to $p^*$ and the continuation payment is $S(p^*) = \overline{S}$ as explained above.

By similar arguments, we can derive $S(p^3)$ that supports $S(p^2)$ in equilibrium and so forth, and put together the following recursive equation to characterize $S(p^n)$ for any

[14] By Bayesian updating, $p^1 = \frac{p^* q}{p^* q + (1-p^*)(1-q)}$.

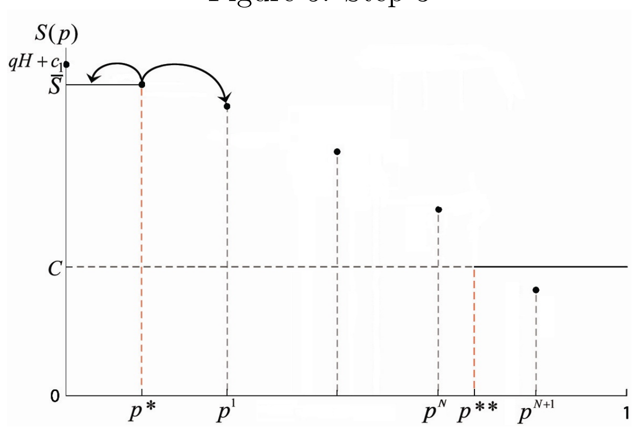
[15] For any $p \in (0, p^*)$, we let the rejection probability be such that right after the rejection, but before the signal, the posterior is exactly $p^*$; therefore, $S(p) = S(p^*)$.

integer $n$:[16]

$$S(p^n) = (1-\delta)(qH + c_1) + \delta q S(p^{n-1}) + \delta(1-q)S(p^{n+1}). \tag{2}$$

Starting from the two initial conditions $S(p^*)$ and $S(p^1)$, the solution to this difference equation can be shown to be strictly decreasing and divergent. Thus, eventually, $S(p^n)$ will drop below $C$, the lowest possible continuation payment in equilibrium. Let $N$ be the largest integer such that $S(p^N) > C$. Figure 5 below summarizes these arguments.

Figure 5: Step 3




**Step 4:** Note that $S(p^N)$ is needed in order to support $S(p^{N-1})$ as an equilibrium continuation payment, but we cannot use $S(p^{N+1})$ to support $S(p^N)$ if the former is less than $C$. Recall that the recursive arguments here are based on player 1 rejecting player 2's demand for sure. Therefore, the critical aspect of the equilibrium is that, at $p^N$, player 2 randomizes such that some demand is accepted while others rejected. The accepted demand must be $C$ from Proposition 1. The only belief level at which the short-run player could be indifferent between $C$ and a losing demand is such that $p^N(1-q)H + (1-p^N)qH - c_2 = C$, yielding

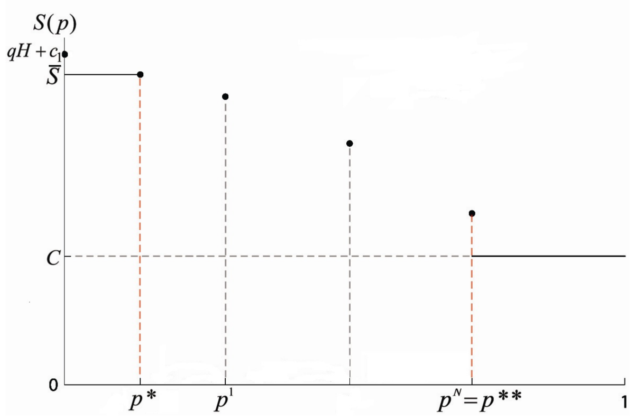
$$p^N = \frac{qH - c_2 - C}{(2q-1)H}.^{17} \tag{3}$$

[16]Note that the unbiased third party assumption, that the precision of third party decision, $q$, is symmetric across player 1 types, implies that the posterior updated from $p^n$ following a bad signal is exactly $p^{n-1}$.

[17]Since we assume that $(2q-1)H > c_1 + c_2$, $p^{**} \in (0, 1)$ is well-defined for any $C \in \left(\underline{C}, \overline{C}\right]$.

This belief is precisely the upper threshold $p^{**}$ that we seek such that $S(p) = C$ for $p > p^{**}$. From $p^N = p^{**}$, we then *backtrack* to find $p^*$; that is, *$N$ consecutive bad signals from $p^{**}$ gives $p^*$.* These arguments are illustrated in Figure 6 below.

Figure 6: Step 4




The exact mixing probability that supports $S(p^N)$, or $S(p^{**})$, is computed in Appendix. Since $p^*$ and $p^{**}$ are now identified, we can also trace the entire continuation payment schedule, which takes the form illustrated in Figure 2 above.

# 4 Gambler's Ruin and Limit Equilibrium Properties

We now introduce gambler's ruin technique to study several key limit properties of the equilibrium.

## 4.1 The probability of reputation building

We say that player 1 *builds his reputation* if the posterior belief moves above $p^{**}$. Our concern is the impacts of increased patience $\delta$ and third party precision $q$ on the reputation building process.

Fix a prior $p_1 \in (p^*, p^{**})$. In this reputation region player 1 always rejects player 2's demands and the movement of beliefs is driven only by conditional i.i.d. third party signals. We first compute the probability that, conditional on type $B$, the posterior $p_t$ exceeds $p^{**}$ before dropping below $p^*$ in equilibrium, denoted by $L(p_1)$.

The process of Bayesian updating can be transformed into a simple random walk.[18] Let $\lambda_t = \log\left(\frac{p_t}{1-p_t}\right)$ and $\lambda = \log\left(\frac{q}{1-q}\right)$. Then, if $p_t \in (p^*, p^{**})$, $\lambda_{t+1} = \lambda_t - \lambda$ upon a bad signal, which happens with probability $q$, and $\lambda_{t+1} = \lambda_t + \lambda$ otherwise. Write $\lambda^* = \log\left(\frac{p^*}{1-p^*}\right)$ and $\lambda^{**} = \log\left(\frac{p^{**}}{1-p^{**}}\right)$. The aforementioned probability $L(p_1)$ is the probability that, starting from $\lambda_1$, $\lambda_t$ exceeds $\lambda^{**}$ before falling below $\lambda^*$.

This is a ruin problem. Intuitively, we are considering a gambler with $\lambda_1 - \lambda^*$ in his pocket entering a sequence of bets, each with stake $\lambda$ and winning probability $1-q$. What are the odds that the gambler wins $\lambda^{**} - \lambda_1$ before going broke? Let $\lceil x \rceil$ denote the smallest integer larger than or equal to $x \in \mathbb{R}$. The following is immediate by taking into account the integer problem in the gambler's ruin result (e.g. Billingsley (1995)).

**Lemma 2** *Starting from a prior $p_1 \in (p^*, p^{**})$, the probability that, conditional on type $B$, the posterior $p_t$ exceeds $p^{**}$ before dropping below $p^*$ is*

$$L(p_1) = \frac{\left(\frac{q}{1-q}\right)^{\left\lceil \frac{\lambda_1 - \lambda^*}{\lambda} \right\rceil} - 1}{\left(\frac{q}{1-q}\right)^{\left\lceil \frac{\lambda^{**} - \lambda_1}{\lambda} \right\rceil + \left\lceil \frac{\lambda_1 - \lambda^*}{\lambda} \right\rceil} - 1}.$$

Note that, even after it falls below $p^*$, the posterior can still go up to $p^{**}$ with a positive probability in the equilibrium. Thus, $L(p_1)$ gives a lower bound on the probability of reputation building. By taking into account player 1's randomization in the region $(0, p^*)$, we can also derive an upper bound. It turns out that, as player 1 becomes increasingly patient, the two bounds collapse into the same limit.

**Proposition 2** *As $\delta$ goes to 1, we have the following:*

- *$p^*$ goes to 0; $p^{**}$ is independent of $\delta$.*
- *The probability of reputation building from a prior $p_1 \in (0, p^{**})$ converges to an interior level:*

$$R(p_1) = \left(\frac{q}{1-q}\right)^{-\left\lceil \log \frac{p^{**}}{1-p^{**}} \frac{1-p_1}{p_1} / \log \frac{q}{1-q} \right\rceil} \in (0,1).$$

[18] See Chamley (2004), for example.

The upper threshold, $p^{**}$, is derived from the short-run player's indifference and, hence, is independent of $\delta$. A patient long-run player puts a large weight on his future reputation and, therefore, as he becomes more patient, player 1 adopts tougher bargaining postures: $p^*$ goes to 0 and the region $(p^*, p^{**})$ expands.

Recall that, in this equilibrium, player 1 reveals his type only in the low reputation region below $p^*$ by accepting a high demand. Surprisingly, even when this lower threshold vanishes to 0 as $\delta$ goes to 1, the overall chance of reputation building in the equilibrium is strictly between 0 and 1. The intuition is similar to that of gambler's ruin. No matter how much the gambler starts with (i.e. $p_1$ is many "steps" away from $p^*$), he wins (i.e. the belief goes above $p^{**}$) with a probability strictly less than 1; likewise, even if the gambler starts with little, his chance of success is strictly bounded above 0.

The gambler's ruin formulation also allows us to consider the impact of increased precision of third party signals.

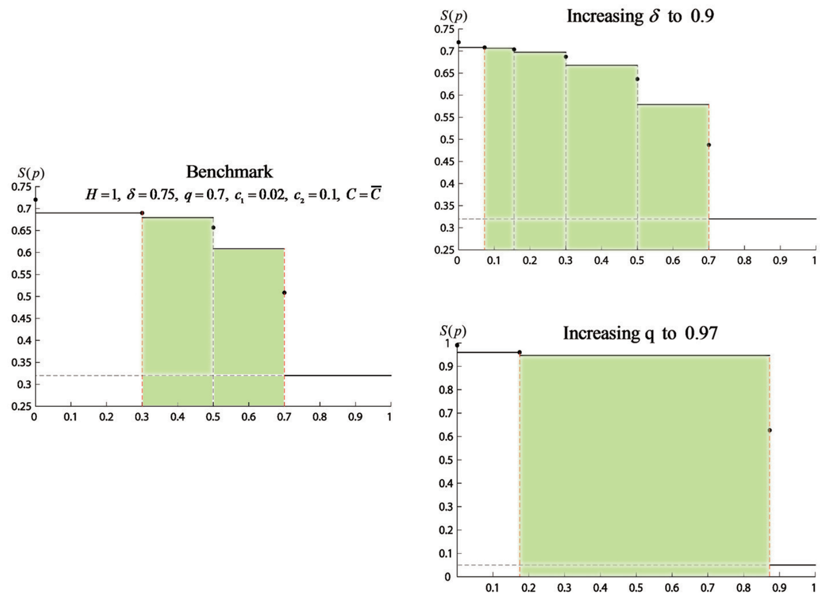
**Proposition 3** *As $q$ goes to* 1*, we have the following:*

- $p^*$ *goes to* 0*;* $p^{**}$ *goes to* $\bar{p} = \frac{H-c_2-C}{H}$.
- *The probability of reputation building from a prior* $p_1 \in (0, \bar{p})$ *converges to* 0.

Increased third party precision generates two opposing effects on the reputation building process. When the third party is very precise, on the one hand, an external signal is very likely to be bad (conditional on the bad type), but on the other hand, a good signal generates a large upward shift in belief. The short-run player is myopic and still makes a high demand at a low belief; this leads the bad type to reject his demand. Hence, we can show that $p^*$ goes to 0 as $q$ goes to 1. Furthermore, when $q$ is close to 1, it can be shown that, from any $p_1 \in (p^*, p^{**})$, $p^{**}$ is just a single jump away and, hence, the lower bound on reputation building probability is $L(p_1) = 1 - q$. We can then demonstrate that the probability indeed converges to 0 as $q$ goes to 1.

*Remark 5.* From Propositions 2 and 3, even though $p^* \to 0$ and the bargaining postures appear similar as $\delta$ or $q$ goes to 1, the dynamics are quite different. As $\delta$ goes to 1, the number of third party signals increases for a fixed level of precision, whereas as $q$ goes to 1 there is only one signal. This is illustrated by a numerical example in Figure 7 below, where the two graphs in the right panel are obtained by increasing $\delta$ or $q$ from a simulation result presented in the left panel.

Figure 7: Comparative statics




## 4.2 Payoffs

The recursive equations in Section 3.3 characterize the equilibrium payments, and the gambler's ruin technique provides a powerful tool to compute the payments in the limit as $\delta \to 1$. Recall that $R(p) = \left(\frac{q}{1-q}\right)^{-\left\lceil \log \frac{p^{**}(1-p)}{(1-p^{**})p} / \log \frac{q}{1-q} \right\rceil}$ is the probability of reputation building in the limit.

**Proposition 4** *For a generic $p \in (0, p^{**})$, player 1's limit equilibrium payment as $\delta \to 1$ is given by $R(p)C + (1 - R(p))(qH + c_1)$*[19]

Intuitively, the bad type expects to pay $qH + c_1$ per period if he fails to build a reputation (note also that the continuation payment upon revelation, $\overline{S} = qH + \delta c_1 - (1-\delta)c_2$, goes to $qH + c_1$ as $\delta \to 1$), which occurs with probability $1 - R(p)$ in the limit, while successful reputation building yields a low expected payment $C$.

[19] We say that a belief $p < p^{**}$ is generic if it cannot be reached from $p^{**}$ after any finite number of consecutive signals. The set of non-generic beliefs is therefore countable.

The result is however more complicated than the intuition suggests. A formal argument requires a rate comparison. For a fixed $\delta$, reputation will settle in finite time almost surely: player 1 will either reveal himself or build his reputation. But, $p^* \to 0$ as $\delta \to 1$ and hence the time it takes for reputation to converge could grow to infinity. We show that the convergence of $\delta$ dominates. Formally, we define a stopping time $\tau(\delta) = \inf\{t : p_t > p^{**} \text{ or } p_t < p^*\}$, and show that $\lim_{\delta \to 1} \mathbf{E}[\delta^{\tau(\delta)}] = 1$.

# 5 Concluding Discussion

Our model can be enriched in many interesting directions to address a variety of real world issues. We believe this to be another merit of the model. We conclude by discussing some potential extensions.

*Coalitional bargaining* In bargaining with multiple short-run players, it is sometimes possible for the players to form a coalition (e.g. class action). Che (1996, 2002) explores the cost sharing effect of a coalition. Extending our model could provide an alternative perspective on the coalitional bargaining problem. How will the long-run bargainer's reputation concern affect coalition formation? There is also a related timing issue regarding the short-run players' incentives on when to enter bargaining.

*Unbiasedness of a third party* In our model, the precision of third party signals is constant. This does not mean that third party decisions are independent. Indeed, they are correlated over time in our model because of the persistent uncertainty about player 1's type. The substance of this assumption is "unbiasedness." For example, in the US, unbiasedness is a primary concern of the jury selection process. One might also enrich our model by considering a biased third party whose decisions explicitly depend on past decisions. An important question would then be to model the third party's decision process when he holds Bayesian beliefs.[20] One possible way to endogenize a bias could be to assume that an otherwise unbiased third party has to exert a cost to make a decision and, therefore, follows the herd if his posterior belief is extreme.

*The (un)observability of demands* We have assumed that the details of bargaining are observable if and only if there is an agreement. (This assumption is consistent with applications; see footnote 8 above.) We can also extend the model by considering voluntary

[20] In one related paper, Che and Yi (1993) assume that a court's decision depends on the previous court's decision in a way that is specified exogenously, assuming away the court's dynamic decision process.

disclosure of an accepted demand and/or voluntary concealment of a rejected demand. Our equilibrium is robust under the following natural specification of beliefs upon observing a confidential agreement or open disagreement: player 2 assigns probability 1 to the bad type. This equilibrium survives refinements such as the intuitive criterion. This argument eliminates any benefit of confidentiality and suggests that other factors not captured in the current model are responsible for confidential agreements observed in real world. For example, a confidenial agreement may reduce the arrival of new disputes. On the other hand, allowing for observability of rejected demands brings a fresh signaling issue.

*Non-Markov equilibria* When player 1's type is known, our model admits a folk theorem: when $p = 0$, any payment in $[qH - c_2, qH + c_1]$ can be supported by a subgame perfect equilibrium with simple trigger strategies. Thus, by simply allowing for non-Markov behavior after the bad type reveals himself, our equilibrium construction can be extended to deliver a wide range of equilibrium payments. More generally, we can construct equilibria in which the players behave differently at the same belief. In particular, we can have an equilibrium where the sequence of observed signals matters. Recall that in the Markov equilibrium player 2 must randomize at the upper reputation threshold $p^{**}$. If we allow for non-Markov strategies, however, an equilibrium can be obtained without such randomization. The formal details of these non-Markov equilibria appear in the Supplementary Material.

*Multiple payoff/behavioral types* Our analysis considers the case in which the long-run player takes one of two possible payoff types. In many product liability cases, a sequence of bargaining takes place in order to settle disputes over a single act for which the defendant is either guity or not guilty. The binary type assumption also motivates the third party whose decisions are binary. Extending our analysis to the case of multiple payoff types and/or to incorporate richer third party signals, nonetheless, is an important direction for future research. Another extension will be to consider multiple insistent cutoffs for the good long-run player.

*Bargaining protocol* The model also considers a simple bargaining protocol within each period: the uninformed player makes a take-it-or-leave-it offer. Such simplicity allows us to focus on the long-run player's dynamic incentives. The one-sided offer by the uninformed player however rules out complex signaling effects. Spier (1992) considers settlement bargaining between a single pair of defendant and plaintiff under more complex bargaining procotols.

# 6 Appendix

## 6.1 Omitted proofs of Section 3.1

### Proof of Lemma 1

First, if $s$ is accepted, the continuation (discounted average expected) payment from accepting $s$ must be at least as good as that from rejecting it.

Since rejected demands are not observable, rejecting any demand results in the same continuation payment. Also, by the insistent strategy assumption on type $G$, accepting any demand strictly above $C$ leads to the same continuation payment at the next period (equal to $qH + c_1$). Then, accepting any $s' \in (C, s)$ must be strictly better than rejecting it since it yields a lower immediate payment.

On the other hand, accepting a demand $s' \leq C$ needs not lead to revelation but the continuation payment at the next period must still be bounded above by $qH + c_1$ and, hence, the same arguments imply that such a demand must also be accepted for sure.

Second, if $s$ is rejected, the continuation payment from rejecting $s$ must be at least as good as that from accepting it. Rejecting $s$ or $s'$ results in identical expected payments, both in the current period and each forthcoming period; on the other hand, while accepting $s'$ and $s$ yield the same continuation payment as of the next period, accepting $s' > s$ involves a strictly higher stage expected payment than accepting $s$. Thus, any $s' > s$ must be rejected for sure.

### Proof of Proposition 1

The proof is by contradiction. We consider the following cases.

*Case 1.* $s < C$ or $s > qH + c_1$.

Any demand $s < C$ is dominated by $C$ since type $G$ accepts $C$ and player 2's stage payoff from type $B$ is $qH - c_2 > (1 - q)H + c_1 \geq C$ should he reject $C$. Therefore, in equilibrium, player 2 will not demand $s < C$. This contradicts the assumption that $s$ is demanded in equilibrium.

If $B$ accepts a demand $s > qH+c_1$, he will reveal his type and the subsequent expected payment is $qH + c_1$ each period. If he rejects $s$, his current period expected payment is $qH + c_1$ while future expected payments are bounded above by $qH + c_1$. Therefore, $s > qH+c_1$, if demanded, will be rejected by $B$ for sure. This contradicts the assumption that $s$ is accepted.

*Case 2.* $s \in (C, qH - c_2)$.

But then, player 2 can profitably deviate by not demanding $s$ and, instead, demanding any $s' > qH + c_1$. Given the insistent strategy, $G$ rejects both $s$ and $s'$ for sure; from Case 1 above, we know that $B$ must also reject $s'$ for sure. But player 2 expects to earn $qH - c_2 > s$ from $B$ by seeking a third party and, therefore, would strictly prefer to have $s'$ rejected than to have $s$ accepted. This is a contradiction.

*Case 3.* $s \in (qH - c_2, qH + c_1]$ and type $B$ rejects $s$ with probability $r \in (0, 1)$.

But then, consider player 2 deviating by demanding $s - \varepsilon > qH - c_2$ instead of $s$ for some small $\varepsilon > 0$. By Lemma 1, such an demand must be accepted by $B$ for sure, while $G$ rejects $s-\varepsilon$. The deviation payoff then amounts to $p((1-q)H-c_2)+(1-p)(s-\varepsilon)$, while the corresponding equilibrium payoff is $p((1-q)H-c_2)+(1-r)(1-p)s+r(1-p)(qH-c_2)$. Thus, such a deviation is profitable if $\varepsilon < r(s - qH + c_2)$. This is a contradiction.

*Case 4.* $s \in (qH - c_2, qH + c_1]$ and type $B$ accepts $s$ with probability 1.

Let $r$ be $B$'s equilibrium strategy, and let $s^*(p) > qH - c_2$ denote the supremum of demands that it accepts with probability 1 at $p$, i.e. $s^*(p) = \sup\{s : r(p, s) = 0\}$. When there is no confusion, we shall also refer to it as $s^*$ to save on notation.

Then, by Lemma 1, $r(p, s') = 0$ for any $s' \in (qH - c_2, s^*)$ and $r(p, s'') = 1$ for any $s'' \in (s^*, \infty)$. Therefore, player 2's payoff from $B$ is $s'$ by demanding $s'$ and $qH - c_2 < s^*$ by demanding $s''$. However, both $s'$ and $s''$ are dominated by $s^* - \frac{s^*-s'}{2}$ which is accepted for sure, yielding a payoff of $s^* - \frac{s^*-s'}{2} > qH - c_2$. Therefore, given our arguments against Cases 1 and 2 above, player 2 will not make a demand other than $C$ or $s^*$ in equilibrium.

Suppose now that player 2 demands $s^*$ with a positive probability. We shall show that this is impossible.

On the one hand, if player 2's equilibrium strategy demands $s^*$ with a positive probability, $B$ must accept it with probability 1 by the same argument as in Case 3; otherwise, player 2 could profitably deviate by demanding $s^* - \varepsilon$ instead of $s^*$ for some small enough $\varepsilon > 0$.

On the other hand, $B$ has an incentive to deviate by rejecting $s^*$ if $\delta > \frac{c_1+c_2}{(2q-1)H+c_1+c_2}$.

As we have already established, in equilibrium, the demand can only be either $C$ or $s^*$ where $C \leq (1-q)H + c_1$. Here $C$ is accepted for sure by both types and $s^*$ is accepted for sure by $B$ while rejected for sure by $G$. It then follows that the equilibrium posterior at the next period after observing rejection in the current period must be 1 and $B$'s continuation payment is

$$(1-\delta)(qH + c_1) + \delta C. \tag{4}$$

But, in equilibrium, acceptance of $s^*$ results in revelation and, hence, the continuation payment

$$(1-\delta)s^* + \delta(qH + c_1). \tag{5}$$

Since $s^* > qH - c_2$ and $\delta > \frac{c_1+c_2}{(2q-1)H+c_1+c_2}$, (5) exceeds (4) and, therefore, the deviation is profitable, a contradiction.

## 6.2 Proof of Theorem 1

We shall first establish that, generically, every Markov equilibrium such that $S(p)$ is non-increasing in $p$ must result in the outcome stated in part 1 of Theorem 1. We then construct an equilibrium explicitly.

For any $p \in (0,1)$, let

$$\Phi^1(p) = \frac{pq}{pq + (1-p)(1-q)} \quad \text{and} \quad \Phi^{-1}(p) = \frac{p(1-q)}{p(1-q) + (1-p)q}.$$

That is, $\Phi^1(p)$ ($\Phi^{-1}(p)$) is the posterior after a good (bad) signal at $p$. It is readily verified that $\Phi^{-1}\left(\Phi^1(p)\right) = p$ for any $p$. Let $\Phi^0(p) = p$ and, for any integer $k > 1$, recursively define $\Phi^k(p) = \Phi^1\left(\Phi^{k-1}(p)\right)$ and $\Phi^{-k}(p) = \Phi^{-1}\left(\Phi^{-(k-1)}(p)\right)$.

We begin by deriving the following characterization of the equilibrium payments.

**Proposition 5** *There exists $\bar{\delta} \in (0,1)$, independent of $C$, such that, for any $\delta > \bar{\delta}$, any Markov equilibrium is such that, for some $p^* \in (0, p^{**})$, $p^{**} = \frac{qH - c_2 - C}{(2q-1)H}$, we have:*

- *For any $p \in (0, p^*)$, $S(p) = qH + \delta c_1 - (1-\delta)c_2 \equiv= S$.*
- *For any $p \in (p^{**}, 1)$, $S(p) = C$.*
- *For any $p \in [p^*, p^{**}]$, $S(p) \in \left[C, \overline{S}\right]$. Moreover, for any positive integer $k$ and any $p \in (\Phi^{-k}(p^{**}), \Phi^{-(k-1)}(p^{**}))$, $S(p) \geq \min\{(1-\delta^k)qH + \delta^k(1-q)H + c_1, \overline{S}\}$.*

**Proof of Proposition 5**

We proceed with the following lemmata.

**Lemma 3** *For any $p \in (0,1)$, $S(p) \in \left[C, \overline{S}\right]$.*

**Proof.** The lower bound, $C$, is immediate since any demand less than $C$ is strictly dominated for player 2 and thus will never occur in equilibrium. For the upper bound, let us consider two cases in turn.

*Case 1.* Every equilibrium demand of player 2 is accepted.

In this case, player 2 must play pure strategy since, given the assumption that each equilibrium demand is accepted, player 2 cannot randomize between a low demand and a high demand. Then, by Proposition 1, the equilibrium demand is either $C$ or $qH - c_2$. If the demand is C, no belief updating occurs and, therefore, $S(p) = C$; if the demand is $qH - c_2$, $B$ reveals himself and hence by the Markov property $S(p) = (1-\delta)(qH - c_2) + \delta(qH + c_1) = \overline{S}$.

*Case 2.* Some equilibrium demand is rejected with a positive probability.

Let $s_*$ be the infimum of the demands that $B$ rejects. By Lemma 1, all demands below $s_*$ will be accepted and all demands above $s_*$ will be rejected by this type.

Note that $B$'s equilibrium payment, $S(p)$, is bounded above by rejecting all demands. In particular, given the definition of $s_*$, the upper bound equals the continuation payment from rejecting an *equilibrium* demand $s_* + \epsilon$, for some $\epsilon \geq 0$.

But, at the same time, since $s_* + \epsilon$ occurs and is rejected in equilibrium, $B$'s equilibrium payment is also bounded above by the continuation payment from accepting it. Therefore, it must be that $S(p) \leq (1-\delta)(s_* + \epsilon) + \delta(qH + c_1)$, where $qH + c_1$ is the maximum possible continuation payment. By the definition of $s_*$, we can take $\epsilon \to 0$ and, hence, obtain

$$S(p) \leq (1-\delta)s_* + \delta(qH + c_1). \tag{6}$$

From (6), we are done if $s_* \leq qH - c_2$. We simply note that it is impossible that $s_* > qH - c_2$. The reasoning is as follows. Suppose not. By the definiton of $s_*$, there exists an equilibrium demand $s \geq s_*$ such that $s$ is rejected and player 2 obtains a payoff of $qH - c_2$. But, by the definition of $s_*$, any $s_* - \epsilon > qH - c_2$ will be accepted by $B$ which gives player 2 a payoff of $s_* - \epsilon > qH - c_2$. Therefore, $s$ cannot be demanded in equilibrium. This is a contradiction. ■

**Lemma 4** *For any $p \in (p^{**}, 1)$, $C$ is demanded and accepted for sure.*

**Proof.** By demanding $C$, player 2 obtains a payoff of at least $C$ since the good type accepts it and he can obtain $qH - c_2 > C$ if the bad type ever rejects the demand. Note that all lower demands are strictly dominated by $C$.

By demanding $qH - c_2$, player 2 obtains at most

$$p((1-q)H - c_2) + (1-p)(qH - c_2) \tag{7}$$

since $G$ will reject it, leading to expected payoff of $(1-q)H - c_2$ for player 2, and $qH - c_2$ is player 2's expected payoff from $B$ regardless of $B$'s response. Note that all demands in $((1-q)H + c_1, qH - c_2)$ are weakly dominated by $qH - c_2$, because $G$ rejects the demand and player 2's payoff is lower than $qH - c_2$ if $B$ ever accepts it.

Now, by Lemma 1 and Proposition 1, any demand greater than $qH - c_2$ is rejected by both types for sure, which gives player 2 a payoff of $p((1-q)H - c_2) + (1-p)(qH - c_2)$. Therefore, we only need to compare $C$ with (7). Since $p > p^{**}$, the former is larger, implying that $C$ must be demanded for sure.

Then, since player 2 plays a pure strategy here, and by $G$'s insistent behavior, accepting the equilibrium demand $C$ cannot reduce the posterior. Thus, acceptance yields a continuation payment $C$ to $B$. On the other hand, rejection yields, at best, a continuation payment $(1-\delta)(qH + c_1) + \delta C > C$, implying that $C$ is accepted for sure. ■

In order to pin down the last part of the claim, we first need the following lemma.

**Lemma 5** *Consider the state space $P \subset [0,1]$ such that $P = P_1 \cup P_2 \cup P_3$. Let $S(p)$ be the discounted average expected payment at $p$ (with discount factor $0 < \delta < 1$).*

*At any $p \in P_3$, with probability $1-q$ the immediate payment is $0$ and the new state becomes $p' = \Phi^1(p)$; with probability $q$, the payment is $H$ and the new state becomes $p'' = \Phi^{-1}(p)$, where $\Phi^1(\cdot)$ and $\Phi^{-1}(\cdot)$ are as defined in the main text above. If $p \in P_1$, $S(p) = S_1 > 0$; If $p \in P_2$, $S(p) = S_2 > 0$.*

*We then have the following: If $qH \geq \min\{S_1, S_2\}$, then $S(p) \geq \min\{S_1, S_2\}$ for any $p \in P_3$.*

**Proof.** Suppose not. Let $S_3 = \inf_{p \in P_3} S(p)$. Then, by assumption, $S_3 < \min\{S_1, S_2\}$. For any small $\varepsilon > 0$, there exists $p^\varepsilon \in P_3$ such that $S(p^\varepsilon) < S_3 + \varepsilon$. We know that

$$\begin{aligned} S(p^\varepsilon) &= (1-\delta)qH + \delta((1-q)S(p') + qS(p'')) \\ &\geq (1-\delta)qH + \delta \min\{S(p'), S(p'')\}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\min\{S(p'), S(p'')\} &\leq \delta^{-1}(S(p^{\varepsilon}) - (1-\delta)qH) \\
&\leq \delta^{-1}[S_3 + \varepsilon - (1-\delta)S_3 + (1-\delta)S_3 - (1-\delta)qH] \\
&< S_3 + \delta^{-1}[\varepsilon + (1-\delta)(S_3 - qH)].
\end{aligned}$$

Taking $\varepsilon$ to 0, we have $\min\{S(p'), S(p'')\} < S_3 + \delta^{-1}(1-\delta)(S_3 - qH)$. However, we know that, by assumption, $S_3 < \min\{S_1, S_2\} \leq qH$. It then follows that $\min\{S(p'), S(p'')\} < S_3$. This contradicts the definition of $S_3$. ■

We are now ready to complete the proof of Proposition 5 with the following lemma.

**Lemma 6** *Define* $\Phi^{-k}(\cdot)$ *as above.*

- *For any positive integer* $k$ *and any* $p \in \left(\Phi^{-k}(p^{**}), \Phi^{-(k-1)}(p^{**})\right)$*, we have*

$$S(p) \geq \min\left\{(1-\delta^k)(qH + c_1) + \delta^k C, \overline{S}\right\}.$$

- *There exists* $p^* \in (0, p^{**})$ *such that, for any* $p \in (0, p^*)$*,* $S(p) = \overline{S}$*.*

**Proof.** We proceed by establishing a series of steps.

Step 1: Fix any $p < p^{**}$, and suppose that player 2 demands $C$ in equilibrium. Then, type $B$ must reject this demand with a positive probability and, hence, the equilibrium posterior after rejection but before the third party signal does not exceed $p$.

*Proof of Step 1.* Suppose to the contrary that $B$ accepts the demand for sure. Player 2's payoff will be $C$. We shall argue that $(1-q)H + c_1$ is strictly dominated and cannot be an equilibrium demand.

Consider another demand $qH - c_2$. If player 1 is $G$ then he will reject it and player 2's payoff will be $(1-q)H - c_2$; if player 1 is $B$ then, whether or not he rejects it, player 2 will expects $qH - c_2$. Therefore, player 2's expected payoff is $p(1-q)H + (1-p)qH - c_2$. Since $p < p^{**}$, this amount is greater than $C$. That is, $qH - c_2$ dominates $(1-q)H + c_1$.

Since $C$ is rejected with a positive probability, all higher demands are rejected for sure by Lemma 1. It follows that in this case rejection reduces the posterior.

Step 2: Fix any $p < p^{**}$. One of the following holds:

(a) $S(p) = \overline{S}$; or

(b) player 1 weakly prefers to reject any equilibrium demand and the equilibrium posterior immediately after rejection (before the third party signal) does not exceed $p$.

*Proof of Step 2.* There are two cases to consider.

*Case 1*: $C$ is demanded with a positive probability in equilibrium.

Then, by Step 1, (b) holds.

*Case 2*: $C$ is not demanded in equilibrium.

In this case only $qH - c_2$ can be possibly accepted by Proposition 1.

(i) If $B$'s equilibrium strategy prescribes that $qH - c_2$ be rejected for sure then the belief will not change after rejection; hence, (b) holds.

(ii) If it prescribes that $qH-c_2$ be accepted with a positive probability then all demands greater than $C$ but less than $qH - c_2$ is accepted for sure, and they are dominated by $qH - c_2$ for player 2 (because only $B$ accepts these demands). There are two further possibilities here.

First, if $qH - c_2$ is not demanded in equilibrium then all equilibrium demands are rejected and, therefore, belief never changes; hence, (b) holds.

Second, if $qH - c_2$ is demanded in equilibrium with a positive probability then $B$'s continuation payment from rejecting any demand is higher than or equal to that from accepting $qH - c_2$. The latter amounts to $(1-\delta)(qH - c_2) + \delta(qH + c_1) = \overline{S}$. But, since $S(p) \leq \overline{S}$ by Lemma 3, it must be that $S(p) = \overline{S}$; hence, (a) holds.

At this point, to ease the exposition, let $p^{-k} = \Phi^{-k}(p^{**})$ for any positive integer $k$.

<u>Step 3</u>: For any positive integer $k$ and any $p \in [p^{-k}, p^{-k+1})$, we have

$$S(p) \geq \min\{(1-\delta^k)(qH + c_1) + \delta^k C, \overline{S}\}.$$

*Proof of Step 3.* We employ induction. First, consider any $p \in [p^{-1}, p^{**})$. By Step 2, we have either $S(p) = \overline{S}$, or an equilibrium demand is rejected and so $S(p)$ is given by the continuation payment from the rejection. In the latter case, clearly, $S(p) \geq (1-\delta)(qH + c_1) + \delta C$. Thus,

$$S(p) \geq \min\{(1-\delta)(qH + c_1) + \delta C, \overline{S}\}. \tag{8}$$

Next, for any $p \in [p^{-k}, p^{-k+1})$, let us assume that

$$S(p) \geq \min\{(1-\delta^k)(qH + c_1) + \delta^k C, \overline{S}\}$$

and show that, for any $p \in [p^{-k-1}, p^{-k})$,

$$S(p) \geq \min\{(1-\delta^{k+1})(qH + c_1) + \delta^{k+1} C, \overline{S}\}.$$

Again, given Step 2, consider the continuation payment when any equilibrium demand here is rejected such that the posterior immediately after rejection does not go above $p$.

Rejection results in the current period expected payment of $qH+c_1$. If the subsequent third party signal is good, the next period's posterior belongs to $[p^{-k}, p^{-k+1})$ and, hence, the corresponding continuation payment must be at least $\min\{(1-\delta^k)(qH+c_1)+\delta^k C, \overline{S}\}$, by assumption. If the third party signal is bad then the next period's posterior must belong to $[p^{-k-2}, p^{-k-1})$. By Lemma 5 (taking $P_3 = [p^{-k-2}, p^{-k-1})$, $P_1 = [p^{-k}, p^{-k+1})$, $P_2 = \{p : S(p) = \overline{S}\}\backslash(P_1 \cup P_3)$ ), the corresponding continuation payment must also be bounded below by $\min\{(1-\delta^k)(qH+c_1)+\delta^k C, \overline{S}\}$.

Thus, we have

$$\begin{aligned} S(p) &\geq \min\{(1-\delta)(qH+c_1)+\delta\left[(1-\delta^k)(qH+c_1)+\delta^k C\right], \overline{S}\} \\ &= \min\{(1-\delta^{k+1})(qH+c_1)+\delta^{k+1}C, \overline{S}\}, \end{aligned}$$

and, given (8), induction closes the proof of Step 3.

Now, let $K$ be the largest integer such that $\overline{S} \geq (1-\delta^K)(qH+c_1)+\delta^K C$. Then, Step 3 immediately implies that, for any $p \in [p^{-k-1}, p^{-k})$, $k \geq K$, we must have

$$S(p) \geq \min\{(1-\delta^{k+1})(qH+c_1)+\delta^{k+1}C, \overline{S}\} = \overline{S}.$$

Since by Lemma 3 we already know that $S(p) \leq \overline{S}$ for any $p \in (0,1)$, it follows that $S(p) = \overline{S}$ for any $p < p^K$. ■

### Refining serious demands and identifying $p^*$

Now, define $\bar{\delta}$ to satisfy $\overline{S} = (1-\delta)(qH+c_1)+\delta q\overline{S}+\delta(1-q)\overline{C}$, where, as before, $\overline{S} = qH+\delta c_1-(1-\delta)c_2$ and $\overline{C} = (1-q)H+c_1$. It is straightforward to see that $\bar{\delta} \in (0,1)$. Fix any $\delta > \bar{\delta}$ and Markov equilibrium such that $S(p)$ that is non-increasing in $p$.[21] Given Proposition 5 above, define $p^* = \sup\{p : S(p) = \overline{S}\}$. We next proceed with the following lemmata.

**Lemma 7** *Player 2 cannot demand $C$ at any $p < p^{**}$.*

[21] Note that, since $\overline{S} > \overline{C}$ for any $\delta$, at any $\delta > \bar{\delta}$ we have $\overline{S} > (1-\delta)(qH+c_1)+\delta\overline{C}$, implying that $\bar{\delta} > \frac{c_1+c_2}{(2q-1)H+c_1+c_2}$.

**Proof.** Fix any $p < p^{**}$. We begin by showing that $C$ cannot be accepted if demanded.

First, this demand cannot be accepted with probability 1 by $B$. To see this, suppose otherwise. But then, we reach a contradiction against the definition of $p^{**}$, where player 2 is indifferent between a losing demand and $C$ that is met with sure acceptance. At $p < p^{**}$, a losing demand is better than sure acceptance of $C$ and, hence, in this case $C$ cannot be demanded in equilibrium.

Then, suppose that $C$ is accepted with an interior probability. If $B$ rejects $C$ with a positive probability, by Lemma 1, he must reject every higher demand for sure. Also, rejected demands are not observable. Thus, his equilibrium continuation payment $S(p)$ is given by the payment from rejecting any demand. Therefore, given Proposition 5,

$$S(p) \geq (1-\delta)(qH + c_1) + \delta C > C. \tag{9}$$

But, since $B$ accepts $C$ with a positive probability, we also have $S(p) \leq (1-\delta)C + \delta S(p)$, where the second term of RHS is the highest possible continuation payment at the next period; acceptance of $C$ cannot reduce reputation so that continuation payment is at most $S(p)$ by $S(p)$ is non-increasing. Thus, $S(p) \leq C$ but this contradicts (9).

Now, suppose that $C$ is demanded with a positive probability. It follows from above that this demand must be met with sure rejection by $B$. On the other hand, $C$ must be accepted for sure by $G$. By Lemma 1, any demand higher than $C$ is rejected for sure by both types. This implies that $C$ strictly dominates all other demands for player 2 and, therefore, $C$ must be demanded with probability 1.

Then, rejection reveals that player 1 is bad and the corresponding equilibrium continuation payment for $B$ is $qH + c_1$. Acceptance of $C$, however, reveals that player 1 is good. Thus, by deviating to accept $C$, type $B$ can obtain continuation payment $C$, which is lower than the equilibrium payment, a contradiction. ■

**Lemma 8** *(i)* $p^* \leq \Phi^{-1}(p^{**})$.

*(ii) Only losing demands are made at any* $p \in (p^*, p^{**})$.

**Proof.** (i) Suppose to the contrary that $p^* \in (\Phi^{-1}(p^{**}), p^{**}]$ (we know from Proposition 5 that $S(p) = C$ for $p > p^{**}$ and note the definition of $p^*$). Then, consider $p^* - \varepsilon > \Phi^{-1}(p^{**})$ for some small $\varepsilon > 0$. By monotonicity of the value function, $S(p^* - \varepsilon) = \overline{S}$. By Proposition 1 and Lemma 7, the only possible serious demand here is $qH - c_2$. Therefore, rejection *per se* (before third party signal) cannot reduce player 1's reputation and, since

$p^* - \varepsilon \in (\Phi^{-1}(p^{**}), p^{**}]$, a subsequent good third party signal must take reputation above $p^{**}$. This implies the following equilibrium requirement

$$S(p^* - \varepsilon) = \overline{S} \leq (1-\delta)(qH + c_1) + \delta q \overline{S} + \delta(1-q)C,$$

where the RHS represents the worst continuation payment from rejecting a demand. However, since $\delta > \overline{\delta}$ the RHS is less than $\overline{S}$, a contradiction.

(ii) Fix any $p \in (p^*, p^{**})$. Given Proposition 1 and Lemma 7, we show that demand $qH - c_2$ cannot be accepted even if it is made. To see this, suppose otherwise. Then, conditional on the demand being made, $B$'s continuation payment from accepting it is equal to $\overline{S}$. But, by the definition of $p^*$, and since $S(p)$ is non-increasing, $S(p) < \overline{S}$. This implies that rejection must also occur in equilibrium and, since rejected demands are not observable, rejecting any demand must yield a continuation payoff strictly less than $\overline{S}$. This in turn implies that accepting $qH - c_2$ cannot be optimal. ■

At this juncture, consider the following recursive equation: for any integer $n$,

$$S_n = (1-\delta)(qH + c_1) + \delta q S_{n-1} + \delta(1-q) S_{n+1} \tag{10}$$

with the initial conditions $S_0 = \overline{S}$ and $S_1$ satisfying $S_0 = (1-\delta)(qH+c_1) + \delta q S_0 + \delta(1-q) S_1$. Simple algebra shows that, given $\delta > \bar{\delta}$, $S_1 > C$. Define $N = \sup\{n \in \mathbb{Z} : S_n > C\}$, where $\mathbb{Z}$ denotes the set of integers.

**Lemma 9** *(i) $S_n$ is strictly decreasing in $n$.*

*(ii) $N$ is finite.*

**Proof.** (i) Notice that $S_0 < qH$ and $S_0$ is a convex combination of $qH + c_1$ and $S_1$. Then $S_1 < S_0$. Suppose $S_n < S_{n-1} < \cdots < S_0 < qH$. From (10), $S_n$ is a convex combination of $qH + c_1$, $S_{n-1}$, and $S_{n+1}$, and hence $S_{n+1} < S_n$. The monotonicity of $S_n$ follows by induction.

(ii) Suppose to the contrary that $N$ is infinite. That is, $S_n > C$ for all $n$. Then, since $S_n$ is strictly decreasing, $S_n$ converges to $S_\infty$ such that $C \leq S_\infty < qH + c_1$. But, from (10), it follows that $S_\infty = qH + c_1$. This is a contradiction. ■

**Lemma 10** *Fix any $p \in (0, p^*)$.*

*(i) Rejection occurs in equilibrium.*

*(ii) Player 1's reputation immediately after rejection is $p^* = \Phi^{-N}(p^{**})$.*

**Proof.** (i) Suppose not. But then, since the accepted demand must be $qH - c_2$, rejection reveals the good type and the bad type's corresponding continuation payment is $(1 - \delta)(qH + c_1) + \delta C$, which is less than $S(p) = \overline{S}$ since $\delta > \bar{\delta}$. Thus, $B$ will want to deviate, a contradiction.

(ii) Fix any $p = p^* - \varepsilon$, $\varepsilon \in (0, p^*)$. We proceed in the following steps.

Step 1: Player 1's reputation immediately after rejection, say, $p^0$, is such that $p^0 \leq p^*$.

*Proof of Step 1.* Suppose not; so, $p^0 > p^*$. There are two cases to consider.

First, suppose that $p^0 \geq p^{**}$. Then, since $S(p) = C$ for any $p \in (p^{**}, 1)$, we have

$$\begin{aligned} S(p) = \overline{S} &= (1-\delta)(qH + c_1) + \delta(1-q)S\left(\Phi^1(p^0)\right) + \delta q S\left(\Phi^{-1}(p^0)\right) \\ &= (1-\delta)(qH + c_1) + \delta(1-q)C + \delta q S\left(\Phi^{-1}(p^0)\right). \end{aligned}$$

But, since $S\left(\Phi^{-1}(p^0)\right) \leq \overline{S}$ and $\delta > \bar{\delta}$, we have a contradiction.

Second, suppose that $p^0 \in (p^*, p^{**})$. By Lemma 8, every equilibrium demand is rejected for sure at $p^0$ and, hence, given the definition of $p^*$ and monotonicity of $S(p)$,

$$S(p^0) = (1-\delta)(qH + c_1) + \delta(1-q)S\left(\Phi^1(p^0)\right) + \delta q S\left(\Phi^{-1}(p^0)\right) < \overline{S}. \tag{11}$$

But, (11) contradicts that

$$S(p^* - \varepsilon) = (1-\delta)(qH + c_1) + \delta(1-q)S\left(\Phi^1(p^0)\right) + \delta q S\left(\Phi^{-1}(p^0)\right) = \overline{S}.$$

Step 2: $S(p^0) = \overline{S}$.

*Proof of Step 2.* Suppose not; so, $S(p^0) < \overline{S}$. Given the definition of $p^*$ and Step 1, it must then be that $p^0 = p^*$. We have

$$S(p^0) = (1-\delta)(qH + c_1) + \delta(1-q)S\left(\Phi^1(p^0)\right) + \delta q \overline{S} < \overline{S}, \tag{12}$$

while, for any $p \in (0, p^*)$,

$$S(p) = (1-\delta)(qH + c_1) + \delta(1-q)S\left(\Phi^1(p^0)\right) + \delta q \overline{S} = \overline{S}. \tag{13}$$

Comparing (13) with (12), we derive a contradiction.

Step 3: $p^0 = \Phi^{-N}(p^{**})$.

*Proof of Step 3.* For expositional ease, let $\Phi^n(p^0) = p^n$. First, we show that $p^1 > p^*$. To see this, from Steps 1-2, we have

$$S(p^0) = (1-\delta)(qH + c_1) + \delta(1-q)S\left(p^1\right) + \delta q \overline{S} = \overline{S},$$

which exactly pins down $S(p^1) < \overline{S}$. Also, since $\delta > \bar{\delta}$, $S(p^1) > C$. Thus, $p^1 \in (p^*, p^{**})$.

Then, given Lemma 8 and Step 2 above, we have

$$\begin{aligned} S(p^1) &= (1-\delta)(qH + c_1) + \delta(1-q)S\left(p^2\right) + \delta q S(p^0) \\ &= (1-\delta)(qH + c_1) + \delta(1-q)S\left(p^2\right) + \delta q \overline{S}, \end{aligned}$$

which pins down $S(p^2)$, and so forth.

But, by Proposition 5, we know that $S(p) \geq C$ for all $p$ and that $S(p) = C$ for all $p > p^{**}$. Also, from Proposition 5 and Lemma 7, we can deduce that $S(p) > C$ for all $p < p^{**}$. Thus, it must be that $S(p^n) = S_n$ only for positive integer $n \leq N$, where $S_n$ solves the recursive equation (10) and $N = \sup\{n \in \mathbb{Z} : S_n > C\}$ as defined above.

Now, suppose that $p^N < p^{**}$. On the one hand, by Lemma 8, rejection gives the equilibrium continuation payment at $p^N$ and, hence,

$$S(p^N) \geq (1-\delta)(qH + c_1) + \delta(1-q)S\left(p^{N-1}\right) + \delta q C. \tag{14}$$

On the other hand, from the recursive equation (10), we have

$$S(p^N) = S_N = (1-\delta)(qH + c_1) + \delta(1-q)S\left(p^{N-1}\right) + \delta q S_{N+1}, \tag{15}$$

where, generically, $S_{N+1} < C$.[22] Thus, (15) contradicts (14).

Step 4: $p^0 = p^*$.

*Proof of Step 4.* Suppose not; so, by Step 1 above, $p^0 < p^*$. We know from Step 3 that $p^0$ is fixed and equal to $\Phi^{-N}(p^{**})$. Moreover, Proposition 1 and Lemma 7 imply that rejection cannot reduce player 1's reputation at $p$. For small $\varepsilon$ such that $p^0 < p^* - \epsilon < p^*$, this contradicts the definition of $p^0$. ■

Thus, it follows from Lemma 10 that, at any $p \in (0, p^*)$, rejection by the bad type must occur with probability $r(p)$ such that $p^* = \frac{p}{p+(1-p)r(p)}$. This implies that $r(p) = \frac{p}{p^*}\frac{1-p^*}{1-p} \leq 1$ as in the claim.

**Lemma 11** *(i)* $S(p^*) = \overline{S}$.

*(ii) At* $p^*$*, rejection occurs with probability 1.*

[22]See the Supplementary Material for the non-generic case.

**Proof.** The first property that $S(p^*) = \overline{S}$ follows immediately from the proof of previous Lemma. To show the second part, suppose otherwise; so, acceptance occurs with a positive probability at $p^*$. We also know from previous Lemma that rejection must also occur at $p^*$. Thus, rejection must itself increase reputation, say, to $p^* + \varepsilon$ for some $\varepsilon > 0$.

It follows that

$$S(p^*) = (1-\delta)(qH + c_1) + \delta q S\left(\Phi^{-1}(p^* + \varepsilon)\right) + \delta(1-q) S\left(\Phi^{1}(p^* + \varepsilon)\right) = \overline{S}. \qquad (16)$$

Note here that, by monotonicity of $S(p)$, $S\left(\Phi^{-1}(p^* + \varepsilon)\right) \leq \overline{S}$ and, from Step 4 of the proof of previous Lemma, $S\left(\Phi^{1}(p^* + \varepsilon)\right) \leq S(p^1)$ since $\Phi^1(p^* + \varepsilon) > p^1$. We also know that

$$\overline{S} = (1-\delta)(qH + c_1) + \delta q \overline{S} + \delta(1-q) S(p^1). \qquad (17)$$

Equating (17) with (16) reveals that, since $S\left(\Phi^{-1}(p^* + \varepsilon)\right) \leq \overline{S}$ and $S\left(\Phi^{1}(p^* + \varepsilon)\right) \leq S(p^1)$, it must be that $S\left(\Phi^{-1}(p^* + \varepsilon)\right) = \overline{S}$ and $S\left(\Phi^{1}(p^* + \varepsilon)\right) = S(p^1)$.

Next, given Lemma 8, it follows from above that

$$\begin{aligned} S(p^* + \varepsilon) &= (1-\delta)(qH + c_1) + \delta q S\left(\Phi^{-1}(p^* + \varepsilon)\right) + \delta(1-q) S\left(\Phi^{1}(p^* + \varepsilon)\right) \\ &= (1-\delta)(qH + c_1) + \delta q \overline{S} + \delta(1-q) S(p^1) \\ &< \overline{S}, \end{aligned} \qquad (18)$$

where the last inequality arises from monotonicity of $S(p)$ and the definition of $p^*$. But, comparing (18) with (17) reveals a contradiction. ■

**Behavior at $p^{**}$**

We have already shown in Lemma 4 in the proof of Proposition 5 above that, for any $p \in (p^{**}, 1)$, $C$ is demanded and accepted for sure. Thus, it only remains to consider behavior at $p^{**}$.

**Lemma 12** *At $p^{**}$, we have the following:*

- *The only possible serious demand is $C$; type $B$ accepts it with probability 1.*
- *A losing demand is made with a positive probability.*
- *Generically, $C$ is demanded with a positive probability.*

**Proof.** Let us proceed in the following steps.

Step 1: Type $B$ accepts $C$ with probability 1.

*Proof of Step 3.* Suppose not; so, $C$ is rejected with a positive probability in equilibrium. Then, acceptance of $C$ must increase reputation and, hence, the corresponding continuation payment must be $C$, which is clearly smaller than the payment from rejection. This is a contradiction.

Step 2: $C$ is the only serious demand.

*Proof of Step 2.* Suppose not; so, given Proposition 1, $qH - c_2$ is demanded and accepted in equilibrium. But then, consider rejecting this demand. Given Step 1, rejection must increase reputation and, hence, the corresponding continuation payment is at most $(1-\delta)(qH + c_1) + \delta q S_{N-1} + \delta(1-q)C$. But this is clearly less than $\overline{S}$, the continuation payment from accepting $qH - c_2$. This is a contradiction.

Step 3: A losing demand is made.

*Proof of Step 3*: It follows from Step 1 that, if $C$ is the only equilibrium demand, $S(p^{**}) = C$. But, this contradicts that $S(p^{**}) = S_N > C$.

Step 4: Suppose that $S_{N+1} < C$ (where $S_{N+1}$ is derived from (10) as above). Then, $C$ is demanded with a positive probability.

*Proof of Step 4.* Suppose that $C$ is not demanded; so, by Step 2, every equilibrium demand is rejected for sure. But then, we obtain

$$\begin{aligned} S(p^{**}) = S_N &= (1-\delta)(qH + c_1) + \delta q S\left(\Phi^{-1}(p^{**})\right) + \delta(1-q) S\left(\Phi^{1}(p^{**})\right) \\ &= (1-\delta)(qH + c_1) + \delta q S_{N-1} + \delta(1-q)C, \end{aligned}$$

where $S_N$ is derived from (10) and the first equality comes from the proof of Lemma 10. But, this contradicts the fact that $S_N = (1-\delta)(qH + c_1) + \delta q S_{N-1} + \delta(1-q) S_{N+1}$ where $S_{N+1} < C$. ■

### Construction

Fix any $\delta > \bar{\delta}$, where $\bar{\delta}$ is as defined above. Let $p^{**} = \frac{qH - c_2 - C}{(2q-1)H}$ as before and $p^{*} = \Phi^{-N}(p^{**})$ as computed for Lemma 10 above. We establish that the following profile constitutes a Markov equilibrium.

First, player 2's strategy is such that:

- At $p = 0$, it demands $qH + c_1$ with probability 1;

- At any $p \in (0, p^{**})$, it demands $qH - c_2$ with probability 1;
- At $p = p^{**}$, it demands $C$ with probability $x$ and $qH - c_2$ with probability $1 - x$, where $x \in (0, 1)$ is defined later;
- At any $p \in (p^{**}, 1]$, it demands $C$ with probability 1.

Second, type $B$'s strategy is as follows:

- At $p = 0$, it accepts a demand $s$ if and only if $s \leq qH + c_1$;
- At any $p \in (0, p^*]$,

  - it rejects/accepts $s$ with probability 1 if $s > qH - c_2$/if $s < qH - c_2$;

  - it rejects $qH - c_2$ with probability $r(p) = \frac{p}{p^*}\frac{1-p^*}{1-p}$, as computed above.
- At any $p \in (p^*, p^{**}]$, it accepts $s$ if and only if $s \leq \max\{\xi(p), C\}$, where $\xi(p)$ is defined later.
- At any $p \in (p^{**}, 1]$, it accepts $s$ if and only if $s \leq C$.

Finally, the belief is updated by Bayes' rule and the equilibrium strategies whenever possible. We also assume that the posterior belief assigns probability 1 to type $B$ after an acceptance of a demand higher than $C$.

**Defining $x$:** At $p^{**}$, player 2 demands $C$ with probability $x$ and $qH - c_2$ with probability $1 - x$; both types of player 1 accept the first demand for sure and reject the second demand with for sure. This implies that the equilibrium posterior at the next period must be such that:

- if $C$ is accepted then the posterior remains at $p^{**}$;
- if rejection occurs, followed by a good signal, the posterior increases to $\Phi^1(p^{**})$; and
- if rejection occurs, followed by a bad signal, the posterior decreases to $\Phi^{-1}(p^{**})$.

Thus, from the characterization arguments above, we have

$$S(p^{**}) \equiv S_N = x\left[(1-\delta)C + \delta S_N\right] + (1-x)X, \tag{19}$$

where $S_N$ is given by (10) above and

$$X \equiv (1-\delta)(qH + c_1) + \delta q S_{N-1} + \delta(1-q)C. \tag{20}$$

**Lemma 13** *There exists a unique $x \in [0, 1)$ that satisfies (19).*

**Proof.** Simple computation yields

$$x = \frac{X - S_N}{X - (1-\delta)C - \delta S_N}.$$

Note first that $S_N \leq X$. This follows from comparing (20) above to the recursive equation

$$S_N = (1-\delta)(qH + c_1) + \delta q S_{N-1} + \delta(1-q)S_{N+1},$$

where, by assumption, $S_{N+1} \leq C$. Also, we have $S_N > (1-\delta)C + \delta S_N$ because, again by assumption, $S_N > C$. Thus, $x \in [0, 1)$. ■

**Equilibrium payments:** At this juncture, we characterize the equilibrium payments of type $B$. The following is clear:

- For $p \in (0, p^*]$, $S(p) = qH + \delta c_1 - (1-\delta)c_2 = \overline{S}$.
- For $p = \Phi^n(p^*)$ with any positive integer $n \leq N$, $S(p) = S_n$.
- For $p > p^{**}$, $S(p) = C$.

Thus, it remains to obtain equilibrium payments for other values of $p \in (p^*, p^{**})$ such that $p \neq \Phi^n(p^*)$ for positive integers $n \leq N$. To this end, consider the following recursive structure: for any integer $n$,

$$W_n = (1-\delta)(qH + c_1) + \delta q W_{n-1} + \delta(1-q)W_{n+1} \tag{21}$$

such that $W_0 = S_0 = \overline{S}$ and $W_{N+1} = C$. By similar arguments to those behind Lemma 9, $W_n$ is strictly decreasing.

**Lemma 14** *Fix any positive integer $n \leq N$ and any $p, p' \in (\Phi^{n-1}(p^*), \Phi^n(p^*))$. Then, $S(p) = S(p') = W_n$ as given by (21) above.*

**Proof.** Every equilibrium demand is rejected for sure at $p \in (p^*, p^{**})$. Then, note that

$$\Phi^{-n}(p) = \Phi^{-n}(p') < p^* \text{ and } \Phi^{-n+1}(p) = \Phi^{-n+1}(p') > p^*;$$
$$\Phi^{N-n+1}(p) = \Phi^{N-n+1}(p') > p^{**} \text{ and } \Phi^{N-n}(p) = \Phi^{N-n}(p') < p^{**}.$$

Thus, it is straightforward to see that $W_n = S(p) = S(p')$. ■

**Defining $\xi(p)$ for $p \in (p^*, p^{**}]$:** Recall that, in specifying player 1's equilibrium strategy earlier, we had deferred the definition of $\xi(p)$ at $p \in (p^*, p^{**}]$. Fix any $p \in (p^*, p^{**}]$, and define $\xi(p)$ as satisfying $(1-\delta)\xi(p)+\delta(qH+c_1) = S(p)$, where $S(p)$ is the equilibrium payment computed above. It is easily seen that $\xi(p) < qH - c_2$.

Let us now show that the above strategy profile and beliefs constitute a Markov equilibrium.

First, given the strategies of both types of player 1 and the definition of $p^{**}$, it is straightforward to establish optimality of player 2 strategy. In particular, note that it is never optimal for player 2 to make a demand $s \in (C, qH - c_2)$.

Second, we check optimality of type $B$'s behavior:

- It is straightforward to check its optimality at $p = 0$.

- Fix any $p \in (0, p^*]$. Suppose first that the demand, $s$, is less than $qH - c_2$. If type $B$ accepts this demand, the continuation payment amounts to $(1-\delta)s + \delta(qH + c_1) < \overline{S}$, while, since rejected demands are not observable, the continuation payment from rejecting continues to be $\overline{S}$. Thus, accepting any $s < qH - c_2$ for sure is optimal. A symmetric argument establishes that rejecting any $s > qH - c_2$ for sure is optimal.

- Fix any $p \in (p^*, p^{**})$. Here, we know that $S(p) < \overline{S}$, but accepting the demand $qH - c_2$ yields precisely $\overline{S} = (1-\delta)(qH - c_2) + \delta(qH + c_1)$ due to revelation. Thus, rejecting the equilibrium demand, $qH - c_2$, is optimal.

- Consider $p = p^{**}$. If $B$ accepts the equilibrium demand $qH - c_2$, he reveals his type and, hence, obtains a continuation payment $\overline{S}$. If he rejects this demand, he obtains $(1-\delta)(qH + c_1) + \delta q S_{N-1} + \delta(1-q)C \equiv X < \overline{S}$, where the last inequality is obtained from the proof of Lemma 13 above. Thus, it is optimal to reject $qH - c_2$.

Next, consider the demand $C$. Rejection, again, yields a continuation payment $X$, while acceptance leads to $(1-\delta)C + \delta S_N$. Since $S_N < X$ and $C < X$, acceptance is optimal.

- For $p \in (p^{**}, 1]$, see Lemma 4 above.

*Remark 6* We can see from this construction that the insistent strategy of type $G$ is in fact optimal given that at any $p < 1$ player 2 never makes a demand belonging to $[0, C) \cup (C, qH - c_2)$. Note also that, by the Markov property, at $p = 1$, $\overline{C}$ must be demanded and accepted if $G$ were a rational type. But, a careful inspection of previous arguments will show that all our arguments remain valid.

## 6.3 Omitted proofs of Section 4

### Proof of Proposition 2

1. We have already established that $p^{**}$ is independent of $\delta$ (see (3) above). By definition, $p^*$ is the posterior after $N$ consecutive bad signals from $p^{**}$. Therefore, to show $p^*$ goes to 0 as $\delta$ goes to 1, it suffices to establish that $N(\delta)$ goes to $+\infty$ as $\delta$ goes to 1.

We first note that $\lim_{\delta\to 1} S(p^n) = S(p^0) = qH + c_1 > C$ for any fixed $n$. This follows directly from the difference equation (10) and its initial conditions. Thus, as $\delta \to 1$, $N$ goes to $+\infty$ by definition.

2. We first establish the upper bound on the probability of reputation building.

**Lemma 15** *Starting from a prior $p_1 \in (p^*, p^{**})$, the probability of reputation building is at most*

$$U(p_1) = 1 - \frac{(1 - L(p_1))(2q - 1)}{(2q - 1) + (1 - q)L(\Phi^1(p^*))}.$$

**Proof.** Consider the lower threshold $p^*$. Let $\rho$ be the probability that type $B$ accepts the equilibrium demand at $\Phi^{-1}(p^*)$; this is

$$\begin{aligned}\rho &= 1 - r(\Phi^{-1}(p^*)) \\ &= 1 - \frac{\Phi^{-1}(p^*)}{1 - \Phi^{-1}(p^*)} \frac{1 - p^*}{p^*} = 1 - \frac{p^*(1 - q)}{(1 - p^*)q} \frac{1 - p^*}{p^*} = \frac{2q - 1}{q}.\end{aligned}$$

Define $\mathcal{R}(p)$ as the probability that, starting from $p$, type $B$ *reveals* his type. Then, we must have the following:

$$\mathcal{R}(p^*) \geq q(\rho + (1 - \rho)\mathcal{R}(p^*)) + (1 - q) \left[1 - L(\Phi^1(p^*))\right] \mathcal{R}(p^*) \tag{22}$$

The RHS is obtained from the following reasoning. From $p^*$, on the one hand, a bad signal (which occurs with probability $q$) takes the posterior down to $\Phi^{-1}(p^*)$, and the probability of immediate revelation there is $\rho$ while rejection takes the posterior back up to $p^*$ and the revelation probability from there on is $\mathcal{R}(p^*)$ by definition. On the other hand, a good signal takes the posterior up to $\Phi^1(p^*)$, from where the posterior will come back to $p^*$ before hitting $p^{**}$ with probability $1 - L(\Phi^1(p^*))$, where $L(\cdot)$ is derived from the gambler's ruin formula as in Lemma 2 above. Re-arranging (22) leads to

$$\mathcal{R}(p^*) \geq \frac{q\rho}{q\rho + (1 - q)L(\Phi^1(p^*))}.$$

Now, fix a prior $p_1 \in (p^*, p^{**})$ and consider $\mathcal{R}(p_1)$. From Lemma 2, the posterior falls below $p^*$ before going above $p^{**}$ with probability $1 - L(p_1)$ and, due to player 1's randomization, immediate revelation occurs with a positive probability once it has fallen below $p^*$. Therefore, we obtain the following lower bound on $\mathcal{R}(p_1)$:

$$\begin{aligned}\mathcal{R}(p_1) &\geq (1-L(p_1))\mathcal{R}(p^*)\\ &\geq \frac{(1-L(p_1))q\rho}{q\rho+(1-q)L(\Phi^1(p^*))}\\ &= \frac{(1-L(p_1))(2q-1)}{(2q-1)+(1-q)L(\Phi^1(p^*))}.\end{aligned}$$

The upper bound on the probability of reputation building immediately follows. ■

Fix a prior $p_1 \in (p^*, p^{**})$. We take the limits of $L(p_1)$ and $U(p_1)$ as in Lemmas 2-15. In the expression of $L(p_1)$, only $\lambda^*$ depends on $\delta$. Since $p^* \to 1$ as $\delta \to 1$, we have $\lambda^* \to -\infty$ as $\delta \to 1$. Therefore,

$$\lim_{\delta\to 1} L(p_1) = \lim_{\lambda^*\to-\infty} \frac{\left(\frac{q}{1-q}\right)^{\left\lceil\frac{\lambda_1-\lambda^*}{\lambda}\right\rceil}-1}{\left(\frac{q}{1-q}\right)^{\left\lceil\frac{\lambda^{**}-\lambda_1}{\lambda}\right\rceil+\left\lceil\frac{\lambda_1-\lambda^*}{\lambda}\right\rceil}-1}.$$

Since $\frac{q}{1-q} > 1$, applying l'Hôpital's rule, we have

$$\lim_{\delta\to 1} L(p_1) = \left(\frac{q}{1-q}\right)^{-\left\lceil\frac{\lambda^{**}-\lambda_1}{\lambda}\right\rceil} = \left(\frac{q}{1-q}\right)^{-\left\lceil\log\frac{p^{**}(1-p_1)}{(1-p^{**})p_1}/\log\frac{q}{1-q}\right\rceil}.$$

Also, note that, when $p_1 = \Phi^1(p^*)$, $\left\lceil\frac{\lambda_1-\lambda^*}{\lambda}\right\rceil = 1$ and $\left\lceil\frac{\lambda^{**}-\lambda_1}{\lambda}\right\rceil = \left\lceil\frac{\lambda^{**}-\lambda^*}{\lambda}\right\rceil - 1$, where the latter follows from the fact that $\Phi^N(p^*) = p^{**}$ with $N = \left\lceil\frac{\lambda^{**}-\lambda^*}{\lambda}\right\rceil$. Therefore,

$$L(\Phi^1(p^*)) = \frac{\left(\frac{q}{1-q}\right)-1}{\left(\frac{q}{1-q}\right)^{\left\lceil\frac{\lambda^{**}-\lambda^*}{\lambda}\right\rceil}-1}.$$

Since $\lambda^* \to -\infty$ as $\delta \to 1$, and $\frac{q}{1-q} > 1$, it is immediate that $\lim_{\delta\to 1} L(\Phi^1(p^*)) = 0$.

It therefore follows that

$$\begin{aligned}\lim_{\delta\to 1} U(p_1) &= \lim_{\delta\to 1}\left[1 - \frac{(1-L(p_1))(2q-1)}{(2q-1)+(1-q)L(\Phi^1(p^*))}\right]\\ &= \lim_{\delta\to 1} L(p_1) = \left(\frac{q}{1-q}\right)^{-\left\lceil\log\frac{p^{**}(1-p_1)}{(1-p^{**})p_1}/\log\frac{q}{1-q}\right\rceil}.\end{aligned}$$

### Proof of Proposition 3

1. The definition of $p^{**}$ immediately implies that $p^{**} \to \frac{H-c_2-C}{H}$ as $q \to 1$. Solving the recursive equation (10) in the main text, we obtain $\lim_{q\to 1} S_1 = H + \delta c_1 - (1-\delta)c_2 = \lim_{q\to 1} S_0$ and $\lim_{q\to 1} S_2 = -\infty$. Thus, there exists some $\bar{q} > 0$ such that $N = 1$ for any $q > \bar{q}$ and, hence, $p^* = \frac{p^{**}(1-q)}{p^{**}(1-q)+(1-p^{**})q}$. Given the limit of $p^{**}$, it follows $p^* \to 0$ as $q \to 1$.

2. If follows from part 1 above that, for any $p_1 \in (p^*, p^{**})$ and $q > \bar{q}$, $\lceil \frac{\lambda_1 - \lambda^*}{\lambda} \rceil = \lceil \frac{\lambda^{**}-\lambda_1}{\lambda} \rceil = 1$ and, hence,

$$L(p_1) = \frac{\left(\frac{q}{1-q}\right) - 1}{\left(\frac{q}{1-q}\right)^2 - 1} = 1 - q \quad \text{and} \quad L(\Phi^1(p^*)) = L(p^{**}) = 1.$$

Plugging $L(p_1)$ and $L(\Phi^1(p^*))$ into $U(p_1)$, we have, for $q > \bar{q}$,

$$U(p_1) = 1 - \frac{q(2q-1)}{(2q-1)+(1-q)} = 2(1-q).$$

Therefore, $\lim_{q\to 1} L(p_1) = \lim_{q\to 1} U(p_1) = 0$.

### Proof of Proposition 4

Recall that type $B$ does not build reputation with probability 1. For any $\delta$, type $B$ reveals himself with an interior probability when the belief drops below $p^*$ and attains a payoff $\overline{S} = qH + \delta c_1 - (1-\delta)c_2$. Furthermore, as $\delta \to 1$, $p^* \to 0$ (Proposition 2). This complicates the explicit computation of type $B$'s limit payment.

Fix $\delta \in (\bar{\delta}, 1)$ (where $\bar{\delta}$ is as in Theorem 1) and any prior $p = p_1 \in (p^*, p^{**})$ such that $p_1 \neq \Phi^{-n}(p^{**})$ for any integer $n$ . Define $\tau = \inf\{t : p_t > p^{**} \text{ or } p_t < p^*\}$ as the first time that, conditional on type $B$, the posterior either exceeds $p^{**}$ or falls below $p^*$. By convention, $\inf \emptyset = +\infty$. Since $p_1 \neq \Phi^{-n}(p^{**})$ we have $p_t \in (p^*, p^{**})$ for any $t < \tau$.

According to our equilibrium, the third party must be called upon before $\tau$. Type $B$'s (one-period) expected payment before $\tau$ is therefore $qH + c_1$. At or after period $\tau$, type $B$'s expected payment is either $C$ or $\overline{S}$, and the former event happens with probability $L(p)$ by Lemma 2. Therefore type $B$'s expected continuation payment at $p = p_1$ is

$$\mathbf{E}[(1-\delta^\tau)(qH + c_1)] + \mathbf{E}\left[\delta^\tau \left(L(p)C + (1-L(p))\overline{S}\right)\right].$$

It follows that, to evaluate type $B$'s limit payment as $\delta \to 1$, we need to evaluate $\lim_{\delta\to1} \mathbf{E}[\delta^\tau]$. This is not immediate since $\tau$ is a function of $\delta$ (through $p^*$ in our equilibrium). We shall show $\lim_{\delta\to1} \mathbf{E}[\delta^\tau] = 1$ in several steps.

Step 1: By convexity of $\delta^\tau$ and Jensen's inequality, $\mathbf{E}[\delta^\tau] \geq \delta^{\mathbf{E}[\tau]}$.

Step 2: As before, denote by $N(\delta)$ the number of steps from $p^{**}$ to $p^*$. From our gambler's ruin formulation, $N(\delta) = \left\lceil \frac{\log \frac{p^{**}}{1-p^{**}} - \log \frac{p^*}{1-p^*}}{\log \frac{q}{1-q}} \right\rceil$. Denote $a = \left\lceil \frac{\log \frac{p^{**}}{1-p^{**}} - \log \frac{p_1}{1-p_1}}{\log \frac{q}{1-q}} \right\rceil$. Note that $p^{**}$ is a constant for $\delta > \bar{\delta}$. Then, by the well-known formula (e.g. Grimmett and Welsh (1986)), the expectation of the stopping time is given by

$$\mathbf{E}[\tau] = \frac{1}{2q-1}\left( N(\delta)\frac{\left(\frac{1-q}{q}\right)^a - 1}{\left(\frac{1-q}{q}\right)^{N(\delta)} - 1} - a \right) \leq \frac{N(\delta)}{2q-1}.$$

Therefore, $\delta^{\mathbf{E}[\tau]} \geq \delta^{\frac{N(\delta)}{2q-1}}$. By Step 1, to show $\lim_{\delta\to1} \mathbf{E}[\delta^\tau] = 1$, it suffices to show $\lim_{\delta\to1} \delta^{N(\delta)} = 1$.

Step 3: In our equilibrium construction, $N(\delta)$ is obtained by a second-order difference equation, which we reproduce here,

$$S_n = (1-\delta)(qH + c_1) + \delta q S_{n-1} + \delta(1-q)S_{n+1} \tag{23}$$

with the initial condition $S_0 = \overline{S}$ and $S_0 = (1-\delta)(qH+c_1)+\delta q S_0 + \delta(1-q)S_1$. Note that $S_n$ is decreasing and divergent, and $N(\delta)$ is defined as the largest $n$ such that $S_n < C$.

We now approximate $N(\delta)$ by considering the following first-order difference equation:

$$\widehat{S}_n = (1-\delta)(qH + c_1) + \delta q(qH + c_1) + \delta(1-q)\widehat{S}_{n+1} \tag{24}$$

with initial condition $\widehat{S}_0 = \overline{S}$. By the same argument as before, $\widehat{S}_n$ is divergent. Define $\widehat{N}(\delta)$ as the largest integer $n$ such that $\widehat{S}_n < C$. Note that (24) is obtained by replacing $S_{n-1}$ with $qH + c_1$ in (23). Since $S_{n-1} < qH + c_1$ for any $n = 1, \ldots$ in (23), $S_n \leq \widehat{S}_n$ for any $n = 0, 1, \ldots$. It follows immediately that $\widehat{N}(\delta) \geq N(\delta)$ (i.e. the sequence $\widehat{S}_n$ falls below $C$ at a slower pace than $S_n$). Therefore, to show $\lim_{\delta\to1} \delta^{N(\delta)} = 1$, it suffices to show $\lim_{\delta\to1} \delta^{\widehat{N}(\delta)} = 1$.

Step 4: The solution to the first order difference equation (24) is given by

$$\widehat{S}_n = \frac{b(1-\rho^n)}{1-\rho} + \rho^n \widehat{S}_0,$$

where $\rho = \frac{1}{\delta(1-q)}$, $b = -\frac{1-\delta(1-q)}{\delta(1-q)}(qH + c_1)$ and $\widehat{S}_0 = \bar{S} = qH + \delta c_1 - (1-\delta)c_2$. Thus, $\widehat{S}_n < C$ is equivalent to

$$n > \frac{\log \frac{C - \frac{b}{1-\rho}}{\widehat{S}_0 - \frac{b}{1-\rho}}}{\log \rho}.$$

Plugging in $\rho$, $b$ and $S^0$, this is equivalent to

$$n > \frac{\log \frac{qH + c_1 - C}{(1-\delta)(c_1+c_2)}}{\log \frac{1}{\delta(1-q)}} = \frac{X + \log(1-\delta)}{Y + \log \delta},$$

where $X = \log(c_1 + c_2) - \log(qH + c_1 - C)$ and $Y = \log(1-q) < 0$. Therefore,

$$\widehat{N}(\delta) = \left\lceil \frac{X + \log(1-\delta)}{Y + \log \delta} \right\rceil.$$

To compute $\lim_{\delta \to 1} \delta^{\widehat{N}(\delta)}$, first consider

$$\begin{aligned} \lim_{\delta \to 1} \log \delta^{\widehat{N}(\delta)} = \lim_{\delta \to 1} \left[ \frac{X + \log(1-\delta)}{Y + \log \delta} \log \delta \right] &= \left( \lim_{\delta \to 1} \frac{1}{Y + \log \delta} \right) \left( \lim_{\delta \to 1} \frac{X + \log(1-\delta)}{(\log \delta)^{-1}} \right) \\ &= \frac{1}{Y} \left( \lim_{\delta \to 1} \frac{X + \log(1-\delta)}{(\log \delta)^{-1}} \right). \end{aligned}$$

Applying l'Hospital's rule twice on the second term on RHS, we obtain

$$\lim_{\delta \to 1} \frac{X + \log(1-\delta)}{(\log \delta)^{-1}} = \lim_{\delta \to 1} \frac{\frac{-1}{1-\delta}}{\frac{-1}{(\log \delta)^2} \frac{1}{\delta}} = \lim_{\delta \to 1} \frac{\delta (\log \delta)^2}{1-\delta} = \lim_{\delta \to 1} \frac{(\log \delta)^2 + 2 \log \delta}{-1} = 0.$$

Therefore, $\lim_{\delta \to 1} \log \delta^{\widehat{N}(\delta)} = 0$, and it follows that $\lim_{\delta \to 1} \delta^{\widehat{N}(\delta)} = 1$.

Now, summarizing the four steps, we have

$$\lim_{\delta \to 1} \mathbf{E}[\delta^\tau] \geq \lim_{\delta \to 1} \delta^{\mathbf{E}[\tau]} \geq \lim_{\delta \to 1} \delta^{\frac{N(\delta)}{2q-1}} \geq \lim_{\delta \to 1} \delta^{\frac{\widehat{N}(\delta)}{2q-1}} = 1.$$

We thus obtain that $\lim_{\delta \to 1} \mathbf{E}[\delta^\tau] = 1$.

Since $p^* \to 0$ as $\delta \to 1$ (Proposition 2), we are ready to compute type $B$'s limit payment at $p = p_1 \in (0, p^{**})$ such that $p_1 \neq \Phi^{-n}(p^{**})$ for any $n = 1, \ldots$. Notice that $\lim_{\delta \to 1} L(p) = R(p)$ by Proposition 2 and $\lim_{\delta \to 1} \overline{S} = qH + c_1$. Therefore,

$$\lim_{\delta \to 1} \left\{ \mathbf{E}[(1-\delta^\tau)(qH + c_1)] + \mathbf{E}\left[ \delta^\tau \left( L(p)C + (1 - L(p))\overline{S} \right) \right] \right\} = R(p)C + (1-R(p))(qH + c_1).$$

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Gambling Reputation:
Repeated Bargaining with Outside Options
*Supplementary Material*[*]


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# 1 Multiple equilibria in the non-generic case

We consider the non-generic case not covered by Theorem 1 in the main text. Fix any $\delta > \bar{\delta}$ as in the proof of Theorem 1. Also, fix any $C \in (\underline{C}, \overline{C}]$. Let $S_0, \ldots, S_N, \ldots$ be the solutions to the recursive equation

$$S_n = (1-\delta)(qH + c_1) + \delta(1-q)S_{n+1} + \delta q S_{n-1} \tag{1}$$

with the initial conditions $S_0 = (1-\delta)(qH+c_1)+\delta(1-q)S_1+\delta q S_0 = qH+\delta c_1-(1-\delta)c_2$, where $N = \sup\{n \in \mathbb{Z} : S_n > C\}$. Define $p^* = \Phi^{-N}(p^{**})$, where $p^{**} = \frac{qH-c_2-C}{(2q-1)H}$.

The case of $S_{N+1} = C$ is non-generic in the following sense. Note that, in order to have $S_K = C$ for some integer $K$, $C$ and $\delta$ must satisfy one of a family of polynomials that are at most countable (recall that such an integer $K$ goes to $\infty$ as $\delta \to 1$). Therefore, for each $C$, the roots in $\delta$ are at most countable.

In the non-generic case, two equilibrium outcomes are possible. The first equilibrium is as reported in Theorem 1 such that, at $p^{**}$, player 2 makes a losing demand for sure and, hence, $S(p^{**}) = S_N$. The other equilibrium is identical to the first equilibrium except for the following:


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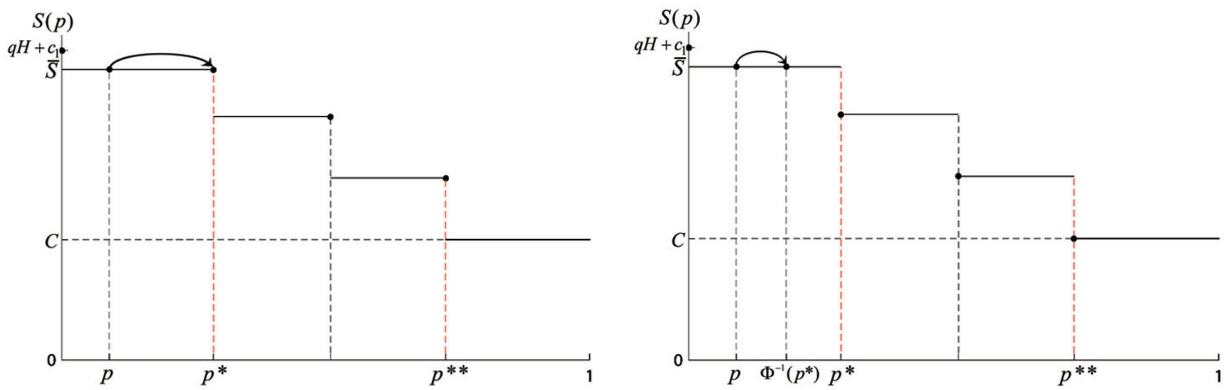
- At any $p \in (0, \Phi^{-1}(p^*))$, type $B$ rejects player 1's demand in a way that posterior immediately after rejection moves to $\Phi^{-1}(p^*)$.

- At any $p \in [\Phi^{-1}(p^*), p^*)$, type $B$ rejects player 1's demand for sure.

- At $p^{**}$, player 2 demands $C$ for sure and type $B$ accepts it for sure. Thus, in this equilibrium, $S(p^{**}) = C$.

The next figure illustrates these two equilibria.

Figure 1: Non-generic equilibria




## 2 When type $G$'s cutoff is $C \in [0, \underline{C}]$

**Proposition 1** *Let $C = \underline{C}$. Fix any $\delta > \frac{c_1+c_2}{(2q-1)H+c_1+c_2}$. There exists a Markov equilibrium. Furthermore, any Markov equilibrium is such that, at any $p \in (0,1)$, a demand $s$ is serious only if $s \in [qH - c_2, qH + c_1]$ and, moreover, every demand is rejected with a positive probability; thus, type $B$'s payment is at least $\overline{S} = qH + \delta c_1 - (1-\delta)c_2$.*

**Proof.** First, It is straightforward to observe that, with $C = \underline{C}$, there exists a Markov equilibrium such that, at any $p \in (0,1)$, player 2 demands $qH + c_1$ and type $B$ rejects it with an arbitrary but interior probability.

Next, fix a Markov equilibrium, and consider any $p \in (0,1)$. Note that

$$p(1-q)H + (1-p)qH - c_2 > (1-q)H - c_2.$$

Thus, player 2 will demand $\underline{C}$ only if type $B$ rejects it for sure. Assume that type $B$ rejects $\underline{C}$ for sure when $\underline{C}$ is demanded. But, it is straightforward to see that accepting such a demand is profitable since acceptance would then show that he is good, a contradiction. Thus, $\underline{C}$ cannot be demanded in equilibrium. On the other hand, it is clear that any demand below $\underline{C}$ will not be made, while given type $G$'s behavior any $s \in (\underline{C}, qH - c_2)$ will be demanded only if it is rejected for sure. Thus, a demand $s$ can be serious only if $s \in [qH - c_2, qH + c_1]$.

Fix any $s \in [qH - c_2, qH + c_1]$. This demand cannot be accepted with probability 1. To see this, suppose otherwise. But then, rejection would show that player 1 is good and, hence, the deviation yields continuation payment $(1-\delta)(qH+c_1)+\delta\underline{C}$, which is less than the equilibrium payment $(1-\delta)s + \delta(qH + c_1)$ given $\delta$. Thus, $s$ must be rejected and the equilibrium payment must be given by rejection.

Since the good type also rejects every equilibrium demand, rejection (and any subsequent third party signal) can never reveal the good type. Moreover, rejection gives one-period expected payment $qH + c_1$. On the other hand, accepting $s$ leads to continuation payment $(1 - \delta)s + \delta(qH + c_1)$. It therefore follows that $S(p) \geq \overline{S}$. ■

**Proposition 2** *Fix $C \in [0, \underline{C})$. Then, the following characterizes all Markov equilibria with sufficiently large $\delta$: player 2 demands a constant demand $s^* \in [qH - c_2, qH + c_1]$ for sure which type $B$ accepts for sure; acceptance leads to a constant demand $qH + c_1$ while rejection (only by type $G$) leads to a constant demand $s' = (\frac{1}{\delta} - 1)s^* + (2 - \frac{1}{\delta})(qH + c_1)$.*

**Proof.** Cases 1-3 in the proof of Proposition 1 (Characterization of serious demands) of the main text remain true. But Case 4 are no longer true with $C < \underline{C}$ since, then, even when $s^*$ is rejected and hence player 1 is known to be type $G$, player 2 will not demand $C$. Since there are only two possible demands in Case 4, $C$ or $s^*$, and $s^*$ is accepted by type $G$ with probability 1, we have the following observations in this case.

(1) Player 2 will never make a demand lower than or equal to $C$. demands lower than $C$ contradict that $B$ plays a cutoff strategy (Lemma 1 in the main text), while a demand $C$ gives player 2 a payoff of at most $pC + (1 - p)(qH - c_2)$. This quantity is strictly less than the payoff from the losing demand $\frac{qH+c_1}{1-\delta}$ because $C < (1 - q)H - c_2$.

(2) The implication of (1) is that type $G$ will always reject player 2's demands.

(3) Therefore, the only demand is $s^* \in (qH - c_2, qH + c_1]$, and it is accepted with probability 1 by type $B$. Then type $B$'s expected payment is

$$(1 - \delta)s^* + \delta(qH + c_1). \tag{2}$$

This payment must not be higher than that from rejection (which leads to the revelation of good type), while the payment from accepting $s^*(p)+\varepsilon$ must not be lower than that from rejection (by the definition of $s^*(p)$) to make rejection incentive compatible. Therefore,

$$(1-\delta)s^* + \delta(qH + c_1) = (1-\delta)(qH + c_1) + \delta s'.$$

where $s$ is the demand made by player 2 when he believes player 1 is type $G$ (note that player 2 can make a very high demand here since the insistent type only accepts a demand below $\underline{C}$ even at $p = 1$). Hence

$$s' = \left(\frac{1}{\delta} - 1\right) s^* + \left(2 - \frac{1}{\delta}\right)(qH + c_1). \tag{3}$$

This implies that $s' \in [s^*(p), qH + c_1]$. Note that type $B$ will accept player 2's demand for sure when $s' < qH + c_1$ in all future periods (but player 2 does not have a deviation because the belief is stuck at 1). It follows from (3) that $s^*$ is independent of belief because $s'$ must be constant (the belief is 1).

(4) Now consider an equilibrium where player 2 demands $s = qH - c_2$. If $s$ is expected to be accepted with probability 1, we need $s' = \left(\frac{1}{\delta} - 1\right)(qH - c_2) + \left(2 - \frac{1}{\delta}\right)(qH + c_1)$ as in (3) above. We now argue that it cannot be accepted by type $B$ with an interior probability. Suppose type $B$ accepts it with an interior probability. Then, because rejection leads to an interior posterior, the lowest possible expected payment for type $B$ after rejection is

$$(1-\delta)(qH + c_1) + (\delta - \delta^2)(qH - c_2) + \delta^2(qH + c_1).$$

This amount is larger than the payment from immediate acceptance of $qH - c_2$. ■

# 3 Non-Markov equilibria

First, we establish a folk theorem for the case of complete information with $p = 0$.

**Proposition 3** *Suppose that player 1's type is known to be $B$. Then, we have the following:*

1. *In any subgame perfect equilibrium, player 1's equilibrium expected payment, $S$, is such that $S \in [qH - c_2, qH + c_1]$.*

2. *Fix any $\delta > \frac{1}{2}$. Then, any $S \in [qH - c_2, qH + c_1]$ can be supported as an equilibrium expected payment of player 1.*

**Proof.** 1. Fix any $\delta$ and any subgame perfect equilibrium.

First, let us show that $S \geq qH - c_2$.Suppose not, so $S < qH - c_2$. Then, since rejecting any offer gives player 1 (one-period) expected payment of $qH + c_1$, acceptance of an offer strictly below $qH - c_2$ must occur at some history on the equilibrium path. Consider player 2 who makes such an offer. But, clearly, this short-run player can improve his expected payoff by instead making any offer at least $qH - c_2$; player 1's rejection gives him payoff $qH - c_2$. Next, let us show that $S \leq qH + c_1$. Suppose not. But, the bad type can guarantee himself payment of $qH + c_1$ by always rejecting.

2. We know that there exists a Markov equilibrium that supports payment $qH + c_1$. Consider any $S \in [qH - c_2, qH + c_1)$ and the following trigger strategy profile:

- At any history in which no deviation from the equilibrium has been observed, player 2 offers $S$ for sure and player 1 accepts an offer if and only if it is less than or equal to $S$.
- At any history in which acceptance of an offer higher than $S$ has been observed, player 2 offers $qH + c_1$ for sure and player 1 accepts an offer if and only if it is less than or equal to $qH + c_1$.
- At any other history, player 2 offers $S$ for sure and player 1 accepts an offer if and only if it is less than or equal to $S$.

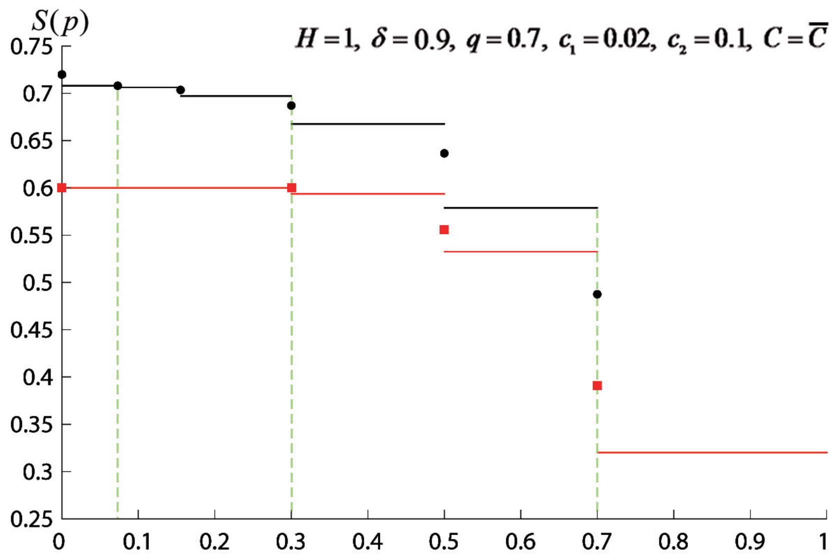
In order to establish that the above profile constitutes a subgame perfect equilibrium, it suffices to consider player 1's incentives when facing a devitating offer $S + \varepsilon$ for small $\varepsilon > 0$. Given the above profile, rejecting the offer yields payment $(1 - \delta)(qH + c_1) + \delta S$, while acceptance leads to $(1 - \delta)(S + \varepsilon) + \delta(qH + c_1)$. Since $\delta > \frac{1}{2}$ and $S < qH + c_1$, it is easily seen that the latter is larger than the former. Thus, player 1 will reject $S + \varepsilon$ for sure. This, in turn, supports optimality of player 2's strategy. ■

## Two constructions of non-Markov equilibrium with incomplete information

Fix any $\delta > \bar{\delta}$ as in the proof of Theorem 1 in the main text. Also, fix any $C \in (\underline{C}, \overline{C}]$.

1. At any history/period $t$ with $p_t > 0$, all players play according to the Markov equilibrium of Theorem 1 for belief $p_t$; at any history with $p^t = 0$, the continuation strategies are given by the equilibrium in which the bad type obtains payment $S^* \in [qH - c_2, qH + c_1)$ (Proposition 3 above). It is straightforward to see that this non-Markov profile only changes the initial condition for the recursive equation (1) above, from $S(p^0) = \overline{S}$ to $(1-\delta)(qH - c_2) + \delta S^*$. We draw below the corresponding equilibrium payments for $S^* = qH - c_2$ (lower lines and squares), together with the equilibrium payments of the Markov equilibrium in the top right panel of Figure 8 (Comparative Statics) in the main text (upper lines and dots).

Figure 2: A non-Markov equilibrium




2. At any history/period $t$ with $p_t < p^{**}$ and $p_t = 1$, all players play according to the Markov equilibrium of Theorem 1 for belief $p_t$. At other histories, the strategies are as follows:

- At any period $t$ such that $p_t \in (p^{**}, 1)$, player 2 offers $C' \in (\underline{C}, \overline{C}]$ and the bad type accept offer $s$ if and only if $s \leq C'$.
- Phase $p^{**}$

- The phase begins at any period $t$ such that $p_t = p^{**}$ and $p_{t-1} < p^{**}$, and lasts for $K+1$ periods.

- In the first $K$ periods of the phase, player 2 offers $C$ for sure and both types of player 1 accept it for sure. Belief upon rejection (off-the-equilibrium) is 0.

- In the last period of the phase, player 2 offers $qH - c_2$ for sure and the bad type reject it for sure.

To establish that the above profile constitutes an equilibrium, consider the equilibrium continuation payment of type $B$ at the beginning of Phase $p^{**}$. It corresponds to

$$(1-\delta^K)C + \delta^K \left[(1-\delta)(qH + c_1) + \delta q S_{N-1} + \delta(1-q)C'\right], \tag{4}$$

where $S_{N-1}$ is derived from the recursive equation (1) above. Note that (4) must equal $S_N$ where

$$S_N = (1-\delta)(qH + c_1) + \delta q S_{N-1} + \delta(1-q)S_{N+1}. \tag{5}$$

If $S_{N+1} < C'$, we have $(1-\delta)(qH + c_1) + \delta q S_{N-1} + \delta(1-q)C' > qS_{N-1} + (1-q)S_{N+1}$; also, $C < qH + c_1$. Then, there exist a positive integer $K$ and $C' \in (\underline{C}, \overline{C}]$ such that (4) equals (5). The players' behavior during the phase are mutually optimal given the stated off-the-equilibrium beliefs.